2.2 TIME DOMAIN SPECIFICATIONS

The desired performance characteristics of control systems are specified in terms of time domain specifications. Systems with energy storage elements cannot respond instantaneously and will exhibit transient responses, whenever they are subjected to inputs or disturbances. The desired performance characteristics of a system of any order may be specified in terms of the transient response to a unit step input signal. The response of a second order system for unit step input with various values of damping ratio is shown in figure 2.2.1.

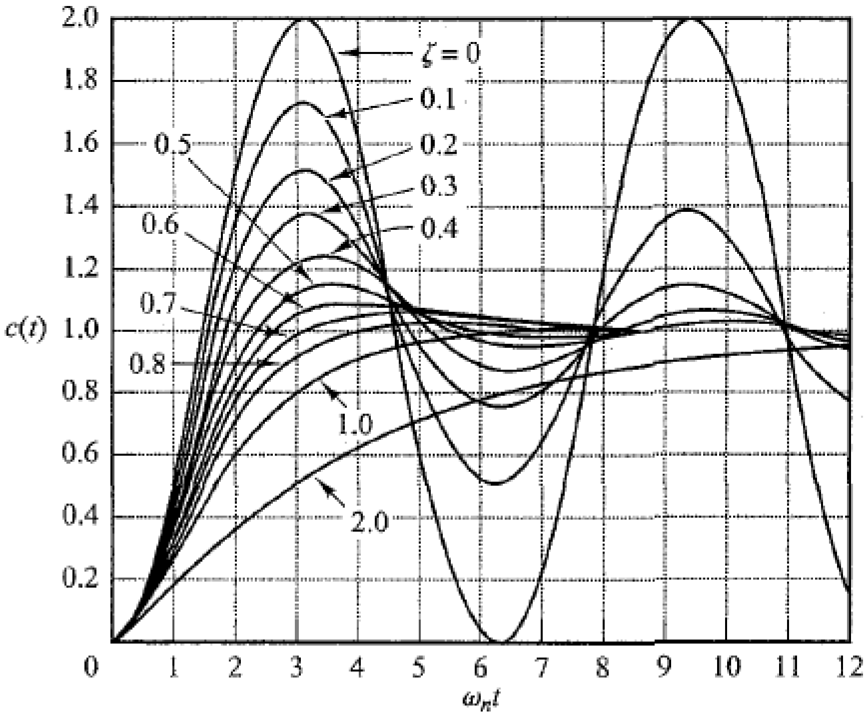

Figure 2.2.1 Time Response

[Source: "Modern Control Engineering" by Katsuhiko Ogata, Page: 229]

The transient response of a system to a unit step input depends on the initial conditions. Therefore, to compare the time response of various systems it is necessary to start with standard initial conditions. The most practical standard is to start with the system at rest and so output and all time derivatives before t=0 will be zero. The transient response of a practical control system often exhibits damped oscillation before reaching steady state. A typical damped oscillatory response of a system is shown in figure 2.2.2.

