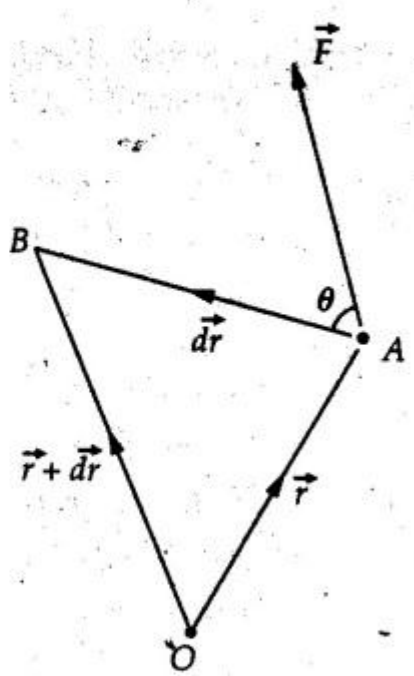
5.3 ENERGY AND MOMENTUM METHODS

The method of work and energy directly relates force, mass, velocity and displacement. The method of impulse and momentum relates force, mass, velocity and time.

1. Work of a Force

• If a force $\vec{F}$ acts on a particle which undergoes a displacement $d\vec{r}$ from A to B at an angle θ with $\vec{F}$ as shown in Fig. then the work done is



$$dU = \vec{F}\, d\vec{r} = F\, ds \cos\theta$$

where $\left|d\vec{r}\right| = ds$ is the magnitude of displacement. If the particle is taken from position 1 to position 2 and the force is variable, the total work done is

$$U_{1\text{-}2} = \int_{s_1}^{s_2} F \cos\theta \, ds$$

• If a graph of F cos versus s is given, the R.H.S represents area under the graph. If the magnitude of force is constant and acts at a constant angle θ with the direction of displacement,

$$U_{1\text{-}2} = \int_{s_1}^{s_2} F \cos\theta \, ds = F \cos\theta \int_{s_1}^{s_2} ds$$

$$U_{1\text{-}2} = F \cos\theta (s_2 - s_1)$$

• The work done by a force is positive when the component of force parallel to displacement (i.e. F cos θ) is in the direction of displacement. When they are in opposite directions, the work done is negative. The unit of work is joule (J).

• Work done by weight:

If *dy* is the upward displacement of an object of weight *w*,

$du = -w \, dy$

Negative sign is due to opposite directions of *w* and *dy*.

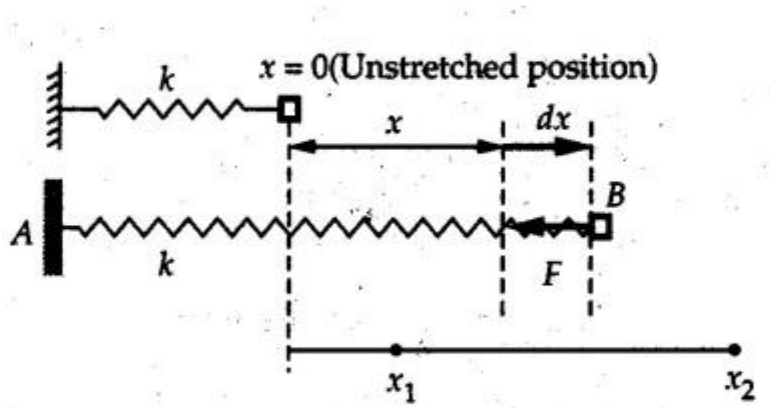
$$U_{1\text{-}2} = -w \int_{y_1}^{y_2} dy = -w (y_2 - y_1)$$

$$U_{1\text{-}2} = -w \, \Delta y$$

• If object moves downwards, Ay will be negative and the work done will be positive.

Work Done by Spring Force :

• Consider a block B connected to a spring as shown in Fig. The spring is undeformed when block is at position *x* = 0.



