2.3 PROPERTIES OF FOURIER TRANSFORM.

Linearity

$$ax_1(t) + bx_2(t) \longleftrightarrow aX_1(j\omega) + bX_2(j\omega)$$

Proof:

$$\mathcal{F}\{ax_1(t) + bx_2(t)\} = \int_{-\infty}^{\infty} (ax_1(t) + bx_2(t))\, e^{-j\omega t} dt$$

$$= a\int_{-\infty}^{\infty} x_1(t)e^{-j\omega t}dt + b\int_{-\infty}^{\infty} x_2(t)e^{-j\omega t}dt = aX_1(j\omega) + bX_2(j\omega)$$

Time Scaling

$$x(at) \longleftrightarrow \frac{1}{a}X\left(\frac{j\omega}{a}\right)$$

Proof:

$$\mathcal{F}\{x(at)\} = \int_{-\infty}^{\infty} x(at)e^{-j\omega t}dt$$

Let $at = \tau$, $t = \frac{\tau}{a}$, $dt = \frac{1}{a}d\tau$, $t \in (-\infty, +\infty) \Rightarrow \tau \in (-\infty, +\infty)$

$$= \int_{-\infty}^{\infty} x(\tau)\exp\left(-j\frac{\omega}{a}\tau\right)\left(\frac{1}{a}d\tau\right) = \frac{1}{a}\int_{-\infty}^{\infty} x(\tau)\exp\left(-j\frac{\omega}{a}\tau\right)d\tau = \frac{1}{a}X\left(\frac{j\omega}{a}\right)$$

Time shift

$$x\left(t-t_0\right) \longleftrightarrow e^{-j\omega t_0} X\left(j\omega\right)$$

Proof:

$$\mathcal{F}\left\{x\left(t-t_0\right)\right\} = \int_{-\infty}^{\infty} x\left(t-t_0\right) e^{-j\omega t} dt$$

Let $\tau = t - t_0$, $t = \tau + t_0$, $dt = d\tau$, $t \in (-\infty, \infty) \Longrightarrow \tau \in (-\infty, +\infty)$

$$= \int_{-\infty}^{\infty} x\left(\tau\right) e^{-j\omega(\tau+t_0)} d\tau = e^{-j\omega t_0} \int_{-\infty}^{\infty} x\left(\tau\right) e^{-j\omega\tau} d\tau = e^{-j\omega t_0} X\left(j\omega\right)$$

Frequency shifting

$$e^{jat} x\left(t\right) \longleftrightarrow X\left(j(\omega - a)\right)$$

Proof:

$$\mathcal{F}\left\{e^{jat} x(t)\right\} = \int_{-\infty}^{\infty} e^{jat} x(t) e^{-j\omega t} dt = \int_{-\infty}^{\infty} x(t) e^{-j(\omega-a)t} dt = X\left(j\left(\omega - a\right)\right)$$

Time Reversal

$$x(-t) \longleftrightarrow X\left(-j\omega\right)$$

Proof:

$$\mathcal{F}\left\{x(-t)\right\} = \int_{-\infty}^{\infty} x\left(-t\right) e^{-j\omega t} dt$$

Let $-t = \tau$, $dt = -d\tau$, $t = -\infty \Rightarrow \tau = \infty$, $t = \infty \Rightarrow \tau = -\infty$

$$= \int_{\infty}^{-\infty} x(\tau)\, e^{j\omega\tau}(-1) d\tau = \int_{-\infty}^{\infty} x(\tau)\, e^{-j(-\omega)\tau} d\tau = X(-j\omega)$$

Differentiation in Time Domain

$$\frac{dx(t)}{dt} \longleftrightarrow j\omega X(j\omega)$$

Proof:

$$\mathcal{F}\left\{\frac{dx(t)}{dt}\right\} = \int_{-\infty}^{\infty} \frac{dx(t)}{dt} e^{-j\omega t} dt$$

$$= \int_{-\infty}^{\infty} e^{-j\omega t} dx(t) = \underbrace{\left[\frac{e^{-j\omega t} x(t)}{-j\omega}\right]_{-\infty}^{\infty}}_{0\ x(\pm\infty)=0} - \int_{-\infty}^{\infty} x(t) de^{-j\omega t}$$

$$= j\omega X(j\omega)$$

Integration

$$\int_{-\infty}^{t} f(\tau) d\tau \longleftrightarrow \frac{1}{j\omega} F(j\omega)$$