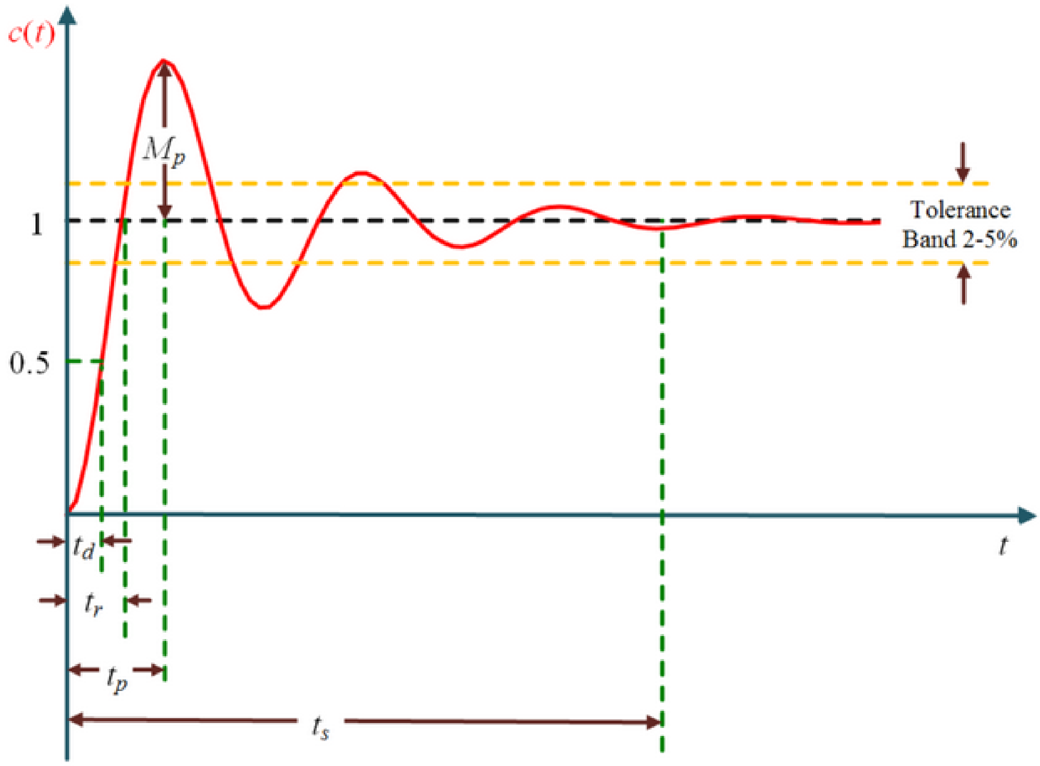

Figure 2.2.2 Transient and steady-state response analyses

[Source: "Modern Control Engineering" by Katsuhiko Ogata, Page: 230]

The transient response characteristics of a control system to a unit step input is specified in terms of the following time domain specifications:

1. Delay time, t_d: It is the time required for the response to reach 50% of the steady state value for the first time.

$$t_d = \frac{1 + 0.7\zeta}{\omega_n}$$

2. Rise time, t_r: It is the time required for the response to reach 100% of the steady state value for under damped systems. However, for over damped systems, it is taken as the time required for the response to rise from 10% to 90% of the steady state value. The unit step response of second order system for underdamped case is given by,

$$c(t) = 1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{(1-\zeta^2)}}\sin(\omega_d t + \theta)$$

At $t = t_r$, $c(t) = c(t_r) = 1$

$$c(t_r) = 1 - \frac{e^{-\zeta\omega_n t_r}}{\sqrt{(1-\zeta^2)}}\sin(\omega_d t_r + \theta) = 1$$

$$\frac{-e^{-\zeta\omega_n t_r}}{\sqrt{(1-\zeta^2)}}\sin(\omega_d t_r+\theta)=0$$

Since $-e^{-\zeta\omega_n t_r}\neq 0$, the term, $\sin(\omega_d t_r+\theta)=0$,

When Φ = 0, π, 2π, 3π,…. sin Φ = 0

$$\omega_d t_r+\theta=\pi$$

$$\omega_d t_r=\pi-\theta$$

$$t_r=\frac{\pi-\theta}{\omega_d}$$

On constructing right angled triangle,

$$\tan\theta=\frac{\sqrt{(1-\zeta^2)}}{\zeta}$$

$$\theta=\tan^{-1}\frac{\sqrt{(1-\zeta^2)}}{\zeta}$$

$$Damped\ frequency,\ \ \omega_d=\omega_n\sqrt{(1-\zeta^2)}$$

$$t_r=\frac{\pi-\tan^{-1}\left(\frac{\sqrt{(1-\zeta^2)}}{\zeta}\right)}{\omega_n\sqrt{(1-\zeta^2)}}$$

3. Peak time, t_p: It is the time required for the response to reach the maximum or peak value of the response. To find the expression for peak time, t_p, differentiate c(t) with respect to 't' and equate to zero.

$$\frac{d}{dt}c(t)|_{t=t_p}=0$$

The unit step response of under damped second order system is given by

$$c(t)=1-\frac{e^{-\zeta\omega_n t}}{\sqrt{(1-\zeta^2)}}\sin(\omega_d t+\theta)$$

Differentiating c(t) with respect to 't',

$$\frac{d}{dt}c(t)=\frac{-e^{-\zeta\omega_n t}}{\sqrt{(1-\zeta^2)}}(-\zeta\omega_n)\sin(\omega_d t+\theta)+\left(\frac{-e^{-\zeta\omega_n t}}{\sqrt{(1-\zeta^2)}}\right)\cos(\omega_d t+\theta)\omega_d$$

Put $\omega_d=\omega_n\sqrt{(1-\zeta^2)}$,

$$\frac{d}{dt}c(t)=\frac{e^{-\zeta\omega_n t}}{\sqrt{(1-\zeta^2)}}(\zeta\omega_n)\sin(\omega_d t+\theta)-\left(\frac{e^{-\zeta\omega_n t}}{\sqrt{(1-\zeta^2)}}\right)\cos(\omega_d t+\theta)\omega_n\sqrt{(1-\zeta^2)}$$

$$= \frac{\omega_n e^{-\zeta\omega_n t}}{\sqrt{(1-\zeta^2)}}\left[\zeta \sin(\,\omega_d t+\theta) - (\sqrt{(1-\zeta^2)})\cos(\omega_d t+\theta)\right]$$

$$= \frac{\omega_n e^{-\zeta\omega_n t}}{\sqrt{(1-\zeta^2)}}[\cos\theta \sin(\,\omega_d t+\theta) - \sin\theta\cos(\omega_d t+\theta)]$$

$$= \frac{\omega_n e^{-\zeta\omega_n t}}{\sqrt{(1-\zeta^2)}}[\sin(\,\omega_d t+\theta-\theta)]$$

$$= \frac{\omega_n e^{-\zeta\omega_n t}}{\sqrt{(1-\zeta^2)}}[\sin(\,\omega_d t)]$$