• When the block is taken from x to $x + dx$, the work done is

$dU = - F\, dx$ [F and dx are in opposite direction]

As $F = kx$ in magnitude,

□ $dU = – kx\, dx$

• If the block is taken from $x = x_1$ to $x = x_2$, the work done is

$$U_{1\text{-}2} = \int_{x_1}^{x_2} dU = \int_{x_1}^{x_2} - kx\, dx = -k\left[\frac{x^2}{2}\right]_{x_1}^{x_2}$$

$$\boxed{U_{1\text{-}2} = \frac{1}{2}kx_1^2 - \frac{1}{2}kx_2^2}$$

• While using the above equation, mistakes in sign can be eliminated by calculating the magnitude of work done using $\frac{1}{2}k\left|x_1^2 - x_2^2\right|$ and then assigning positive or negative sign depending upon the directions of spring force and displacement.

• Of particular interest, is a special case when either x_1 or x_2 is zero. Then the magnitude of work done is

$$U_{1\text{-}2} = \frac{1}{2}kx^2$$

where x represents either elongation or compression of the spring, i.e., deformation.

• This work done on the spring is stored in it as potential energy called the spring potential energy.

2. Principle of Work and Energy

• Consider a particle of mass m subjected to a single force *F*. Then,

$$F = ma$$

But $$a = \frac{dv}{dt} = v\frac{dv}{dx}$$

$$\therefore \quad F = mv\frac{dv}{dx}$$

$$F\,dx = mv\,dv$$

$$\int_{x_1}^{x_2} F\,dx = m\int_{v_1}^{v_2} v\,dv$$

$$\therefore \quad U_{1\text{-}2} = \frac{1}{2}mv_2^2 - \frac{1}{2}mv_1^2$$

• The quantity 1/2 mv^2 is defined as the kinetic energy of particle. Its S.I unit is joule(J).

$$\therefore \quad U_{1\text{-}2} = K.E_2 - K.E_1$$

$$\therefore \quad K.E_1 + U_{1\text{-}2} = K.E_2$$

The above equation represents the work-energy principle.

General Procedure for Solving Problems

1) Draw F.B.D. of the object.

ii) Resolve all forces into two mutually perpendicular directions - parallel and perpendicular to displacement.

iii) Identify velocities at initial position 1 and final position 2 to write KE_1, and KE_2.

iv) Calculate $U_{1\text{-}2}$, ie, the work done by all forces parallel to the direction of displacement. If force is in the direction of displacement then the work done will be positive. If force is in a direction opposite to displacement, work done is negative. Work done by forces perpendicular to displacement is zero.

v) Use the work-energy principle to find required quantities.

Note: The work energy principle cannot be used to find acceleration or forces for which the work done is zero.

Solved Examples

1. A body of mass 'm' is projected up 25° inclined plane with initial velocity 15 m/s. If $\mu_s = 0.28$ and $\mu_k = 0.25$, determine how far the body will move up the plane and time required to reach the highest point.

Solution:

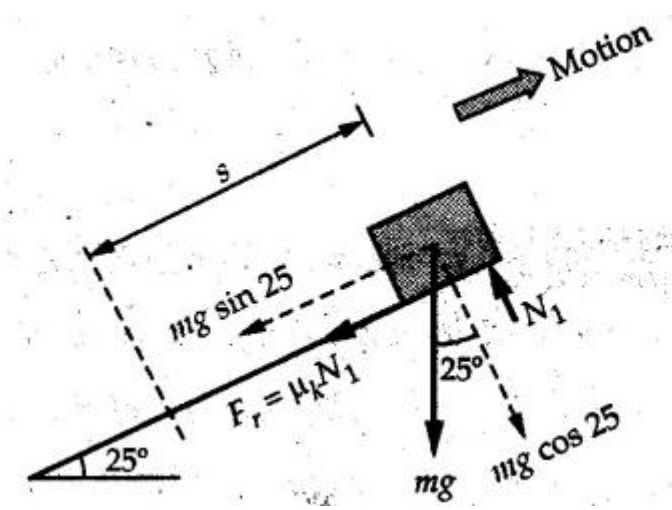
Let s = Distance travelled up the plane. By work-energy principle,

$K.E_1$ + Work done by all forces = $K.E_2$

$$K.E_1 = \frac{1}{2}m \times 15^2$$

$$K.E_2 = 0$$

From F.B.D. of the body shown in Fig.



