VOGEL APPROXIMATION METHOD

INTRODUCTION

VAM is an improved version of the least cost method that generally produces better solutions. The steps involved in this method are:

Step : 1 : Find the cells having smallest and next to smallest cost in each row and write the difference (called penalty) along the side of the table in row penalty.

Step : 2 : Find the cells having smallest and next to smallest cost in each column and write the difference (called penalty) along the side of the table in each column penalty.

Step : 3 : Select the row or column with the maximum penalty and find cell that has least cost in selected row or column. Allocate as much as possible in this cell. If there is a tie in the values of penalties then select the cell where maximum allocation can be possible

Step : 4 : Adjust the supply & demand and cross out (strike out) the satisfied row or column.

Step : 5 : Repeat this steps until all supply and demand values are 0.

Problem :1 : Find Solution using Voggel's Approximation method

	D1	D2	D3	D4	Supply
S1	19	30	50	10	7
S2	70	30	40	60	9
S3	40	8	70	20	18
Demand	5	8	7	14	

Solution:

Problem Table is

	*D*1	*D*2	*D*3	*D*4	Supply
*S*1	19	30	50	10	7
*S*2	70	30	40	60	9
Demand	5			14	

Σ Supply=Σ Demand → The given transportation problem is balanced.

	*D*1	*D*2	*D*3	*D*4	Supply	Row Penalty
*S*1	19	30	50	10	7	9=19-10
*S*2	70	30	40	60	9	10=40-30
*S*3	40	8	70	20	18	12=20-8
Demand	5	8	7	14		
Column Penalty	21=40-19	22=30-8	10=50-40	10=20-10		

The maximum penalty, 22, occurs in column *D*2.
The minimum *cij* in this column is *c*32 = 8.
The maximum allocation in this cell is min(18,8) = **8**.
It satisfy demand of *D*2 and adjust the supply of *S*3 from 18 to 10 (18 - 8 = 10).

	*D*1	*D*2	*D*3	*D*4	Supply	Row Penalty
*S*1	19	30	50	10	7	9=19-10
*S*2	70	30	40	60	9	20=60-40
*S*3	40	8**(8)**	70	20	10	20=40-20
Demand	5	0	7	14		
Column Penalty	21=40-19	--	10=50-40	10=20-10		

The maximum penalty, 21, occurs in column *D*1.

The minimum *cij* in this column is *c*11 = 19.

The maximum allocation in this cell is min(7,5) = **5**.

It satisfy demand of *D*1 and adjust the supply of *S*1 from 7 to 2 (7 - 5 = 2).

	*D*1	*D*2	*D*3	*D*4	Supply	Row Penalty
*S*1	19**(5)**	30	50	10	2	40=50-10
*S*2	70	30	40	60	9	20=60-40
*S*3	40	8**(8)**	70	20	10	50=70-20
Demand	0	0	7	14		
Column Penalty	--	--	10=50-40	10=20-10		

The maximum penalty, 50, occurs in row *S*3.

The minimum *cij* in this row is *c*34 = 20.

The maximum allocation in this cell is min(10,14) = **10**.

It satisfy supply of *S*3 and adjust the demand of *D*4 from 14 to 4 (14 - 10 = 4).

	*D*1	*D*2	*D*3	*D*4	Supply	Row Penalty
*S*1	19**(5)**	30	50	10	2	40=50-10
*S*2	70	30	40	60	9	20=60-40
*S*3	40	8**(8)**	70	20**(10)**	0	--
Demand	0	0	7	4		
Column Penalty	--	--	10=50-40	50=60-10		

The maximum penalty, 50, occurs in column *D*4.

The minimum *cij* in this column is *c*14 = 10.

The maximum allocation in this cell is min(2,4) = **2**.

It satisfy supply of *S*1 and adjust the demand of *D*4 from 4 to 2 (4 - 2 = 2).

	*D*1	*D*2	*D*3	*D*4	Supply	Row Penalty
*S*1	19**(5)**	30	50	10**(2)**	0	--
*S*2	70	30	40	60	9	20=60-40
*S*3	40	8**(8)**	70	20**(10)**	0	--
Demand	0	0	7	2		
Column Penalty	--	--	40	60		

The maximum penalty, 60, occurs in column *D*4.

The minimum *cij* in this column is *c*24 = 60.

The maximum allocation in this cell is min(9,2) = **2**.

It satisfy demand of *D*4 and adjust the supply of *S*2 from 9 to 7 (9 - 2 = 7).

