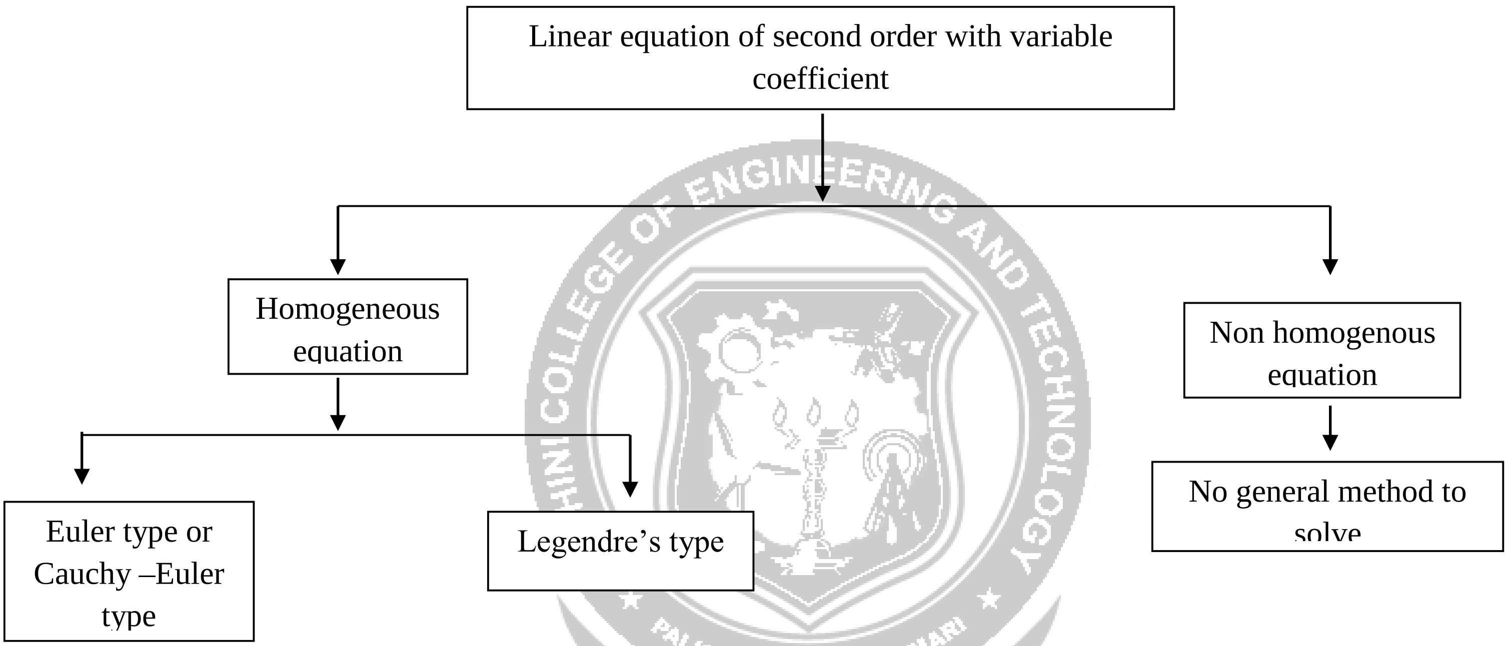
Homogeneous equation of Euler's and Legendre's type

The general form of linear equation of second order is given by $\frac{d^2y}{dx^2} + p\frac{dy}{dx} + Qy = R$

Where P,Q and R are functions of x only



Homogeneous equation of Euler type (Cauchy type)

Linear differential equation with variable coefficients:

An equation of the form

$$\frac{x^n d^n y}{dx^n} + a_1 x^{n-1}\frac{d^{n-1}y}{dx^{n-1}} + a_2 x^{n-2}\frac{d^{n-2}y}{dx^{n-2}} + \cdots \dots . + a_n y = f(x)$$

