2.2. MARGINAL AND CONDITIONAL DISTRIBUTIONS

Problems on Conditional distribution:

1. The two dimensional random variable (X, Y) has the joint probability mass function $f(x, y) = \frac{x+2y}{27}, x = 0, 1, 2 ; y = 0, 1, 2$. Find the conditional distribution of Y for $X = x$. Also find the conditional distribution of Y given $X = 1$

Solution:

We know that the conditional probability distribution of Y for $X = x$ is

$f\left(\frac{y}{x}\right) = \frac{f(x,y)}{f(x)}$ where $f(x, y)$ is the joint probability function of X and y.

To find $f(x, y)$:

Marginal Distributions

Given $f(x, y) = \frac{x+2y}{27}, x = 0,1,2 ; y = 0,1,2.$

		X			$P_Y(y) = P(Y = y)$
		0	1	2	
Y	0	0 $P(0,0)$	$\frac{1}{27}$ $P(1,0)$	$\frac{2}{27}$ $P(2,0)$	$\frac{3}{27}$ $P(Y=0)$
	1	$\frac{2}{27}$ $P(0,1)$	$\frac{3}{27}$ $P(1,1)$	$\frac{4}{27}$ $P(2,1)$	$\frac{9}{27}$ $P(Y=1)$
	2	$\frac{4}{27}$ $P(0,2)$	$\frac{5}{27}$ $P(1,2)$	$\frac{6}{27}$ $P(2,2)$	$\frac{15}{27}$ $P(Y=2)$
$P_X(x) = P(X = x)$		$\frac{6}{27}$ $P(X=0)$	$\frac{9}{27}$ $P(X=1)$	$\frac{12}{27}$ $P(X=2)$	1

The Conditional Probability of Y for $X = x$ is given by $f\left(\frac{y}{x}\right) = \frac{f(x,y)}{f(x)}$

By using the above table we get the conditional probability of Y for $X = x$ as follows

When $x = 0$,

$$P[Y = 0/X = 0] = \frac{P[X = 0, Y = 0]}{P[X = 0]} = \frac{0}{6/27} = 0$$

$$P[Y = 1/X = 0] = \frac{P[X = 0, Y = 1]}{P[X = 0]} = \frac{2/27}{6/27} = \frac{1}{3}$$

$$P[Y = 2/X = 0] = \frac{P[X= 0, Y= 2]}{P[X= 0]} = \frac{4/_{27}}{6/_{27}} = \frac{2}{3}$$

When $x = 1$,

$$P[Y = 0/X = 1] = \frac{P[X= 1, Y= 0]}{P[X= 1]} = \frac{1/_{27}}{9/_{27}} = \frac{1}{9}$$

$$P[Y = 1/X = 1] = \frac{P[X= 1, Y= 1]}{P[X= 1]} = \frac{3/_{27}}{9/_{27}} = \frac{1}{3}$$

$$P[Y = 2/X = 1] = \frac{P[X= 1, Y= 2]}{P[X= 1]} = \frac{5/_{27}}{9/_{27}} = \frac{5}{9}$$

When $x = 2$,

$$P[Y = 0/X = 2] = \frac{P[X= 2, Y= 0]}{P[X= 2]} = \frac{2/_{27}}{12/_{27}} = \frac{1}{6}$$

$$P[Y = 1/X = 2] = \frac{P[X= 2, Y= 1]}{P[X= 2]} = \frac{4/_{27}}{12/_{27}} = \frac{1}{3}$$

$$P[Y = 2/X = 2] = \frac{P[X= 2, Y= 2]}{P[X= 2]} = \frac{6/_{27}}{12/_{27}} = \frac{1}{2}$$

Table of $f(y/x)$			
X \ Y	0	1	2
0	0	$\frac{1}{3}$	$\frac{2}{3}$
1	$\frac{1}{9}$	$\frac{1}{3}$	$\frac{5}{9}$