	*D*1	*D*2	*D*3	*D*4	Supply	Row Penalty
*S*1	19**(5)**	30	50	10**(2)**	0	--
*S*2	70	30	40	60**(2)**	7	40
*S*3	40	8**(8)**	70	20**(10)**	0	--
Demand	0	0	7	0		
Column Penalty	--	--	40	--		

The maximum penalty, 40, occurs in row *S*2.

The minimum *cij* in this row is *c*23 = 40.

The maximum allocation in this cell is min(7,7) = **7**.

It satisfy supply of *S*2 and demand of *D*3.

Initial feasible solution is

	$D1$	$D2$	$D3$	$D4$	Supply	Row Penalty
$S1$	19**(5)**	30	50	10**(2)**	7	9 | 9 | 40 | 40 | -- | -- |
$S2$	70	30	40**(7)**	60**(2)**	9	10 | 20 | 20 | 20 | 20 | 40 |
$S3$	40	8**(8)**	70	20**(10)**	18	12 | 20 | 50 | -- | -- | -- |
Demand	5	8	7	14		
Column Penalty	21 21 -- -- -- --	22 -- -- -- -- --	10 10 10 10 40 40	10 10 10 50 60 --		

The minimum total transportation cost =19×5+10×2+40×7+60×2+8×8+20×10=779

Here, the number of allocated cells = 6 is equal to m + n - 1 = 3 + 4 - 1 = 6

∴ This solution is non-degenerate.

Optimality test using modi method...

Allocation Table is

	$D1$	$D2$	$D3$	$D4$	Supply
$S1$	19 **(5)**	30	50	10 **(2)**	7
$S2$	70	30	40 **(7)**	60 **(2)**	9
$S3$	40	8 **(8)**	70	20 **(10)**	18
Demand	5	8	7	14	

Iteration-1 of optimality test

1. Find u_i and v_j for all occupied cells(i,j), where $c_{ij}=u_i+v_j$

Substituting, $v_4=0$, we get

$c_{14}=u_1+v_4 \Rightarrow u_1=c_{14}-v_4 \Rightarrow u_1=10-0 \Rightarrow u_1=10$

$c_{11}=u_1+v_1 \Rightarrow v_1=c_{11}-u_1 \Rightarrow v_1=19-10 \Rightarrow v_1=9$

$c_{24}=u_2+v_4 \Rightarrow u_2=c_{24}-v_4 \Rightarrow u_2=60-0 \Rightarrow u_2=60$

$c_{23}=u_2+v_3 \Rightarrow v_3=c_{23}-u_2 \Rightarrow v_3=40-60 \Rightarrow v_3=-20$

$c_{34}=u_3+v_4 \Rightarrow u_3=c_{34}-v_4 \Rightarrow u_3=20-0 \Rightarrow u_3=20$

$c_{32}=u_3+v_2 \Rightarrow v_2=c_{32}-u_3 \Rightarrow v_2=8-20 \Rightarrow v_2=-12$

	D_1	D_2	D_3	D_4	Supply	u_i
S_1	19 **(5)**	30	50	10 **(2)**	7	$u_1=10$
S_2	70	30	40 **(7)**	60 **(2)**	9	$u_2=60$
S_3	40	8 **(8)**	70	20 **(10)**	18	$u_3=20$
Demand	5	8	7	14		
v_j	$v_1=9$	$v_2=-12$	$v_3=-20$	$v_4=0$		

Find d_{ij} for all unoccupied cells(i,j), where $d_{ij}=c_{ij}-(u_i+v_j)$

$d_{12}=c_{12}-(u_1+v_2)=30-(10-12)=32$

$d_{13}=c_{13}-(u_1+v_3)=50-(10-20)=60$

$d_{21}=c_{21}-(u_2+v_1)=70-(60+9)=1$

$d_{22}=c_{22}-(u_2+v_2)=30-(60-12)=-18$

$d_{31}=c_{31}-(u_3+v_1)=40-(20+9)=11$

$d_{33}=c_{33}-(u_3+v_3)=70-(20-20)=70$

	D_1	D_2	D_3	D_4	Supply	u_i
S_1	19 **(5)**	30 [32]	50 [60]	10 **(2)**	7	$u_1=10$
S_2	70 [1]	30 [-18]	40 **(7)**	60 **(2)**	9	$u_2=60$
S_3	40 [11]	8 **(8)**	70 [70]	20 **(10)**	18	$u_3=20$
Demand	5	8	7	14		
v_j	$v_1=9$	$v_2=-12$	$v_3=-20$	$v_4=0$		

Now choose the minimum negative value from all d_{ij} (opportunity cost) = d_{22} = [-18]

and draw a closed path from S_2D_2.