Where $a_1, a_2, \dots, a_n$ are constants and $f(x)$ is a function of x

Equation (1) can be reduced to linear differential equation with constant coefficients by putting substitution

$x = e^t$ (or) t= log x

Now $\frac{dy}{dx} = \frac{dy}{dt}\,\frac{dt}{dx} = \frac{1}{x}\,\frac{dy}{dt}$

$$x\frac{dy}{dx} = \frac{dy}{dt}\ (or)\ \ x\frac{dy}{dx} = Dy \quad where\ D = \frac{d}{dt}$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left[\frac{1}{x}\quad\frac{dy}{dt}\right]$$

$$= \frac{-1}{x^2}\frac{dy}{dt} + \frac{1}{x}\frac{d}{dx}\left(\frac{dy}{dt}\right)$$

$$= \frac{-1}{x^2}\frac{dy}{dt} + \frac{1}{x}\frac{d}{dx}\left(\frac{dy}{dt}\right)\frac{dt}{dx}$$

$$= \frac{-1}{x^2}\frac{dy}{dt} + \frac{1}{x}\,\frac{d^2y}{dt^2}\left(\frac{1}{x}\right)$$

$$= \frac{-1}{x^2}\frac{dy}{dt} + \frac{1}{x^2}\,\frac{d^2y}{dt^2}$$

$$\frac{x^2d^2y}{dx^2} = \frac{d^2y}{dt^2} - \frac{dy}{dt}$$

$$= (D^2 - D)y$$

$$= D(D-1)y$$

Similarly $x^3\frac{d^3y}{dx^3} = D(D-1)(D-2)y$ and so on

Problems based on Cauchy's type

Example1

Solve $(\boldsymbol{x^2D^2 + 4xD + 2})\boldsymbol{y} = \boldsymbol{x^2 + \frac{1}{x^2}}$

Solution:

Put t = log$x \Rightarrow x = e^t$

$$xD = D$$

$$x^2D^2 = D(D-1)$$

$$\therefore [D(D-1) + 4D + 2]y = (e^t)^2 + \frac{1}{(e^t)^2}$$

$$(D^2 - D + 4D + 2)y = e^{2t} + e^{-2t}$$

$$(D^2 + 3D + 2)y = e^{2t} + e^{-2t}$$

Auxiliary Equation is $m^2 + 3m + 2 = 0$

$$(m+1)(m+2) = 0$$

$$m = -1, -2$$

C.F = $Ae^{-t} + Be^{-2t}$

$P.I_1 = \frac{1}{D^2+3D+2}\ e^{2t}$ Replace D by 2

$$= \frac{1}{4+6+2}\ e^{2t}$$

$$= \frac{1}{12}\ e^{2t}$$

$P.I_2 = \frac{1}{D^2+3D+2}\ e^{-2t}$ Replace D by -2

$$= \frac{1}{0}\ e^{-2t}$$

$$= \frac{t}{2D+3}\ e^{-2t}$$

$= -te^{-2t}$ Replace D by -2

The general solution is y = C.F+ $P.I_1 + P.I_2$

$$= Ae^{-t} + Be^{-2t} + \frac{e^{2t}}{12} - te^{-2t}$$

$$= Ax^{-1} + Bx^{-2} + \frac{x^2}{12} - x^{-2}\log x$$

Example:2

Solve $(x^2D^2 - xD + 4)y = \sin(\log x)$

Solution:

Put $t = log\, x \Rightarrow x = e^t$

$xD = D$

$x^2D^2 = D(D-1)$

$[D(D-1) - D + 4]y = \sin t$

$(D^2 - 2D + 4D)y = \sin t$

Auxiliary Equation is $m^2 - 2m + 4 = 0$

$$m = \frac{2 \pm \sqrt{4-16}}{2}$$

$$= \frac{2 \pm \sqrt{-12}}{2}$$

$$= \frac{2 \pm 2i\sqrt{3}}{2} \qquad = 1 \pm i\sqrt{3}$$

$$\alpha = 1 \quad \beta = \sqrt{3}$$

C.F = $e^t\left[A\, cos\, \sqrt{3}\, t + B \sin \sqrt{3}\ t\right]$

$P.I = \frac{1}{D^2 - 2D + 4}\ sint$ Replace $D^2\ by - 1$

$$= \frac{1}{-1-2D+4}\ sint$$

$$= \frac{1}{3-2D}\ sint$$

$$= \frac{3+2D}{(3+2D)(3-2D)}\ sint$$

$$= \frac{(3+2D)}{9-4D^2}\ sint \qquad \text{Replace } D^2\ by-1$$

$$= \frac{3\sin t+2D(\sin t)}{9-16(-1)}\ sint$$

$$= \frac{3\sin t+2\cos t}{25}$$

The general solution is y = C.F +P.I

$$y = e^t\left[Acos\sqrt{3}t + Bsin\sqrt{3}t\right] + \frac{3\sin t+2\cos t}{25}$$

$$= x\left[Acos\left(\sqrt{3}\log x\right) + Bsin\left(\sqrt{3}\log x\right)\right] + \frac{3\sin(\log x)+2\cos(\log x)}{25}$$

Example: 3

Solve $(x^2D^2 - 2xD - 4)y = 16(\log x)^2$

Solution:

Put $t = log \Rightarrow x = e^t$

$$[D(D-1) - 2D - 4]y = 16t^2$$

$$(D^2 - 3D - 4)y = 16t^2$$

$$A.E\ is\ m^2 - 3m - 4 = 0$$

$$m = -1,4$$

$$\text{C.F} = Ae^{-t} + Be^{4t}$$

$$P.I = \frac{1}{D^2-3D-4} 16t^2$$

$$= 16 \frac{1}{-4\left[1-\left(\frac{D^2-3D}{4}\right)\right]} t^2$$

$$= -4\left[1-\left(\frac{D^2-3D}{4}\right)\right]^{-1} t^2$$

$$= -4\left[1+\frac{D^2-3D}{4}+\left(\frac{D^2-3D}{4}\right)^2\right] t^2$$

$$= -4\left[1+\frac{D^2}{4}-\frac{3D}{4}+\frac{9D^2}{16}\right] t^2$$

$$= -4\left[t^2+\frac{2}{4}-\frac{6t}{4}+\frac{9}{16}(2)\right]$$

$$= -4\left[t^2+\frac{2}{4}-\frac{6t}{4}+\frac{9}{8}\right]$$

$$= -\frac{1}{2}(8t^2-12t+13)$$

The general solution is $y =$ C.F+P.I

$$y = Ae^{-t}+Be^{4t}-\frac{1}{2}(8t^2-12t+13)$$

$$= Ax^{-1}+Bx^4-\frac{1}{2}[8(\log x)^2-12(\log x)+13]$$

Example:4

Solve $(x^2D^2-xD+4)y = x^2\sin(\log x)$

Solution:

