UNIT- III

ANALYTIC FUNCTIONS

INTRODUCTION

The theory of functions of a complex variable is the most important in solving a large number of Engineering and Science problems. Many complicated integrals of real function are solved with the help of a complex variable.

Complex Variable

$x + iy$ is a complex variable and it is denoted by z.

$(i.e.)z = x + iy \ where \ i = \sqrt{-1}$

Function of a complex Variable

If $z = x + iy$ and $w = u + iv$ are two complex variables, and if for each value of z in a given region R of complex plane there corresponds one or more values of w is said to be a function z and is denoted by $w = f(z) = f(x + iy) = u(x, y) + iv(x, y)$where $u(x, y)$and $v(x, y)$are real functions of the real variables x and y.

Note:

(i) single-valued function

If for each value of z in R there is correspondingly only one value of w, then w is called a single valued function of z.

Example: $w = z^2, w = \frac{1}{z}$

$w = z^2$					$w = \frac{1}{z}$				
z	1	2	-2	3	z	1	2	-2	3
w	1	4	4	9	w	1	$\frac{1}{2}$	$\frac{1}{-2}$	$\frac{1}{3}$

(ii) Multiple – valued function

If there is more than one value of w corresponding to a given value of z then w is called multiple – valued function.

Example: $w = z^{1/2}$

$w = z^{1/2}$			
z	4	9	1
w	$-2,2$	$3,-3$	$1,-1$

(iii) The distance between two points $z$and z_o is $|z - z_o|$

(iv) The circle C of radius δ with centre at the point $z_o$can be represented by $|z - z_o| = \delta$.

(v) $|z - z_o| < \delta$ represents the interior of the circle excluding its circumference.

(vi) $|z - z_o| \leq \delta$ represents the interior of the circle including its circumference.

(vii) $|z - z_o| > \delta$ represents the exterior of the circle.

(viii) A circle of radius 1 with centre at origin can be represented by $|z| = 1$

Neighbourhood of a point z_o

Neighbourhood of a point z_o, we mean a sufficiently small circular region [excluding the points on the boundary] with centre at z_o.

$$(i.e.)\ |z - z_o| < \delta$$

Here, δ is an arbitrary small positive number.

Limit of a Function

Let $f(z)$ be a single valued function defined at all points in some neighbourhood of point z_o.

Then the limit of $f(z)$ as z approaches $z_o is\ w_o$.

$$(i.e.)\ \lim_{z \to z_o} f(z) = w_o$$

Continuity

If $f(z)$ is said to continuous at $z = z_o$ then

$$\lim_{z \to z_o} f(z) = f(z_o)$$

If two functions are continuous at a point their sum, difference and product are also continuous at that point, their quotient is also continuous at any such point $[dr \neq 0]$

Example: 1 State the basic difference between the limit of a function of a real variable and that of a complex variable. [A.U M/J 2012]

Solution:

In real variable, $x \to x_0$ implies that x approaches x_0 along the X-axis (or) a line parallel to the X-axis.

In complex variables, $z \to z_0$ implies that z approaches z_0 along any path joining the points $z\ and\ z_0$ that lie in the z-plane.

Differentiability at a point

A function $f(z)$ is said to be differentiable at a point, $z = z_0$ if the limit

$$f(z_0) = \underset{\Delta z \to 0}{\text{Lt}} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z} \text{ exists.}$$

This limit is called the derivative of $f(z)$at the point $z = z_0$

If $f(z)$ is differentiable at z_0, then $f(z)$is continuous at z_0. This is the necessary condition for differentiability.

Example: 2 If $f(z)$ is differentiable at z_0, then show that it is continuous at that point.

Solution:

As $f(z)$ is differentiable at z_0, both $f(z_0)$ and $f'(z_0)$exist finitely.

$$\text{Now, } \lim_{z \to z_o} |f(z) - f(z_o)| = \lim_{z \to z_o} \frac{f(z) - f(z_o)}{z - z_0}(z - z_0)$$

$$= \lim_{z \to z_o} \frac{f(z) - f(z_o)}{z - z_0} \lim_{z \to z_o}(z - z_0)$$

$$= f'(z_0).0 = 0$$

Hence, $\lim_{z \to z_o} f(z) = \lim_{z \to z_o} f(z_o) = f(z_o)$

As $f(z_0)$ is a constant.

This is exactly the statement of continuity of $f(z)$ at z_0.

Example: 3 Give an example to show that continuity of a function at a point does not imply the existence of derivative at that point.

Solution:

Consider the function $w = |z|^2 = x^2 + y^2$

This function is continuous at every point in the plane, being a continuous function of two real variables. However, this is not differentiable at any point other than origin.

Example: 4 Show that the function $\mathbf{f(z)}$ is discontinuous at $z = 0$, given that $f(z) = \frac{2xy^2}{x^2+3y^4}$, when $z \neq 0$ and $\mathbf{f(0) = 0}$.

Solution:

Given $f(z) = \frac{2xy^2}{x^2+3y^4}$,

Consider $\lim_{z \to z_o} [f(z)] = \lim_{\substack{y=mx \\ x\to 0}} [f(z)] = \lim_{x\to 0} \frac{2x(mx)^2}{x^2+3(mx)^4} = \lim_{x\to 0} \left[\frac{2m^2x}{1+3m^4x^2}\right] = 0$

$$\lim_{\substack{y^2=x \\ x\to 0}} [f(z)] = \lim_{x\to 0} \frac{2x^2}{x^2+3x^2} = \lim_{x\to 0} \frac{2x^2}{4x^2} = \frac{2}{4} = \frac{1}{2} \neq 0$$

$\therefore f(z)$ is discontinuous

Example: 5 Show that the function $f(z)$ is discontinuous at the origin $(z = 0)$, given that

$$f(z) = \frac{xy(x-2y)}{x^3+y^3}, \text{ when } z \neq 0$$

$$= 0 \qquad , \text{when } z = 0$$

Solution:

$$\text{Consider } \lim_{z \to z_o} [f(z)] = \lim_{\substack{y=mx \\ x \to 0}} [f(z)] = \lim_{x \to 0} \frac{x(mx)(x-2(mx))}{x^3+(mx)^3}$$

$$= \lim_{x \to 0} \frac{m(1-2m)x^3}{(1+m^3)x^3} = \frac{m(1-2m)}{1+m^3}$$

Thus $\lim_{z \to 0} f(z)$ depends on the value of m and hence does not take a unique value.

$\therefore \lim_{z \to 0} f(z)$ does not exist.

$\therefore f(z)$ is discontinuous at the origin.