At t = t_p, $\frac{d}{dt}c(t) = 0$

$$\frac{\omega_n e^{-\zeta\omega_n t_p}}{\sqrt{(1-\zeta^2)}}\left[\sin(\,\omega_d t_p)\right] = 0$$

Since, $e^{-\zeta\omega_n t_p} \neq 0$, the term, $\left[\sin(\,\omega_d t_p)\right] = 0$

When Φ = 0, π, 2π, 3π,…. sin Φ = 0

$$\omega_d t_p = \pi$$

$$t_p = \frac{\pi}{\omega_d}$$

On substituting, we get,

$$t_p = \frac{\pi}{\omega_n\sqrt{(1-\zeta^2)}}$$

4. Peak overshoot, M_p: It is defined as the difference between the peak value of the response and the steady state value. Iris usually expressed in percent of the steady state value. If the time for the peak is t_p, percent peak overshoot is given by,

Maximum percent overshoot = $\frac{c(t_p)-c(\infty)}{c(\infty)}$

$$At\ t = \infty, c(t) = c(\infty) = 1 - \frac{e^{-\zeta\omega_n\infty}}{\sqrt{(1-\zeta^2)}}\sin(\omega_d t+\theta) = 1-0 = 1$$

$$At\ t = t_p, c(t) = c(t_p) = 1 - \frac{e^{-\zeta\omega_n t_p}}{\sqrt{(1-\zeta^2)}}\sin\left(\omega_d t_p+\theta\right)$$

$$= 1 - \frac{e^{-\zeta\omega_n \frac{\pi}{\omega_n\sqrt{(1-\zeta^2)}}}}{\sqrt{(1-\zeta^2)}}\sin\left(\omega_d\frac{\pi}{\omega_d}+\theta\right)$$

$$= 1 - \frac{e^{-\zeta\frac{\pi}{\sqrt{(1-\zeta^2)}}}}{\sqrt{(1-\zeta^2)}}\sin(\pi+\theta)$$

$$= 1 - \frac{e^{-\zeta\frac{\pi}{\sqrt{(1-\zeta^2)}}}}{\sqrt{(1-\zeta^2)}}\sin(\theta) = 1 + \frac{e^{\frac{\zeta\pi}{\sqrt{(1-\zeta^2)}}}}{\sqrt{(1-\zeta^2)}}\sqrt{(1-\zeta^2)}$$

$$\%M_p = \frac{c(t_p) - c(\infty)}{c(\infty)}$$

$$\%M_p = e^{\frac{\pi\zeta}{\sqrt{(1-\zeta^2)}}} \times 100$$

5. Settling time, t_s: It is the time taken by the response to reach and stay within a specified error. It is usually expressed as percentage of final value. The usual tolerable error is 2% and 5% of the final value.

The response of second order system has two components. They are

a. Decaying exponential component, $\frac{e^{-\zeta\omega_n t}}{\sqrt{(1-\zeta^2)}}$

b. Sinusoidal component, $\sin(\omega_d t + \theta)$

In these terms, the decaying component term dampens or reduces the oscillations produced by sinusoidal component. Hence, the settling time is decided by the exponential component. The settling time can be found out by equating exponential component to percentage tolerance errors.

For 2% tolerance error band, at t = t_s, $\frac{e^{-\zeta\omega_n t_s}}{\sqrt{(1-\zeta^2)}} = 0.02$

For least values of ζ, $e^{-\zeta\omega_n t_s} = 0.02$

On taking natural logarithm on both sides, we get,

$$-\zeta\omega_n t_s = \ln(0.02) = -4$$

$$t_s = \frac{4}{\zeta\omega_n} = 4T$$

For 5% tolerance error band, at t = t_s, $\frac{e^{-\zeta\omega_n t_s}}{\sqrt{(1-\zeta^2)}} = 0.05$

For least values of ζ, $e^{-\zeta\omega_n t_s} = 0.05$

On taking natural logarithm on both sides, we get,

$$-\zeta\omega_n t_s = \ln(0.02) = -3$$

$$t_s = \frac{3}{\zeta\omega_n} = 3T$$

$$Settling\ time, t_s = \frac{4}{\zeta\omega_n}\ for\ 2\%\ error$$

$$Settling\ time, t_s = \frac{3}{\zeta\omega_n}\ for\ 5\%\ error$$

The performance of a system is usually evaluated in terms of the following qualities:

- How fast it is able to respond to the input?
- How fast it is reaching the desired output?
- What is the error between the desired output and the actual output, once the transients die down and steady state is achieved?
- Does it oscillate around the desired value?
- Is the output continuously increasing with time or is it bounded?
- These are the specifications to be given for the design of a controller for a given system.