Proof:

Consider $g(t) = \int_{-\infty}^{t} f(\tau) d\tau$, $\lim_{t\to\pm\infty} g(t) = 0$

$$\frac{dg(t)}{dt} = f(t)$$

$$\mathcal{F}\left\{\frac{dg(t)}{dt}\right\} = j\omega G(j\omega) \quad \overset{\frac{d}{dt}}{\underset{\int}{\longleftrightarrow}} \quad \mathcal{F}\left\{f(t)\right\} = F(j\omega)$$

$$\begin{cases} G(j\omega) = \mathcal{F}\left\{g(t)\right\} = \mathcal{F}\left\{\int_{-\infty}^{t} f(\tau)d\tau\right\} \\ j\omega G(j\omega) = F(j\omega) \end{cases}$$

$$\mathcal{F}\left\{\int_{-\infty}^{t} f(\tau)d\tau\right\} = \frac{1}{j\omega}F(j\omega)$$

Complex Conjugation

$$\text{if} \quad \mathcal{F}[x(t)] = X(j\omega), \quad \text{then} \ \mathcal{F}[x^*(t)] = X^*(-j\omega)$$

Proof: Taking the complex conjugate of the inverse Fourier transform, we get

$$x^*(t) = [\frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}d\omega]^* = \frac{1}{2\pi}\int_{-\infty}^{\infty} X^*(j\omega)e^{-j\omega t}d\omega$$

Replacing ω by $-\omega'$we get the desired result:

$$x^*(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X^*(-\omega')e^{j\omega' t}d\omega' = \mathcal{F}^{-1}[X^*(-\omega)]$$

We further consider two special cases:

If $x(t) = x^*(t)$ is real, then

$$\mathcal{F}[x(t)] = X(j\omega) = X_r(j\omega) + jX_i(j\omega)$$

$$\mathcal{F}[x^*(t)] = X^*(-\omega) = X_r(-\omega) - jX_i(-\omega)$$

i.e., the real part of the spectrum is even (with respect to frequency ω), and the imaginary part is odd:

$$\begin{cases} X_r(j\omega) = X_r(-j\omega) \\ X_i(j\omega) = -X_i(-j\omega) \end{cases}$$

If $x(t) = -x^*(t)$ is imaginary, then

$$\mathcal{F}[x(t)] = X(j\omega) = X_r(j\omega) + jX_i(j\omega)$$

: $$\mathcal{F}[-x^*(t)] = -X^*(-j\omega) = -X_r(-j\omega) + jX_i(-j\omega)$$

i.e., the real part of the spectrum is odd, and the imaginary part is even:

$$\begin{cases} X_r(j\omega) = -X_r(-j\omega) \\ X_i(j\omega) = X_i(-j\omega) \end{cases}$$

Symmetry (or Duality)

$$\text{if} \quad \mathcal{F}[x(t)] = X(j\omega), \;\; \text{then} \;\; \mathcal{F}[X(t)] = 2\pi\, x(-j\omega)$$

Or in a more symmetric form:

$$\text{if} \quad \mathcal{F}[x(t)] = X(f), \;\; \text{then} \;\; \mathcal{F}[X(t)] = x(-f)$$

Proof:

As $F[x(t)] = X(j\omega)$, we have

$$x(t) = \mathcal{F}^{-1}[X(j\omega)] = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}d\omega$$

Letting $t' = -t$, we get

$$x(-t') = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{-j\omega t'}d\omega$$

Interchanging t' and ω we get:

$$2\pi x(-\omega) = \int_{-\infty}^{\infty} X(t')e^{-j\omega t'}dt' = \mathcal{F}[X(t)]$$

or

$$x(-f) = \int_{-\infty}^{\infty} X(t')e^{-j2\pi f t'}dt' = \mathcal{F}[X(t)]$$

In particular, if the signal is even:

$$x(t) = x(-t)$$

then we have

$$\text{if} \quad \mathcal{F}[x(t)] = X(f), \quad \text{then} \quad \mathcal{F}[X(t)] = x(f)$$

For example, the spectrum of an even square wave is a sinc function, and the spectrum of a sinc function is an even square wave.