Put $t = log\ x \Rightarrow x = e^t$

$xD = D$

$$x^2D^2 = D(D-1)$$

$$[D(D-1) - D + 4]y \quad = e^{2t}\sin t$$

$$(D^2 - 2D + 4)y = e^{2t}\sin t$$

Auxiliary Equation is $\quad m^2 - 2m + 4 = 0$

$$m = \frac{2 \pm \sqrt{4-16}}{2}$$

$$= \frac{2 \pm 2i\sqrt{3}}{2}$$

$$= 1 \pm i\sqrt{3} \qquad \alpha = 1 \;\; \beta = \sqrt{3}$$

$$\text{C.F} = e^{t}\left[A\,cos\,\sqrt{3}\,t + B\sin\sqrt{3}\;\,t\right]$$

$$P.I = \frac{1}{D^2-2D+4} e^{2t}\; sint$$ Replace $D\; by\; D+2$

$$= e^{2t}\frac{1}{(D+2)^2 - 2(D+2) + 4}\; sint$$

$$= e^{2t}\frac{1}{D^2+4D+4-2D-4+4}\; sint$$

$$= e^{2t}\frac{1}{D^2+2D+4}\; sint$$

$$= e^{2t}\frac{1}{-1+2D+4}\; sint$$

$$= e^{2t}\frac{1}{3+2D}\; sint$$

$$= e^{2t}\frac{3-2D}{(3-2D)(3+2D)}\sin t$$

$$= e^{2t}\frac{3-2D}{9-4D^2}\sin t$$ Replace $D^2\; by -1$

$$= \frac{e^{2t}(3\sin t - 2\cos t)}{13}$$

The general solution is $y =$ C.F+P.I

$$y = e^{t}\left[Acos\sqrt{3}t + Bsin\,\sqrt{3}t\right] + e^{2t}\frac{[3\sin t - 2\cos t]}{13}$$

$$= x\left[Acos\left(\sqrt{3}\log x\right) + Bsin\,\left(\sqrt{3}\log x\right)\right] + \frac{x^2(3\sin(\log x) - 2\cos(\log x))}{13}$$

Example:5

Solve $(x^2D^2 - 3xD + 4)y = \mathrm{x}^2\cos(\log x)$

Solution:

Put $log\,x = t \Rightarrow x = e^{t}$

$xD = D$

$x^2D^2 = D(D-1)$

$[D(D-1) - 3D + 4]y = e^{2t}\sin t$

$(D^2 - 4D + 4)y = e^{2t}\sin t$

Auxiliary Equation is $m^2 - 4m + 4 = 0$

$m = 2, 2$

C.F = $(At + B)e^{2t}$

$P.I = \frac{1}{D^2 - 4D + 4}\, e^{2t}sint$

$= \frac{1}{(D-2)^2}\, e^{2t}sint$ Replace $D\ by\ D + 2$

$= e^{2t}\frac{1}{(D+2-2)^2}\, sin\,t$

$= e^{2t}\frac{1}{D^2}\sin t$ Replace $D\ by - 1$

$$= e^{2t}\frac{1}{-1}\sin t \ = -e^{2t}\sin t$$

The general solution is $y =$ C.F + P.I

$$y = (At + B)e^{2t} - e^{2t}\cos t$$

$$= (Alog\ x + B)x^2 - x^2\cos\log x$$

Example :6

Solve $(\boldsymbol{x^2D^2 - xD + 1)y = \left(\frac{\log x}{x}\right)^2}$

Solution:

Put $t = log\ x \Rightarrow x = e^t$

$$xD = D$$

$$x^2D^2 = D(D-1)$$

$$[D(D-1) - D + 1]y = \left(\frac{t}{e^t}\right)^2$$

$$(D^2 - 2D + 1)y = t^2e^{-2t}$$

Auxiliary Equation is $m^2 - 2m + 1 = 0$

$$m = 1, 1$$

C.F = $(At + B)e^t$

$$P.I = \frac{1}{D^2 - 2D + 1}\ e^{-2t}t^2$$

$$= \frac{1}{(D-1)^2}\ e^{-2t}t^2 \qquad \text{Replace} \quad D\ by\ D-2$$

$$= e^{-2t}\frac{1}{(D-2-1)^2}\ t^2$$

$$= e^{-2t}\frac{1}{(D-3)^2}t^2$$

$$= e^{-2t}\frac{1}{9\left(1-\frac{D}{3}\right)^2}t^2$$

$$= \frac{e^{-2t}}{9}\left(1-\frac{D}{3}\right)^2 t^2$$

$$= \frac{e^{-2t}}{9}\left[1+\frac{2D}{3}+3\left(\frac{D}{3}\right)^2\right]t^2$$