2	$\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{2}$

The conditional distribution of Y given $X = 1$ is given in the table

Y	Table of $f(y/x = 1)$	
0	$P[Y = 0/X = 1] = \frac{P[X = 1, Y = 0]}{P[X = 1]} = \frac{1/27}{9/27} = \frac{1}{9}$	
1	$P[Y = 1/X = 1] = \frac{P[X = 1, Y = 1]}{P[X = 1]} = \frac{3/27}{9/27} = \frac{1}{3}$	
2	$P[Y = 2/X = 1] = \frac{P[X = 1, Y = 2]}{P[X = 1]} = \frac{5/27}{9/27} = \frac{5}{9}$	

2. The joint probability mass function of X and Y is

X \ Y	0	1	2
0	0.10	0.04	0.02
1	0.08	0.20	0.06
2	0.06	0.14	0.30

Find the M.D.F of X and Y. Also $P(X \leq 1, Y \leq 1)$ and check if X and Y are independent.

Solution:

The marginal distributions are given in the table below

X \ Y	0	1	2	$P(X = x)$
0	0.10	0.04	0.02	0.16
1	0.08	0.20	0.06	0.34
2	0.06	0.14	0.30	0.50
$P(Y = y)$	0.24	0.38	0.38	1

Now, $P(X \leq 1, Y \leq 1) = p(0,0) + p(1,0) + p(0,1) + p(1,1)$

= 0.1 + 0.08 + 0.04 + 0.2 = 0.42

To test X and Y are independent

P(X = 0) P (Y = 0) = 0.16 X 0.24 $\neq$ 0.1

$\therefore P(X = 0)P(Y = 0) \neq P(X = 0, Y = 0)$

$\therefore$ X and Y are independent.

Joint Probability distribution function for Continuous random variables of X and Y:

The joint probability distribution function of a two dimension as random variables (X, Y) is denoted by $F_{XY}(x, y)$ and is given by

$f_{XY}(x, y) \geq 0$ and $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{XY}(x, y)\, dxdy = 1$

Properties of joint Distribution Functions:

1. $F(-\infty, y) = 0 = F(x, \infty) and\ F(-\infty, \infty) = 1$

2. $P(a_1 < X < b_1, a_2 < Y < b_2) = F(b_1, b_2) + F(a_1, a_2) - F(a_1, b_2) - F(b_1, a_2)$

Marginal Probability density function:

The marginal probability density function of the two random variables X and Y are defined as follows

$f(x) = f_X(x) = \int_{-\infty}^{\infty} f_{XY}(x, y)dy$ (Marginal pdf of X)

$f(y) = f_Y(y) = \int_{-\infty}^{\infty} f_{XY}(x, y)dx$ (Marginal pdf of Y)

Problems under Marginal density function

1. The bivariate random variable X and Y has the pdf $f(x,y) = K\,x^2(8-y), x<y<2x, 0 \le x \le 2$. Find K.

Solution:

We know that if $f(x,y)$ is a pdf then $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{XY}(x,y)\,dxdy = 1$

$$\Rightarrow \int_0^2\int_x^2 \;K\,x^2(8-y)\,dxdy = 1$$

$$\Rightarrow K\int_0^2 x^2\left(8y-\frac{y^2}{2}\right)_x^{2x} dx = 1$$

$$\Rightarrow K\int_0^2 x^2\left(16x-\frac{4x^2}{2}-8x+\frac{x^2}{2}\right)_x^{2x} dx = 1$$

$$\Rightarrow K\int_0^2\left(16x^3-2x^4-8x^3+\frac{x^4}{2}\right)_x^{2x} dx = 1$$

$$\Rightarrow K\int_0^2\left(8x^3-\frac{3x^4}{2}\right)_x^{2x} dx = 1$$

$$\Rightarrow K\left[\frac{8x^4}{4}-\frac{3}{2}\frac{x^5}{5}\right]_0^2 = 1$$

$$\Rightarrow K\left[32-\frac{48}{5}\right] = 1$$

$$\Rightarrow K\left(\frac{112}{5}\right) = 1$$

$$\Rightarrow K = \frac{5}{112}$$

2. Find "K" if the joint probability density function of a bivariate random variable X and Y has the pdf $f(x,y) = K\,(1-x)(1-y), 0 < x < 4, 1 < y < 5$