Multiplication theorem

$$\int_{-\infty}^{\infty} x(t)y^*(t)dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)Y^*(j\omega)d\omega$$

Proof:

$$\int_{-\infty}^{\infty} x(t)y^*(t)dt = \int_{-\infty}^{\infty} x(t)[\frac{1}{2\pi}\int_{-\infty}^{\infty} Y^*(j\omega)e^{-j\omega t}d\omega]dt$$

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} Y^*(j\omega)[\int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt]d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)Y^*(j\omega)d\omega$$

Parseval's equation

$$\int_{-\infty}^{\infty} |x(t)|^2 dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} |X(j\omega)|^2 d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_X(j\omega) d\omega = \text{Total energy in } x(t)$$

where $$S_X(j\omega) \triangleq |X(j\omega)|^2$$

Is the energy density function representing how the signal's energy is distributed along the frequency axes. The total energy contained in the signal is obtained by integrating $S(j\omega)$ over the entire frequency axes.

The Parseval's equation indicates that the *energy* or *information* contained in the signal is reserved, i.e., the signal is represented equivalently in either the time or frequency domain with no energy gained or lost.

Correlation

The *cross-correlation* of two real signals $x(t)$ and $y(t)$ is defined as

$$R_{xy}(t) \triangleq \int_{-\infty}^{\infty} x(\tau) y(\tau - t) d\tau = \int_{-\infty}^{\infty} x(t+\tau) y(\tau) d\tau$$

Specially, when $x(t) = y(t)$, the above becomes the *auto-correlation* of signal $x(t)$

$$R_x(t) \triangleq \int_{-\infty}^{\infty} x(\tau) x(\tau - t) d\tau$$

Assuming $F[x(t)] = X(j\omega)$, we have $F[x(t-\tau)] = X(j\omega)e^{-j\omega\tau}$ and according to multiplication theorem, $R_x(\tau)$ can be written as

$$R_x(\tau) : \int_{-\infty}^{\infty} x(t) x(t-\tau) dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega) X^*(j\omega) e^{j\omega\tau} d\omega$$

$$: \frac{1}{2\pi}\int_{-\infty}^{\infty}|X(j\omega)|^2 e^{j\omega\tau}d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty}S_X(j\omega)e^{j\omega\tau}d\omega = \mathcal{F}^{-1}[S_X(j\omega)]$$

i.e.,

$$\mathcal{F}[R_x(t)] = S_X(j\omega)$$

that is, the auto-correlation and the energy density function of a signal $x(t)$ are a Fourier transform pair.

Convolution Theorems

The convolution theorem states that convolution in time domain corresponds to multiplication in frequency domain and vice versa:

$$\mathcal{F}[x(t) * y(t)] = X(j\omega)\,Y(j\omega) \qquad (a)$$

$$\mathcal{F}[x(t)\,y(t)] = X(j\omega) * Y(j\omega) \qquad (b)$$

Proof of (a):

$$\mathcal{F}[x(t) * y(t)] \quad : \quad \int_{-\infty}^{\infty}[\int_{-\infty}^{\infty}x(\tau)y(t-\tau)d\tau]e^{-j\omega t}dt$$

$$: \quad \int_{-\infty}^{\infty}x(\tau)[\int_{-\infty}^{\infty}y(t-\tau)e^{-j\omega t}dt]d\tau$$

$$: \quad \int_{-\infty}^{\infty}x(\tau)e^{-j\omega\tau}[\int_{-\infty}^{\infty}y(t-\tau)e^{-j\omega(t-\tau)}d(t-\tau)]d\tau$$

$$: \quad X(j\omega)\,Y(j\omega)$$

Proof of (b):

$$\mathcal{F}[x(t)\,y(t)] \quad : \quad \int_{-\infty}^{\infty}x(t)\,y(t)e^{-j\omega t}dt$$

$$: \quad \int_{-\infty}^{\infty}[\frac{1}{2\pi}\int_{-\infty}^{\infty}X(j\omega')e^{j\omega' t}d\omega']\,y(t)e^{-j\omega t}dt$$

: $$\frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega')[\int_{-\infty}^{\infty} y(t)e^{j\omega' t}e^{-j\omega t}dt]d\omega'$$

: $$\frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega')[\int_{-\infty}^{\infty} y(t)e^{-j(\omega-\omega')t}dt]d\omega'$$

: $$\frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega')Y(j(\omega-\omega'))d\omega' = X(j\omega) * Y(j\omega)$$