$$= \frac{e^{-2t}}{9}\left[t^2+\frac{2D}{3}(t^2)+\frac{3}{9}D^2(t^3)\right]$$

$$= \frac{e^{-2t}}{9}\left[\frac{3t^2+4t+2}{3}\right]$$

$$= \frac{e^{-2t}}{27}[3t^2+4t+2]$$

The general solution is y = C.F + P.I

$$y = (At+B)e^t + \frac{e^{2t}}{27}(3t^2+4t+2)$$

$$= (Alog\ x+B)x + \frac{1}{27x^2}[3(\log x)^2 - 4\log x + 2]$$

Legendre’s Linear Differential Equation

An equation of the form

$$(\boldsymbol{ax}+\boldsymbol{b})^{\boldsymbol{n}}\frac{\boldsymbol{d^n y}}{\boldsymbol{dx^n}} + \boldsymbol{k_j}(\boldsymbol{ax}+\boldsymbol{b})^{\boldsymbol{n-1}}\frac{\boldsymbol{d^{n-1}y}}{\boldsymbol{dx^{n-1}}} + \ldots + \boldsymbol{x_n y} = \boldsymbol{0} \ldots (\boldsymbol{1})$$

Where $k's$ are constant and Q is a function of x is called such equations are reduced by using substitution

$$ax+b=e^t$$

$$t = \log(ax+b)$$

$$(ax+b)D = a\,D$$

$$(ax+b)^2D^2 = a^2D(D-1) \text{ and so on.}$$

After making these substitution in (1) it reduces to a linear differential equation with constant coefficients

Problems based on Legendre's Linear Differential Equation

Example: 1

Solve $(3x+2)^2y'' + 3(3x+2)y' - 36y = 3x^2 + 4x + 3$

Solution:

Put t = log (3x+2)

$$3x+2 = e^t; \quad x = \frac{e^t-2}{3}$$

(3 x +2) D = 3D

$$(3x+2)^2\,D^2 = 9D(D-1)$$

$$[9D(D-1) + 9D - 36]y \quad = 3\left(\frac{e^t-2}{3}\right)^2 + 4\left(\frac{e^t-2}{3}\right) + 1$$

$$[9D^2 - 9D + 9D - 36]y \quad = 3\frac{(e^{2t}-4e^t+4)}{9} + 4\left(\frac{e^t-2}{3}\right) + 1$$

$$[9D^2 - 36]y \quad = \frac{(e^{2t}-4e^t+4e^t+4-8+3)}{3}$$

$$= \frac{e^{2t}-1}{3}$$

$$(D^2-4)y = \frac{e^{2t}-1}{27}$$

$$(D^2-4)y = \frac{1}{27}e^{2t} - \frac{1}{27}$$

Auxiliary Equation is $\quad m^2 - 4 = 0$

$$m^2 = 4$$

$$m = \pm 2$$

C.F $= Ae^{2t} + Be^{-2t}$

$$P.I = \frac{1}{D^2-4}\,\frac{1}{27}e^{2t} \qquad \text{Replace D by 2}$$

$$= \frac{1}{0}\frac{1}{27}\,e^{2t}$$

$$= \frac{t}{2D}\frac{1}{27}\,e^{2t}$$

$$= \frac{t}{108}\,e^{2t}$$

$$P.I_2 \quad = \frac{1}{D^2-4}\left(\frac{-1}{27}\right) \qquad \text{Replace } D \ by \ 0$$

$$= \frac{1}{-4}\left(\frac{-1}{27}\right)$$

$$= \frac{1}{108}$$

The general solution is y = C.F + P.I

$$y = Ae^{2t} + Be^{-2t} + \frac{t}{108}e^{2t} + \frac{1}{108}$$

$$y = A(3\,x+2)^2 + B(3x+2)^{-2} + (3x+2)^2\frac{\log x}{108} + \frac{1}{108}$$

Example: 2

Solve $\boldsymbol{(1+x)^2\frac{d^2y}{dx^2} + (1+x)\frac{dy}{dx} + y = 4\cos[\log(1+x)]}$

Solution:

Put t = log (1+x)