Closed path is $S_2D_2 \to S_2D_4 \to S_3D_4 \to S_3D_2$

Closed path and plus/minus sign allocation

	D_1	D_2	D_3	D_4	Supply	u_i
S_1	19 **(5)**	30 [32]	50 [60]	10 **(2)**	7	$u_1=10$
S_2	70 [1]	30 [-18] **(+)**	40 **(7)**	60 **(2)** **(-)**	9	$u_2=60$
S_3	40 [11]	8 **(8)** **(-)**	70 [70]	20 **(10)** **(+)**	18	$u_3=20$
Demand	5	8	7	14		
v_j	$v_1=9$	$v_2=-12$	$v_3=-20$	$v_4=0$		

Minimum allocated value among all negative position **(-)** on closed path = 2

Substract 2 from all **(-)** and Add it to all **(+)**

	D_1	D_2	D_3	D_4	Supply
S_1	19 **(5)**	30	50	10 **(2)**	7
S_2	70	30 **(2)**	40 **(7)**	60	9
S_3	40	8 **(6)**	70	20 **(12)**	18
Demand	5	8	7	14	

Repeat the step 1 to 4, until an optimal solution is obtained.

Iteration-2 of optimality test

Find u_i and v_j for all occupied cells(i,j), where $c_{ij}=u_i+v_j$

Substituting, $u_1=0$, we get

$c_{11}=u_1+v_1 \Rightarrow v_1=c_{11}-u_1 \Rightarrow v_1=19-0 \Rightarrow v_1=19$

$c_{14}=u_1+v_4 \Rightarrow v_4=c_{14}-u_1 \Rightarrow v_4=10-0 \Rightarrow v_4=10$

$c_{34}=u_3+v_4 \Rightarrow u_3=c_{34}-v_4 \Rightarrow u_3=20-10 \Rightarrow u_3=10$

$c_{32}=u_3+v_2 \Rightarrow v_2=c_{32}-u_3 \Rightarrow v_2=8-10 \Rightarrow v_2=-2$

$c_{22}=u_2+v_2 \Rightarrow u_2=c_{22}-v_2 \Rightarrow u_2=30+2 \Rightarrow u_2=32$

$c_{23}=u_2+v_3 \Rightarrow v_3=c_{23}-u_2 \Rightarrow v_3=40-32 \Rightarrow v_3=8$

	$D1$	$D2$	$D3$	$D4$	Supply	ui
$S1$	19 **(5)**	30	50	10 **(2)**	7	$u1=0$
$S2$	70	30 **(2)**	40 **(7)**	60	9	$u2=32$
$S3$	40	8 **(6)**	70	20 **(12)**	18	$u3=10$
Demand	5	8	7	14		
vj	$v1=19$	$v2=-2$	$v3=8$	$v4=10$		

Find dij for all unoccupied cells(i,j), where $dij=cij-(ui+vj)$

$d12=c12-(u1+v2)=30-(0-2)=32$

$d13=c13-(u1+v3)=50-(0+8)=42$

$d21=c21-(u2+v1)=70-(32+19)=19$

$d24=c24-(u2+v4)=60-(32+10)=18$

$d31=c31-(u3+v1)=40-(10+19)=11$

$d33=c33-(u3+v3)=70-(10+8)=52$

	$D1$	$D2$	$D3$	$D4$	Supply	ui
$S1$	19 **(5)**	30 [32]	50 [42]	10 **(2)**	7	$u1=0$
$S2$	70 [19]	30 **(2)**	40 **(7)**	60 [18]	9	$u2=32$
$S3$	40 [11]	8 **(6)**	70 [52]	20 **(12)**	18	$u3=10$
Demand	5	8	7	14		
vj	$v1=19$	$v2=-2$	$v3=8$	$v4=10$		

Since all $dij \geq 0$. So final optimal solution is arrived.

	$D1$	$D2$	$D3$	$D4$	Supply
$S1$	19 **(5)**	30	50	10 **(2)**	7
$S2$	70	30 **(2)**	40 **(7)**	60	9
$S3$	40	8 **(6)**	70	20 **(12)**	18
Demand	5	8	7	14	

The minimum total transportation cost $=19\times5+10\times2+30\times2+40\times7+8\times6+20\times12=743$

Problem : 2 : Find Solution using Vogel's Approximation method

	D1	D2	D3	D4	Supply
S1	11	13	17	14	250
S2	16	18	14	10	300
S3	21	24	13	10	400
Demand	200	225	275	250	

Solution:

Problem Table is

	$D1$	$D2$	$D3$	$D4$	Supply
$S1$	11	13	17	14	250
$S2$	16	18	14	10	300
$S3$	21	24	13	10	400
Demand	200	225	275	250	

Σ Supply=Σ Demand $\rightarrow$ The given transportation problem is balanced.

