1.7 MOMENTS

Definition:

The rth moment about origin is $\mu'_r = E[x'_r]$

First moment about origin $\mu'_1 = E[X]$

$= E(X^2) - [E(X)]^2$

Variance $\sigma^2 = \mu'_2 - (\mu'_1)^2$

The rth moment about mean is $\mu_r = E[(X-\mu)^r], where\ \mu\ is\ mean\ of\ X.$

$$\mu_1 = E[(X-\mu)^1]$$

$$= E[X] - E[\mu] = \mu - \mu = 0$$

$$\therefore \mu_1 = 0$$

$$\mu_2 = E[(X-\mu)^2]$$

$$= E[X^2 + \mu^2 - 2X\mu]$$

$$= E[X^2] + \mu^2 - 2E[X]\mu$$

$$= E(X^2) + [E(X)]^2 - 2E(X)E(X)$$

$$= E(X^2) + [E(X)]^2 - 2[E(X)]^2$$

$$= E(X^2) - [E(X)]^2 = \sigma^2$$

$$\therefore \mu_2 = \sigma^2$$

If the probability density of X is given $f(x) = \begin{cases} 2(1-x) & 0 < x < 1 \\ 0 & otherwise \end{cases}$

Find its r^th^ moment about origin. Hence find evaluate $E[(2X+1)^2]$

Solution:

The r^th^ moment about origin is given by

$$\mu'_r = E[x'_r] = \int_0^1 x^r f(x)dx$$

$$= \int_0^1 x^r \, 2(1-x)dx$$

$$= 2\int_0^1 (x^r - x^{r+1})dx$$

$$= 2\left[\frac{x^{r+1}}{r+1} - \frac{x^{r+1+1}}{r+2}\right]_0^1$$

$$= 2\left[\frac{1}{r+1} - \frac{1}{r+2}\right]$$

$$= 2\left[\frac{(r+2)-(r+1)}{(r+2)(r+1)}\right] = \frac{2}{r^2+3r+2}$$

$$E[(2X+1)^2] = E[4X^2 + 4X + 1]$$

$$= 4E[X^2] + 4E[X] + 1 = 4\mu'_2 + 4\mu'_1 + 1$$

$$= 4\frac{2}{2^2+3(2)+2} + 4\frac{2}{2^2+3(2)+2} + 1$$

$$= \frac{8}{12} + \frac{8}{6} + 1 = 3$$

$$\therefore E[(2X+1)^2] = 3$$

1.7 MOMENT GENERATIVG FUNCTION (MGF)

Let X be a random variable. Then the MGF of X is $M_X(t) = E[e^{tx}]$

If X is a discrete random variable, then the MGF is given by

$$M_X(t) = \sum_x e^{tx} p(x)$$

If X is a continuous random variable, then the MGF is given by

$$M_X(t) = \int_{-\infty}^{\infty} e^{tx} f(x)dx$$

Define MGF and why it is called so?

Sol: Let X be a random variable. Then the MGF of X is $M_x(t) = E[e^{tX}]$ Let X be a continuous random variable. Then

$M_X(t) = \int_{-\infty}^{\infty} e^{tx} f(x)dx = \int_{-\infty}^{\infty} \left[1 + \frac{tx}{1!} + \frac{t^2x^2}{2!} + \cdots \frac{t^r x^r}{r!} + \cdots\right] f(x)dx$

$= \int_{-\infty}^{\infty} \left[f(x) + \frac{tx}{1!} f(x) + \frac{t^2x^2}{2!} f(x) + \cdots \frac{t^r x^r}{r!} f(x) + \cdots\right] dx$

$= \int_{-\infty}^{\infty} f(x)dx + \frac{t}{1!}\int_{-\infty}^{\infty} xf(x)dx + \frac{t^2}{2!}\int_{-\infty}^{\infty} x^2 f(x)dx \ldots + \frac{t^r}{r!}\int_{-\infty}^{\infty} x^r f(x)dx + \cdots$

$M_X(t) = 1 + \frac{t}{1!}\mu_1' + \frac{t^2}{2!}\mu_2' + \cdots \frac{t^r}{r!}\mu_r' + \cdots \ldots \ldots$