$(1+x) = e^t$

$(1+x)^2 D^2 = D(D-1)$

$[D(D-1) + D + 1]y = 4\cos t$

$[D^2+1]y \quad = 4\cos t$

Auxiliary Equation is $m^2 + 1 = 0$

$m^2 = -1$

$m = \pm i$

$\alpha = 0, \beta = 1$

C.F $= A cos\ t + B sin\ t$

$P.I = \frac{1}{D^2+1} 4\cos t$ Replace D $by -1$

$= \frac{1}{-1+1} 4\cos t$

$= \frac{t}{2D} 4\cos t$

$= 2t\sin t$

The general solution is y = C.F+P.I

$y = A cos\ t + B sin\ t + 2t\ sin\ t$

$y = A cos \log(1+x) + B sin \log(1+x) + 2\log(1+x)\sin\log(1+x)$

Example:3

Solve $(\boldsymbol{x}+\mathbf{2})^2\frac{\boldsymbol{d^2y}}{\boldsymbol{dx^2}}-(\boldsymbol{x}+\mathbf{2})\frac{\boldsymbol{dy}}{\boldsymbol{dx}}+\boldsymbol{y}=\mathbf{3}\boldsymbol{x}+\mathbf{4}$

Solution:

Put $x+2 \; = e^t$

$t=\log(x+2)$

$x=e^t-2$

$(x+2)D=D$

$(x+2)^2D^2=D(D-1)$

$[D(D-1)-D+1]y=3(e^t-2)+4$

$[D^2-2D+1]y \; = 3e^t-2$

Auxiliary Equation is $m^2-2m+1=0$

$(m-1)^2=0$

$m=1,1$

C.F = $(At+B)e^t$

$=[Alog\,(x+2)+B](x+2)$

$P.I_1=\frac{1}{(D-1)^2}\,3e^t$ Replace $D\; by\; 1$

$=3\frac{1}{(1-1)^2}\,e^t$

$=3\frac{1}{0}\,e^t$

$=\frac{3t}{2(D-1)}\,e^t$ Replace $D\; by\; 1$

$$= \frac{3t^2}{2} e^t$$

$$P.I_2 = \frac{1}{(D-1)^2} \left(-2e^{0t}\right) \qquad \text{Replace } D \text{ by } 0$$

$$= -2$$

The general solution is $y = C.F + P.I_1 + P.I_2$

$$y = [A\log(x+2) + B](x+2) + \frac{3}{2}[\log(x+2)]^2(x+2) - 2$$

Example:4

Solve$(2x+2)^2 \frac{d^2y}{dx^2} - (2x+3)\frac{dy}{dx} + 2y = 6x$

Solution:

Put 2x+3 $= e^t$

$$t = \log(2x+3)$$

$$(2x+3)D = 2D$$

$$(2x+3)^2 D^2 = 4D(D-1)$$

$$[4D(D-1) - 2D - 12]y = 6\left[\frac{e^t - 3}{2}\right]$$

$$[4D^2 - 4D - 2D - 12]y = 3e^t - 9$$

$$[4D^2 - 6D - 12]y = 3e^t - 9$$

$$\left[D^2 - \frac{3}{2}D - 3\right]y = \frac{1}{4}\left(3e^t - 9\right)$$

Auxiliary Equation is $m^2 - \frac{3}{2}m - 3 = 0$

$$m = \frac{\frac{3}{2} \pm \sqrt{\frac{9}{4}+12}}{2}$$

$$m = \frac{\frac{3}{2} \pm \sqrt{\frac{57}{4}}}{2} = \frac{3 \pm \sqrt{57}}{4}$$