Solution:

We know that if $f(x,y)$ is a pdf then $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{XY}\,(x,y)\,dxdy = 1$

$$\Rightarrow \int_1^5\int_0^4 K\,(1-x)(1-y)\,dxdy = 1$$

$$\Rightarrow K\int_1^5\left(\frac{(1-x)^2}{-2}\right)_0^4 (1-y)\,dy = 1$$

$$\Rightarrow -\frac{K}{2}\int_1^5 (9-1)\,(1-y)dy = 1$$

$$\Rightarrow \frac{-8K}{2}\int_1^5 (1-y)dy = 1$$

$$\Rightarrow -4K\left(\frac{(1-x)^2}{-2}\right)_1^5 = 1$$

$$\Rightarrow 2K[16-0] = 1$$

$$\Rightarrow 32K = 1$$

$$\Rightarrow K = \frac{1}{32}$$

3. The joint pdf of random variable X and Y is given by $f(x,y) = Kxye^{-(x^2+y^2)}, x > 0, y > 0$. Find the value of K and prove also that X and Y are independent.

Solution:

We know that if $f(x,y)$ is a p.d.f, then $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{XY}(x,y)\,dxdy = 1$

Here $f(x,y) = Kxye^{-(x^2+y^2)}, x > 0, y > 0$

We know that $\int_0^{\infty}\int_0^{\infty} Kxye^{-(x^2+y^2)}\,dydx = 1$

$$\Rightarrow \int_0^{\infty}\int_0^{\infty} Kxye^{-x^2} e^{-y^2} dydx = 1$$

$$\Rightarrow K\int_0^{\infty} ye^{-y^2} dy \int_0^{\infty} xe^{-x^2}\, dx = 1$$

$$\Rightarrow K\frac{1}{2}\frac{1}{2} = 1$$

$$\Rightarrow K = 4$$

To prove X and Y are independent we have to prove that $f(x).f(y) = f(x,y)$

Now, $f(x) = f_X(x) = \int_{-\infty}^{\infty} f(x,y)dy$

$$= \int_0^{\infty} Kxye^{-(x^2+y^2)}\, dy$$

$$= 4xe^{-x^2}\int_0^{\infty} ye^{-y^2} dy$$

$$= 4xe^{-x^2}\left(\frac{1}{2}\right)$$

$$= 2xe^{-x^2}, x > 0$$

Similarly $f(y) = f_Y(y) = 2ye^{-y^2}, y > 0$

Now, $f(x).f(y) = 2xe^{-x^2}\, 2ye^{-y^2}$

$$= 4xye^{-x^2}e^{-y^2}$$

$$= 4xye^{-(x^2+y^2)}$$

$$= f(x,y)$$

$\therefore$ X and Y are independent.

4. The joint probability density function of a two dimensional random variable (X,Y) is given by $f(x,y) = xy^2 + \frac{x^2}{8}, 0 \le x \le 2, 0 \le y \le 1$ Compute (i) $P\left(X > 1/Y < \frac{1}{2}\right)$ (ii) $P\left(Y < \frac{1}{2}/X > 1\right)$ (iii) $P(X < Y)$ (iv) $P(X + Y \le 1)$

Solution:

Given $f(x,y) = xy^2 + \frac{x^2}{8}, 0 \le x \le 2, 0 \le y \le 1$

Now $P\left(X > 1/Y < \frac{1}{2}\right) = \frac{P\left(X>1, Y<\frac{1}{2}\right)}{P(Y<\frac{1}{2})} = \int_0^{\frac{1}{2}}\int_1^2 (xy^2 + \frac{x^2}{8})dx$

$$= \int_0^{\frac{1}{2}} (\frac{x^2y^2}{2} + \frac{x^3}{24})_1^2 dy$$

$$= \int_0^{\frac{1}{2}} \left(2y^2 + \frac{1}{3}\right) - (\frac{y^2}{2} + \frac{1}{24}) \quad dy$$

$$= \int_0^{\frac{1}{2}} (2y^2 + \frac{1}{3} - \frac{y^2}{2} - \frac{1}{24}) \, dy$$

$$= \int_0^{\frac{1}{2}} \left(\frac{3y^2}{2} + \frac{7}{24} \right) dy = \frac{5}{24}$$

$$\Rightarrow P\left(X > 1/Y < \frac{1}{2}\right) = \frac{5}{24}$$

$$P(Y < \frac{1}{2}) = \int_0^{\frac{1}{2}} \int_1^2 (xy^2 + \frac{x^2}{8}) dxdy$$

$$= \int_0^{\frac{1}{2}} (\frac{x^2y^2}{2} + \frac{x^3}{24})_0^2 dy$$

$$= \int_0^{\frac{1}{2}} \left(2y^2 + \frac{1}{3}\right) dy$$

$$= (\frac{2y^3}{3} + \frac{y}{3})_0^{\frac{1}{2}} = \frac{1}{4}$$

$$\therefore P\left(Y < \frac{1}{2}\right) = \frac{1}{4}$$

$$P(X > 1) = \int_0^1 \int_1^2 (xy^2 + \frac{x^2}{8}) dxdy$$

$$= \int_0^1 (\frac{x^2y^2}{2} + \frac{x^3}{24})_1^2 dy$$

$$= \int_0^1 \left(\frac{3y^2}{2} + \frac{7}{24}\right) dy$$

$$= \left[\frac{3}{2}\frac{y^3}{3} + \frac{7y}{24}\right]_0^1 = \frac{19}{24}$$

(i) $P\left(X > 1 \,/\, Y < \frac{1}{2}\right) = \frac{P(X>1, Y<\frac{1}{2})}{P(Y<\frac{1}{2})}$

$$= \frac{5}{24} \times 4 = \frac{5}{6}$$

(ii) $P\left(Y < \frac{1}{2} / X > 1\right) = \frac{P(Y<\frac{1}{2}, X>1)}{P(X>1)} = \frac{5}{24} \times \frac{24}{19} = \frac{5}{19}$

(iii) $P(X < Y) = \int_0^1 \int_0^y (xy^2 + \frac{x^2}{8}) dxdy$

$$= \int_0^1 (\frac{x^2y^2}{2} + \frac{x^3}{24})_0^y dy$$

$$= \int_0^1 \left(\frac{y^4}{2} + \frac{y^3}{24}\right) dy$$

$$= \left[\frac{y^5}{10} + \frac{y^4}{96}\right]_0^1 = \frac{53}{480}$$

(iv) $P(X + Y \leq 1) = \int_0^1 \int_0^{1-y} (xy^2 + \frac{x^2}{8}) dxdy$

$$= \int_0^1 (\frac{x^2y^2}{2} + \frac{x^3}{24})_0^{1-y} dy$$

$$= \int_0^1 \left(\frac{(1-y)^2y^2}{2} + \frac{(1-y)^3}{24}\right) dy$$

$$= \int_0^1 \left(\frac{(1-y)^2y^2}{2} + \frac{(1-y)^3}{24}\right) dy$$

$$= \int_0^1 \left(\frac{1}{2}(y^2 + y^4 - 2y^3) + \frac{1}{24}(1-y)^3\right) dy$$

$$= \frac{1}{2}(\frac{y^3}{3} + \frac{y^5}{5} - \frac{2y^4}{4})_0^1 + \frac{1}{24}\left(\frac{(1-y)^4}{-4}\right)_0^1 = \frac{13}{480}$$

5. The joint density function of X and Y is $f(x,y) = \begin{cases} e^{-(x+y)} & 0 \le x, y \le \infty \\ 0 & otherwise \end{cases}$.