$\because M_X(t)$ generates moments therefore it is moment generation function

NOTE

If X is a discrete RV and if $M_X(t)$ is known, then $\mu_r' = \left[\frac{d^r}{dt^r}[M_X(t)]\right]_{t=0}$

If X is a continuous RV and if $M_X(t)$ is known, then μ_r'

$$= r! \times \text{ coeff of } t^r \text{ in } M_X(t)$$

PROBLEMS UNDER MGF OF DISCRETE RANDOM VARIABLE

$$M_X(t) = \sum_x e^{tx} p(x)$$

If X is a discrete RV and if $M_X(t)$ is known, then $\mu_r' = \left[\frac{d^r}{dt^r}[M_X(t)]\right]_{t=}$

1. Let *X* be the number occur when a die is thrown. Find the MGF and hence find Mean and Variance of *X*.

Solution:

x	1	2	3	4	5	6
$p(x)$	1/6	1/6	1/6	1/6	1/6	1/6

(i) $M_X(t) = \sum_{x=1}^{6} e^{tx} p(x)$

$= e^t P(1) + e^{2t} P(2) + e^{3t} P(3) + e^{4t} P(4) + e^{5t} P(5) + e^{6t} P(6)$

$= e^t \frac{1}{6} + e^{2t} \frac{1}{6} + e^{3t} \frac{1}{6} + e^{4t} \frac{1}{6} + e^{5t} \frac{1}{6} + e^{6t} \frac{1}{6}$

$M_X(t) = \frac{1}{6}[e^t + e^{2t} + e^{3t} + e^{4t} + e^{5t} + e^{6t}]$

(ii) $E(X) = \left[\frac{d}{dt} M_X(t)\right]_{t=0} = \frac{1}{6}[e^t + 2e^{2t} + 3e^{3t} + 4e^{4t} + 5e^{5t} + 6e^{6t}]_{t=0}$

$= \frac{1}{6}[1 + 2 + 3 + 4 + 5 + 6] = \frac{21}{6}$

$E(X) = 3.5$

$E(X^2) = \left[\frac{d^2}{dt^2}[M_X(t)]\right]_{t=0}$

$$= \frac{1}{6}[e^t + 4e^{2t} + 9e^{3t} + 16e^{4t} + 25e^{5t} + 36e^{6t}]$$

$$= \frac{1}{6}(1 + 4 + 9 + 16 + 25 + 36) = \frac{91}{6}$$

=15.1

(iii) Variance of $X = E(X^2) - [E(X)]^2 = 15.1 - 12.25$

$$\sigma_X = 2.85$$

2. Find the moment generating function for the distribution

where $(X = x) = \begin{cases} \frac{2}{3}; x = 1 \\ \frac{1}{3}; x = 2 \\ 0 \text{ ; otherwise} \end{cases}$ **. Also find its mean & variance,**

Sol: The probability distribution of X is given by

x	1	2
$p(x)$	$2/3$	$1/3$

$$M_X(t) = E[e^{tx}] = \sum_{x=1}^{2} e^{tx}p(x)$$

$$= e^t p(X = 1) + e^{2t}p(X = 2) = e^t \frac{2}{3} + e^{2t}\frac{1}{3}$$

$$M_X(t) = \frac{1}{3}(2e^t + e^{2t})$$

$$E(X) = M_X'(0)$$

$$= \left[\frac{d}{dt}\left[\frac{1}{3}(2e^t + e^{2t})\right]\right]_{t=0} \qquad = \frac{1}{3}(2e^t + 2e^{2t})$$

E(X)= $\frac{4}{3}$

$$E(X^2) = M_X''(0) = \left[\frac{d^2}{dt^2}\left[\frac{1}{3}(2e^t + e^{2t})\right]\right]_{t=0}$$

$$= \left[\frac{d}{dt}\left[\frac{1}{3}(2e^t + 2e^{2t})\right]\right]_{t=0}$$

$$= \left[\frac{1}{3}(2e^t + 4e^{2t})_{t=0}\right] = \frac{6}{3} = 2$$

Variance of $X = E(X^2) - [E(X)]^2 = 2 - \left(\frac{4}{3}\right)^2$

$$\text{Var}(X) = \frac{2}{9}$$

3. Let X be a RV with PMF $P(x) = \left(\frac{1}{2}\right)^x$; $x = 1, 2, 3$, ... Find MGF and hence find mean and variance of X.