Let a $= \frac{3+\sqrt{57}}{4}$, $b = \frac{3-\sqrt{57}}{4}$

C.F = $Ae^{at} + Be^{bt}$

$$= A(2x+3)^a + B(2x+3)^b$$

$$P.I_1 = \frac{1}{D^2 - \frac{3}{2}D - 3}\left(\frac{3e^t}{4}\right)$$

$$= \frac{3}{4}\ \frac{1}{1-\frac{3}{2}-3}\ e^t$$ Replace $D\ by\ 1$

$$= \frac{3}{4}\left(\frac{-2}{7}\right)e^t$$

$$= \frac{-3}{14}\ e^t$$

$$= \frac{-3}{14}\ (2x+3)$$

$$P.I_2 = \frac{1}{D^2 - 3D - 3}\ \left(\frac{-9}{4}e^{0t}\right)$$ Replace $D\ by\ 0$

$$= \frac{3}{4}$$

The general solution is $y = C.F + P.I_1 + P.I_2$

$$y = [A(2x+3)^a + B(2x+3)^b] - \frac{3}{14}(2x+3) + \frac{3}{4}$$

Where $a = \frac{3+\sqrt{57}}{4}$; $b = \frac{3-\sqrt{57}}{4}$

Example: 5

Solve $(\mathbf{1}+\boldsymbol{x})^2\frac{\boldsymbol{d^2y}}{\boldsymbol{dx^2}}+(\mathbf{1}+\boldsymbol{x})\frac{\boldsymbol{dy}}{\boldsymbol{dx}}+\boldsymbol{y}=\mathbf{2}\,\mathbf{sin}[\mathbf{log}(\mathbf{1}+\boldsymbol{x})]$

Solution:

Put $1+\ x\ \ =e^t$

$t=\log(1+x)$

$(1+x)\frac{dy}{dx}=Dy$

$(1+x)^2\frac{d^2y}{dx^2}=D(D-1)y$

$[D(D-1)+D+1]y=2\sin t$

$[D^2+1]y\quad=2\sin t$

Auxiliary Equation is $m^2+1=0$

$m=\pm i$

C.F = $Acos\ t+Bsin\ t$

$=Acos\ [\log(1+x)]+Bsin\ [\log(1+x)]$

$P.I=\frac{1}{D^2+1}\ 2\sin t$ Replace $D^2 by-1$

$=2\ \frac{1}{0}\sin t$

$=\frac{2t}{2D}\sin t$

$=\frac{t}{D}\sin t$

$=-tcos\ t$

$=-\log(1+x)\ cos[\log(1+x)]$

The general solution is $y = C.F + P.I$

$$y = Acos[\log(1+x)] + Bsin[\log(1+x)] - \log(1+x)cos[\log(1+x)]$$

Example:6

Solve $(2x-1)^2 \frac{d^2y}{dx^2} - 4(2x-1)\frac{dy}{dx} + 8y = 8x$

Solution:

Put $2x - 1 \; = e^t$

$t = \log(2x-1)$

$(2x-1)\frac{dy}{dx} = 2Dy$

$(2x-1)^2 \frac{d^2y}{dx^2} = 4D(D-1)y$

$[4D(D-1) - 8D + 8]y = 8\left(\frac{e^t - 1}{2}\right)$

$[4D^2 - 4D - 8D + 8]y \; = 4(e^t + 1)$

$[4D^2 - 12D + 8]y \; = 4(e^t + 1)$

$\div \, by \;\; 4$ $\quad (D^2 - 3D + 2)y = e^t + 1$

Auxiliary Equation is $m^2 - 3m + 2 = 0$

$m = 2, 1$

C.F = $Ae^{2t} + Be^t$

$= A(2x-1)^2 + B(2x-1)$

$P.I_1 = \frac{1}{D^2 3D + 2} \; e^t$

$= \frac{t}{2D-3} e^{t}$ Replace $D\ by$ 1

$= -te^{t}$

$P.I_2 = \frac{1}{D^2-3D+2} e^{0t}$ Replace $D\ by$ 0

$= \frac{1}{2}$

The general solution is $y = C.F + P.I_1 + P.I_2$

$$y = A(2x-1)^2 + B(2x-1) - (2x-1)\log(2x-1) + \frac{1}{2}$$