Are X and Y independent. Find (i) $P(X < 1)$ (ii) $P(X + Y < 1)$

Solution:

Given $f(x,y) = \begin{cases} e^{-(x+y)} & 0 \le x, y \le \infty \\ 0 & otherwise \end{cases}$

The Marginal pdf of X is $f_X(x) = f(x) = \int_{-\infty}^{\infty} f(x,y)dy$

$$= \int_0^{\infty} e^{-(x+y)}\, dy$$

$$= e^{-x} \int_0^{\infty} e^{-y}\, dy$$

$$= e^{-x}[-e^{-y}]_0^{\infty}$$

$$= e^{-x}[0+1] = e^{-x}$$

Marginal pdf of 'x' is $f(x) = e^{-x}$

The Marginal pdf of Y is $f_Y(y) = f(y) = \int_{-\infty}^{\infty} f(x,y)dx$

$$= \int_0^{\infty} e^{-(x+y)}\, dx$$

$$= e^{-y} \int_0^{\infty} e^{-x}\, dx$$

$$= e^{-y}[-e^{-x}]_0^{\infty}$$

$$= e^{-y}[0+1] = e^{-y}$$

Marginal pdf of 'Y' is $f(y) = e^{-y}$

Now $f(x).f(y) = e^{-x}.e^{-y} = e^{-(x+y)}$

But $f(x,y) = e^{-(x+y)}$

$$\therefore f(x,y) = f(x).f(y)$$

Hence X and Y ae independent.

(i) $P(X < 1) = \int_0^1 \int_0^\infty e^{-(x+y)} \, dy\, dx$

$$= \int_0^1 e^{-x} \left[\frac{e^{-y}}{-1}\right]_0^\infty dx$$

$$= \int_0^1 e^{-x} \left[\frac{0-1}{-1}\right] dx$$

$$= \int_0^1 e^{-x}\, dx$$

$$= [-e^{-x}]_0^1$$

$$= -e^{-1} + e^0 = 1 - e^{-1}$$

(ii) $P(X + Y < 1) = \int_0^1 \int_0^{1-x} e^{-(x+y)}\, dy\, dx$

$$= \int_0^1 e^{-x} \left[\frac{e^{-y}}{-1}\right]_0^{1-x} dx$$

$$= -\int_0^1 e^{-x} [e^{x-1} - e^0] dx$$

$$= -\int_0^1 e^{-x}[e^{x-1} - e^0]\, dx$$

$$= -\int_0^1 [e^{-x+x-1} - e^{-x}]dx$$

$$= -[e^{-1}x + e^{-x}]_0^1$$

$$= -[e^{-1} + e^{-1} - 0 - e^0]$$

$$= -[2e^{-1} - 1] = 1 - 2e^{-1}$$

6. If the joint pdf of X and Y is given by $\boldsymbol{f(x,y) = \begin{cases} \frac{1}{8}(6-x-y) & 0<x<2, 2<y<4 \\ 0 & otherwise \end{cases}}$ **Find** $\boldsymbol{(i) P\,(X<1 \cap Y<3)\ (ii)\ P(\,X<1\ /Y<3)}$

Solution:

We know that $(i)\ P\,(X<1\ \cap Y<3) = \int_0^1 \int_2^3 f(x,y\,)\ dy\,dx$

$$= \int_0^1 \int_2^3 \frac{1}{8}\,(6-x-y)dy\,dx$$

$$= \frac{1}{8}\int_0^1 [6y - xy - \frac{y^2}{2}]_2^3\ dx$$

$$= \frac{1}{8}\int_0^1 \{(\,18 - 3x - \frac{9}{2}) - (\,12 - 2x - \frac{4}{2})\}\,dx$$

$$= \frac{1}{8}\int_0^1 (\,8 - x - \frac{9}{2})\,dx$$

$$= \frac{1}{8}\int_0^1 (\,\frac{7-2x}{2}\,)\,dx$$

$$= \frac{1}{16}[7x - x^2]_0^1 = \frac{6}{16} = \frac{3}{8}$$

$\therefore P(X < 1 \cap Y < 3) = \frac{3}{8}$ ………………….. (1)