Sol:

(i) $M_X(t) = E[e^{tX}]$

$= \sum_{x=1}^{\infty} e^{tx} p(x)$

$= \sum_{x=1}^{\infty} e^{tx}\left(\frac{1}{2}\right)^x = \sum_{x=1}^{\infty}\left(\frac{e^t}{2}\right)^x$

$= \left[\frac{e^t}{2} + \left(\frac{e^t}{2}\right)^2 + \left(\frac{e^t}{2}\right)^3 + \cdots\right] = \frac{e^t}{2}\left(1 + \frac{e^t}{2} + \left(\frac{e^t}{2}\right)^2 + \cdots\right)$

$= \frac{e^t}{2}\left(1 - \frac{e^t}{2}\right)^{-1} = \frac{e^t}{2}\frac{e^t}{2-e^t}$

$M_X(t) = \frac{e^t}{2-e^t}$

$(ii)\ E(X) = \left[\frac{d}{dt}[M_X(t)]\right]_{t=0}$

$= \left[\frac{d}{dt}\left(\frac{e^t}{2-e^t}\right)\right]_{t=0}$

$= \left[\frac{(2-e^t)e^t - e^t(0-e^t)}{(2-e^t)^2}\right]_{t=0}$

$$= \left[\frac{2e^t - e^{2t} + e^{2t}}{(2-e^t)^2}\right]_{t=0} \quad \because d\left(\frac{u}{v}\right) = \frac{vu' - uv'}{v^2}$$

$$= \left[\left[\frac{2e^t}{(2-e^t)^2}\right]_{t=0}\right] = \frac{2}{1}$$

$$E(X) = 2$$

$$E(X^2) = \left[\frac{d^2}{dt^2}[M_X(t)]\right]_{t=0}$$

$$= \left[\frac{d}{dt}\left(\frac{2e^t}{(2-e^t)^2}\right)\right]_{t=0}$$

$$= \left[\frac{(2-e^t)^2 2e^t - 2e^t 2(2-e^t)(-e^t)}{(2-e^t)^4}\right]_{t=0} = \frac{2+4}{1} = 6$$

(iii) Variance $= E(X^2) - [E(X)]^2 = 6 - 4$

$$\text{Var}(X) = 2$$

PROBLEMS UNDER MGF OF DISCRETE RANDOM VARIABLE

$$M_X(t) = \int_{-\infty}^{\infty} e^{tx} f(x) dx$$

If X is a continuous RV and if $M_X(t)$ is known, then μ_r'

$$= r! \times \text{ coeff of } t^r \text{ in } M_X(t)$$

1. If a random variable "X" has the MGF, $M_X(t) = \frac{2}{2-t}$, find the variance of X.

Solution:

Given $M_X(t) = \frac{2}{2-t} = 2(2-t)^{-1}$

$$M_X'(t) = -2(2-t)^{-2}(-1)$$

$$= 2(2-t)^{-2}$$

$$M_X'(t=0) = 2(2-0)^{-2} = \frac{2}{4} = \frac{1}{2}$$

$$M_X''(t) = -4(2-t)^{-3}(-1)$$

$$= 4(2-t)^{-3}$$

$$M_X''(t=0) = 4(2-0)^{-3} = \frac{4}{8} = \frac{1}{2}$$

$$Var(X) = E(X^2) - E[(X)]^2$$

$$= \frac{1}{2} - \frac{1}{4} = \frac{1}{4}$$

2. **Let X be a RV with PDF $f(x) = ke^{-2x}, x \geq 0$. Find (i) k, (ii) MGF, (iii) Mean and (iv) variance**