$$\sum F_y = 0 \Rightarrow N_1 = mg\cos 25$$

$$\therefore \quad F_r = \mu_k mg\cos 25$$

$$\therefore \frac{1}{2}m \times 15^2 - (mg\sin 25) \times s - (\mu_k\, mg\cos 25) \times s = 0$$

$$\frac{1}{2} \times 15^2 - 9.81\sin 25 \times s - 0.25 \times 9.81\cos 25 \times s = 0$$

$$\therefore \quad \boxed{s = 17.665 \text{ m}}$$

Distance = (Average velocity) × Time

$$\therefore \quad t = \frac{17.665}{\left(\frac{15+0}{2}\right)}$$

$$\boxed{t = 2.355 \text{ s}}$$

2.A 1/8 kg bullet travelling at 150 m/sec. will penetrate 125 mit into a fixed block of wood. Find the velocity with which it would emerge if fired through a fixed board 60 mm thick. The resistance being supposed to be uniform and to have same value in each case.

Solution:

Let R be the resistance force. By work energy principle,

$K.E_1$ + Work done by all forces = $K.E_2$

$$\text{For the block, } K.E_1 = \frac{1}{2}\left(\frac{1}{8}\right)(150^2) = 1406.25 \text{ J}$$

$$\text{Work done by } R = -R\,(0.125)$$

The work done is negative as R acts opposite to displacement.

$K.E_2 = 0$

$\Rightarrow$ 1406.25 - R (0.125) = 0

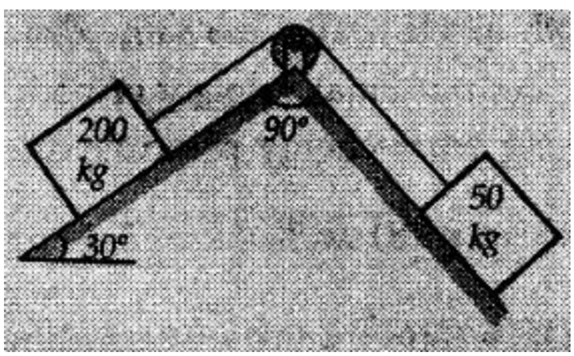
$\Rightarrow$ R = 11250 N

For the board, let v be the velocity with which bullet emerges

Then, $$1406.25 - (11250)(60 \times 10^{-3}) = \frac{1}{2}\left(\frac{1}{8}\right)v^2$$

$$\therefore \qquad \boxed{v = 108.17 \text{ m/s}}$$

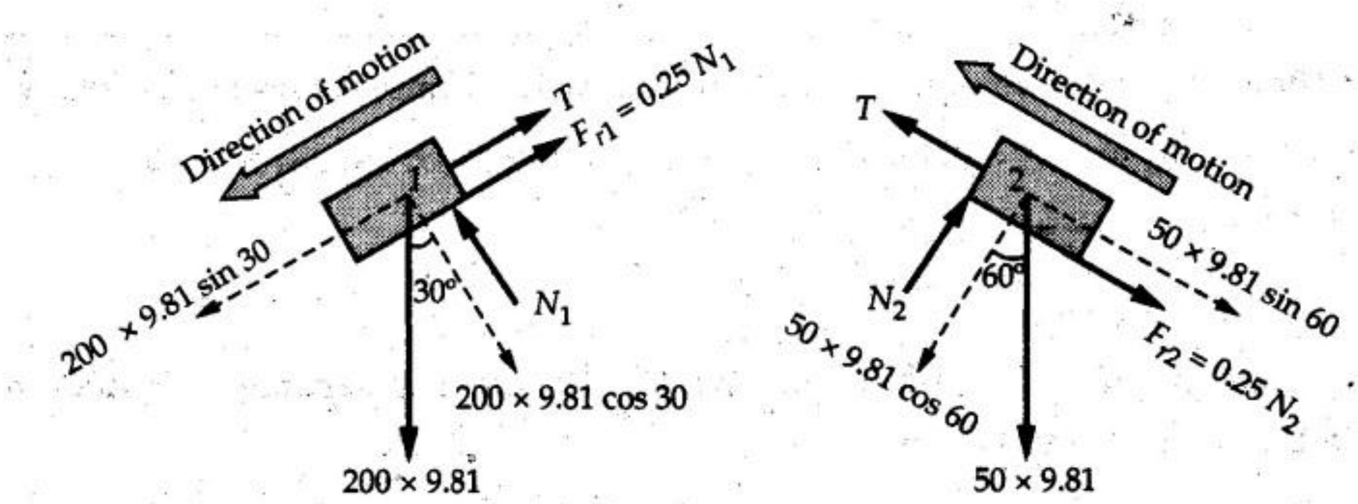
3.A block and pulley system is shown in Fig. The pulley is frictionless. Find the tension in the cable and the velocity of 50 kg block after it has moved a distance of 1.5 m when the system starts from rest. Neglect the mass of the pulley. Take the coefficient of kinetic friction between the blocks and the planes as 0.25. use the principle of work and energy.