(ii) $P(X < 1/ < 3) = \frac{P(X<1 \cap Y<3)}{P(Y<3)}$ ………… (2)

$$= \int_0^1 \int_2^3 \frac{1}{8}(6 - x - y)dy\, dx$$

$$= \frac{1}{8}\int_0^2 [6y - xy - \frac{y^2}{2}]_2^3\, dx$$

$$= \frac{1}{8}\int_0^2 \{(18 - 3x - \frac{9}{2}) - (12 - 2x - 2)\}\, dx$$

$$= \frac{1}{8}\int_0^2 (8 - x - \frac{9}{2})\, dx$$

$$= \frac{1}{8}\int_0^2 \{(\frac{16-9}{2}) - x\}\, dx$$

$$= \frac{1}{8}[\int_0^2 (\frac{7}{2} - x)dx] = \frac{1}{8}[\frac{7}{2}x - \frac{x^2}{2}]_0^2 = \frac{5}{8}$$

$\therefore P(Y < 3) = \frac{5}{8}$ ………………… (3)

Substituting (1) and (3) in (2) we get $P(X < 1/Y < 3) = \frac{\frac{3}{8}}{\frac{5}{8}} = \frac{3}{5}$

7. If the joint distribution function of X and Y is given by

$\boldsymbol{f(x,y) = \begin{cases} (1-e^{-x})(1-e^{-y}) & for\ x>0, y>0 \\ 0 & otherwise \end{cases}}$ **(i) Find the marginal densities of X &Y.**

(ii) Are X and Y independent? (iii)$P(1 < X < 3, 1 < Y < 2)$

Solution:

Given $f(x,y) = (1-e^{-x})(1-e^{-y})$

$$= 1 - e^{-x} - e^{-y} + e^{-(x+y)}$$

The joint pdf is given by $f(x,y) = \frac{\partial^2 F(x,y)}{\partial x \partial y}$

$$= \frac{\partial^2}{\partial x \partial y}[1 - e^{-x} - e^{-y} + e^{-(x+y)}]$$

$$= \frac{\partial}{\partial x}\left[e^{-y} + e^{-(x+y)}\right] = 0 + e^{-(x+y)}$$

$$\therefore f(x,y) = \begin{cases} (1-e^{-x})(1-e^{-y}) & for\ x>0, y>0 \\ 0 & otherwise \end{cases}$$

$f(x,y) = e^{-(x+y)}$

The marginal pdf of X is $f_{X(x)} = f(x) = \int_{-\infty}^{\infty} f(x,y)dy$

$$= \int_0^\infty e^{-(x+y)}dy$$

$$= [-e^{-(x+y)}]_0^\infty$$

$$= -e^{-\infty} + e^{-x} = e^{-x} \quad \ldots\ldots\ldots\ldots(1)$$

The marginal pdf of Y is $f_{Y(y)} = f(y) = \int_{-\infty}^{\infty} f(x,y)dy$

$$= \int_0^{\infty} e^{-(x+y)}dx$$

$$= [-e^{-(x+y)}]_0^{\infty}$$

$$= -e^{-\infty} + e^{-y} = e^{-y} \ldots\ldots\ldots\ldots\ldots (2)$$

(ii) From (1) and (2) we get

$$f(x).f(y) = e^{-x}e^{-y} = e^{-(x+y)} = f(x,y)$$

Hence X and Y are independent.

(iii) P (1 < X < 3, 1< Y < 2) = P(1 < X < 3) . P (1< Y < 2) (Since X and Y are independent)

$$= \int_1^3 e^{-x}\, dx . \int_1^2 e^{-y}\, dy$$

$$= (-e^{-x})_1^3 \ (-e^{-y})_1^2$$

$$= (- e^{-3} + e^{-1}) (-e^{-2} + e^{-1})$$

$$= e^{-5} - e^{-4} - e^{-3} + e^{-2}$$