Sol: Given: $f(x) = ke^{-2x}; 0 \leq x < \infty$

i) To find k:

$$\int_0^\infty f(x)dx = 1 \Rightarrow \int_0^\infty ke^{-2x}dx = 1k\left[\frac{e^{-2x}}{-2}\right]_0^\infty = 1 \Rightarrow \frac{k}{-2}(e^{-\infty} - 1)$$

$$= 1$$

$$\frac{k}{-2}(0-1) = 1 \Rightarrow \quad \frac{k}{2} = 1$$

$$k = 2$$

(ii) $M_X(t) = E[e^{tx}] \qquad = \int_0^\infty e^{tx} f(x)dx$

$$= 2\int_0^\infty e^{tx}e^{-2x}dx = 2\int_0^\infty e^{tx-2x}dx$$

$$= 2\int_0^\infty e^{-(2-t)x}dx = 2\left[\frac{e^{-(2-t)x}}{-(2-t)}\right]_0^\infty = 2\left(0 + \frac{1}{2-t}\right)$$

$M_X(t) = \frac{2}{2-t}$

(iii) To find Mean and Variance:

$$M_X(t) = \frac{2}{2\left(1-\frac{t}{2}\right)} = \left(1-\frac{t}{2}\right)^{-1} = 1 + \frac{t}{2} + \frac{t^2}{2^2} + \cdots$$

Coefficient of $t = \frac{1}{2}$ Coefficient of $t^2 = \frac{1}{2^2}$

Mean $E(X) = \mu_1' = 1! \times$ coefficient of $t \Rightarrow E(X) = \frac{1}{2}$

$$E(X^2) = 2! \times \text{coefficient of } t^2 = 2 \times \frac{1}{2^2} = \frac{1}{2}$$

(iv) Variance $= E(X^2) - [E(X)]^2 = \frac{1}{2} - \frac{1}{4} = \frac{2-1}{4}$

$$\text{Var}(X) = \frac{1}{4}$$

3. Let X be a continuous RV with PDF $f(x) = Ae^{\frac{-x}{3}}; x \geq 0$. Find

(i) A, (ii) MGF,(iii) Mean and (iv) variance

Sol: Given: $f(x) = Ae^{\frac{-x}{3}}; 0 \leq x \leq \infty$

(i) To find A :

$$\int_0^\infty f(x)dx = 1 \Rightarrow \int_0^\infty Ae^{\frac{-x}{3}}dx = 1A\left[\frac{e^{\frac{-x}{3}}}{-\frac{1}{3}}\right]_0^\infty = 1 \Rightarrow -3A(0-1) = 1$$

$$3A = 1 \Rightarrow A = \frac{1}{3}$$

$$\therefore f(x) = \frac{1}{3}e^{\frac{-x}{3}}; 0 \leq x \leq \infty$$

(ii) $M_X(t) = E[e^{tx}] = \int_0^\infty e^{tx}f(x)dx = \frac{1}{3}\int_0^\infty e^{tx}e^{\frac{-x}{3}}dx$

$$= \frac{1}{3}\int_0^\infty e^{tx-\frac{x}{3}}dx = \frac{1}{3}\int_0^\infty e^{-\left(\frac{1}{3}-t\right)x}dx = \frac{1}{3}\left[\frac{e^{-\left(\frac{1}{3}-t\right)x}}{-\left(\frac{1}{3}-t\right)}\right]_0^\infty$$

$$= \frac{1}{3}\left[0 + \frac{1}{\frac{1}{3}-t}\right] = \frac{1}{3}\frac{1}{\frac{1-3t}{3}}$$

$$= (1-3t)^{-1}$$

(iii) To find mean and variance:

$M_X(t) = (1-3t)^{-1}$

$= 1 + 3t + 9t^2 + 27t^3 + \cdots$

coefficient of $t = 3$ coefficient of $t^2 = 9$

E(X)=1! X coefficient of t in $M_X(t) = 1$ X 3

Mean = 3

E(X)=2! X coefficient of t^2 in $M_X(t)$

= 2X9 =18

(iv)Variance = $E(X^2) - [E(X)]^2$ = 18-9

Var (X) =9

4. Let X be a continuous RV with PDF $f(x)$

$$= \begin{cases} x & ; 0 < x < 1 \\ 2 - x & ; 1 < x < 2 \\ 0 & ; \text{elsewhere} \end{cases}$$

. Find (i) MGF, (ii) Mean and variance.