Solution:

• To use principle of work and energy, draw free body diagrams of the two blocks and resolve all forces parallel and perpendicular to the direction of motion as shown in Fig. 9.3.5 (a).

• The work done by forces acting perpendicular to the displacement is zero. For forces acting parallel to the displacement, work done is product of force and displacement. if the force is in the direction of displacement, work done is positive.



• If the force is opposite to direction of displacement, work done is negative.

For 200 kg block,

$N_1 = 200 \times 9.81 \cos 30$

 $F_{r1} = 0.25 \times 200 \times 9.81 \cos 30$

For 50 kg block,

$N_2 = 50 \times 9.81 \cos 60$

 $F_{r2} = 0.25 \times 50 \times 9.81 \cos 60$

• By work-energy principle,

Initial kinetic energy + Work done by all forces = Final kinetic energy

For both blocks, initial velocity is zero. Hence their initial kinetic energy is zero.

Both blocks travel the same distance of 1.5 m.

• Let v = Final velocity of both blocks.

Using work-energy principle for 200 kg block,

$$0 + [200 \times 9.81 \sin 30 \times 1.5 - T \times 1.5 - F_{r1} \times 1.5] = \frac{1}{2} \times 200 v^2$$

$$\therefore T \times 1.5 + 100 v^2 = 200 \times 9.81 \sin 30 \times 1.5 - 0.25 \times 200 \times 9.81 \cos 30 \times 1.5 \quad \text{... (1)}$$

• Using work-energy principle for 50 kg block,

$$0+[T\times1.5-50\times9.81\sin60\times1.5-F_{r2}\times1.5]=\frac{1}{2}\times50v^2$$

$$\therefore\ T\times1.5-25v^2=50\times9.81\sin60\times1.5+0.25\times50\times9.81\cos30\times1.5 \quad \text{... (2)}$$

From equations (1) and (2),

$$\therefore\quad \boxed{T=500.12\text{ N}}$$

and $$v^2=0.8414$$

$$\therefore\quad \boxed{v=0.9173\text{ m/s}}$$

Note: The velocity can be obtained by considering the complete system as follows:

0 + [200 × 9.81 × sin 30 × 1.5 - 50 × 9.81 × sin 60 × 1.5 - 0.25 × 200 × 9.81 cos 30 × 1.5 -0.25 × 50 × 9.81 cos 60 × 1.5] = 1/2 × 250v^2

 v = 0.9173 m/s

As *T* is an internal force when the two objects are not separated, it does not appear in the work-energy equation.

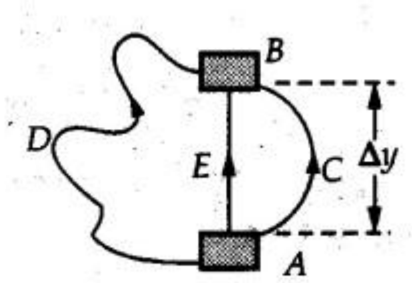
To find *T*, free body diagram of one of the blocks has to be used.

4. Conservative Forces and Potential Energy

• If the work done by a force does not depend on the path along which the particle is taken from one point to another, the force is called conservative force. In such a case the work done depends on the initial and final positions of the particle.

• For example, the work done by weight (U_{1-2} = -W Δy) depends only on the difference of height Δy.

• Even if the object is taken from A to B along different paths ACB, AEB or ADB, the work done remains same. Similarly, the work done by spring force $\left(U_{1-2}=\frac{1}{2}kx_1^2-\frac{1}{2}kx_2^2\right)$ depends only on positions x_1 and x_2 and hence spring force is a conservative force.