Sol: Since X is defined in the region $0 < x < 2, X$ is a continuous RV.

$M_X(t) = E[e^{tX}] = \int_0^2 e^{tx} f(x)dx$

$= \int_0^1 e^{tx} x dx + \int_1^2 e^{tx}(2-x)dx = \int_0^1 x e^{tx} dx + \int_1^2 (2-x)e^{tx} dx$

$= \left[x\left(\frac{e^{tx}}{t}\right) - 1\left(\frac{e^{tx}}{t^2}\right)\right]_0^1 + \left[(2-x)\frac{e^{tx}}{t} - (-1)\frac{e^{tx}}{t^2}\right]_1^2$

$= \left[1\left(\frac{e^t}{t}\right) - 1\left(\frac{e^t}{t^2}\right) - \left(\frac{-1}{t^2}\right)\right] + \left[0 + \frac{e^{2t}}{t^2} - \frac{e^t}{t} - \frac{e^t}{t^2}\right]$

$$= \left[\frac{e^t}{t} - \frac{e^t}{t^2} + \frac{1}{t^2} + \frac{e^{2t}}{t^2} - \frac{e^t}{t} - \frac{e^t}{t^2}\right] = \frac{1}{t^2} - \frac{2e^t}{t^2} + \frac{e^{2t}}{t^2}$$

$$M_X(t) = \frac{1 - 2e^t + e^{2t}}{t^2}$$

To find Mean and Variance:

$$M_X(t) = \frac{1}{t^2}\left[1 - 2\left(1 + \frac{t}{1!} + \frac{t^2}{2!} + \frac{t^3}{3!} + \frac{t^4}{4!} + \cdots\right) + \left(1 + \frac{2t}{1!} + \frac{2^2t^2}{2!} + \frac{2^3t^3}{3!} + \frac{2^4t^4}{4!} + \cdots\right)\right]$$

$$\mu_r' = r! \times \text{coefficient of } t^r$$

$$\text{Coefficient of } t = -\frac{2}{3!} + \frac{2^3}{31} = \frac{-2}{6} + \frac{8}{6} = 1$$

$$\text{Coefficient of } t^2 = -\frac{2}{4!} + \frac{2^4}{4!} = \frac{14}{24} = \frac{7}{12}$$

$$\mu_1' = 1! \times \text{coefficient of } t$$

$$\mu_1' = 1$$

Mean $= 1$

$$\mu_2' = 2! \times \text{coefficient of } t^2; \mu_2' = 2 \times \frac{7}{12} = \frac{7}{6}$$

Variance $= \mu_2' - (\mu_1')^2 = \frac{7}{6} - 1$

Var(X) $= \frac{1}{6}$

5. **Let X be a continuous random variable with PDF $f(x)\frac{1}{2a}; -a < x < a$. Then find the M.G.F of X.**

Sol: X is a continuous random variable defined in $-a < x < a$.

$$M_x(t) = E[e^{tx}]$$

$$= \int_{-a}^{a} e^{tx} f(x) dx$$

$$= \int_{-a}^{a} e^{tx} \frac{1}{2a} dx$$

$$= \frac{1}{2a}\left(\frac{e^{tx}}{t}\right)^{a} \quad e^{x} - e^{-x} = 2\sin hx$$

$$= \frac{1}{2a}\left(\frac{e^{ta} - e^{-ta}}{t}\right)$$

$$= \frac{1}{2a}\frac{2\sinh at}{t}$$

$$M_X(t) = \frac{\sinh at}{at}$$

Discrete Distributions:

(i) Binomial Distribution

(ii) Poisson Distribution

(iii) Geometric Distribution

Continuous Distributions:

(i) Exponential Distribution

(ii) Uniform Distribution

(iii) Normal Distribution