• Friction is a non-conservative force. The work done by friction depends on the path.

• Potential energy of a particle is due to the position of the DC particle (Kinetic energy is due to velocity).

• It is defined for the conservative forces for which the work done depends only on the initial and final position like the gravitational force and the spring force.

• In mechanics we come across two types of potential energies the gravitational potential energy and the potential energy in a spring (the spring potential energy).

Gravitational Potential Energy (G.P.E) :

• G.P.E. = $mgh = Wh$

where h is measured from a reference level. Although, any reference level can be taken, it is more convenient to take lower of the two positions as reference so that gravitational potential energy at the lower position becomes zero.

Spring Potential Energy (S.P.E)

• $$S.P.E. = \frac{1}{2}kx^2$$

where k = Spring constant and

x = Deformation of spring.

5. Conservation of Energy

• Principle of conservation of energy states that under the action of only conservative forces, the sum of kinetic energy and potential energy of a particle remains constant.

• If a particle under the action of only conservative forces is taken from position 1 to position 2,

$$K.E_1 + G.P.E_1 + S.P.E_1 = K.E_2 + G.P.E_2 + S.P.E_2$$

• Conservation of energy implies that during the motion of particle, the kinetic energy will be converted to potential energy or potential energy to kinetic energy so as to keep the total energy constant.

• An important result which can be remembered while solving problems is the velocity attained by an object falling freely under gravity through height '*h*', starting from a state of rest. Its potential energy gets converted to kinetic energy.

$$mgh = \frac{1}{2}mv^2$$

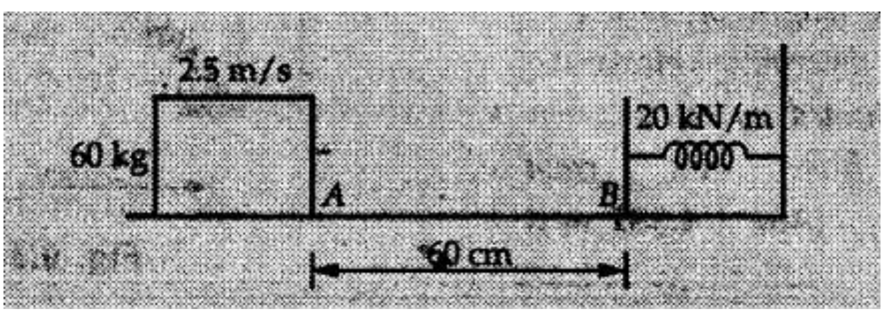
$$\therefore \quad h = \frac{v^2}{2g}$$

or

$$\therefore \quad v = \sqrt{2gh}$$

Solved Examples

A package of mass 60 kg, moving with velocity 2.5 m/s on a surface of friction μ hits a spring of constant 20 kN/m and compressed initially by 12.0 cm. If block compresses the spring further to the maximum of 4.0 cm and the block is initially 60 cm away from end of spring, determine μy and velocity of block when it just starts pressing spring (see Fig.)



Solution:

Given: m = 60 kg, v_1 = 2.5 m/s,

k = 20 kN/m = 20 × 10^3 N/m, x_1 = 12 cm = 0.12 m,

x_2 = 12 + 4 = 16 cm = 0.16 m, s = 60 + 4 = 64 cm = 0.64 m

After compression the package comes to rest, hence $v_2 = 0$.

By conservation of energy,

$$\frac{1}{2}mv_2^2 - \frac{1}{2}mv_1^2 = (WD)_{friction} + (WD)_{spring}$$

$$0 - \frac{1}{2} \times 60 \times 2.5^2 = -\mu_k N \times s + \frac{1}{2}k(x_1^2 - x_2^2)$$

$$-187.5 = -\mu_k \times 60 \times 9.81 \times 0.64 + \frac{1}{2} \times 20 \times 10^3 \times (0.12^2 - 0.16^2)$$

$$-187.5 = -376.704\,\mu_k - 112$$

$$\therefore \quad \mu_k = \mathbf{0.2} \qquad \textbf{...Ans.}$$