	$D1$	$D2$	$D3$	$D4$	Supply	Row Penalty
$S1$	11	13	17	14	250	2=13-11
$S2$	16	18	14	10	300	4=14-10
$S3$	21	24	13	10	400	3=13-10
Demand	200	225	275	250		
Column Penalty	5=16-11	5=18-13	1=14-13	0=10-10		

The maximum penalty, 5, occurs in column $D1$.

The minimum cij in this column is $c11 = 11$.

The maximum allocation in this cell is min(250,200) = **200**.

It satisfy demand of $D1$ and adjust the supply of $S1$ from 250 to 50 (250 - 200 = 50).

	*D*1	*D*2	*D*3	*D*4	Supply	Row Penalty
*S*1	11**(200)**	13	17	14	50	1=14-13
*S*2	16	18	14	10	300	4=14-10
*S*3	21	24	13	10	400	3=13-10
Demand	0	225	275	250		
Column Penalty	--	5=18-13	1=14-13	0=10-10		

The maximum penalty, 5, occurs in column *D*2.

The minimum *cij* in this column is *c*12 = 13.

The maximum allocation in this cell is min(50,225) = **50**.

It satisfy supply of *S*1 and adjust the demand of *D*2 from 225 to 175 (225 - 50 = 175).

	*D*1	*D*2	*D*3	*D*4	Supply	Row Penalty
*S*1	11**(200)**	13**(50)**	17	14	0	--
*S*2	16	18	14	10	300	4=14-10
*S*3	21	24	13	10	400	3=13-10
Demand	0	175	275	250		
Column Penalty	--	6=24-18	1=14-13	0=10-10		

The maximum penalty, 6, occurs in column *D*2.

The minimum *cij* in this column is *c*22 = 18.

The maximum allocation in this cell is min(300,175) = **175**.

It satisfy demand of *D*2 and adjust the supply of *S*2 from 300 to 125 (300 - 175 = 125).

	*D*1	*D*2	*D*3	*D*4	Supply	Row Penalty
*S*1	11**(200)**	13**(50)**	17	14	0	--
*S*2	16	18**(175)**	14	10	125	4=14-10
*S*3	21	24	13	10	400	3=13-10
Demand	0	0	275	250		
Column Penalty	--	--	1=14-13	0=10-10		

The maximum penalty, 4, occurs in row *S*2.

The minimum *cij* in this row is *c*24 = 10.

The maximum allocation in this cell is min(125,250) = **125**.

It satisfy supply of *S*2 and adjust the demand of *D*4 from 250 to 125 (250 - 125 = 125).

	*D*1	*D*2	*D*3	*D*4	Supply	Row Penalty
*S*1	11**(200)**	13**(50)**	17	14	0	--
*S*2	16	18**(175)**	14	10**(125)**	0	--
*S*3	21	24	13	10	400	3=13-10
Demand	0	0	275	125		
Column Penalty	--	--	13	10		

The maximum penalty, 13, occurs in column *D*3.

The minimum *cij* in this column is *c*33 = 13.

The maximum allocation in this cell is min(400,275) = **275**.

It satisfy demand of *D*3 and adjust the supply of *S*3 from 400 to 125 (400 - 275 = 125).

	*D*1	*D*2	*D*3	*D*4	Supply	Row Penalty
*S*1	11**(200)**	13**(50)**	17	14	0	--
*S*2	16	18**(175)**	14	10**(125)**	0	--
*S*3	21	24	13**(275)**	10	125	10
Demand	0	0	0	125		
Column Penalty	--	--	--	10		

The maximum penalty, 10, occurs in row *S*3.

The minimum *cij* in this row is *c*34 = 10.

The maximum allocation in this cell is min(125,125) = **125**.

It satisfy supply of *S*3 and demand of *D*4.

Initial feasible solution is

	*D*1	*D*2	*D*3	*D*4	Supply	Row Penalty
*S*1	11**(200)**	13**(50)**	17	14	250	2 | 1 | -- | -- | -- | -- |
*S*2	16	18**(175)**	14	10**(125)**	300	4 | 4 | 4 | 4 | -- | -- |
*S*3	21	24	13**(275)**	10**(125)**	400	3 | 3 | 3 | 3 | 3 | 10 |
Demand	200	225	275	250		
Column Penalty	5 -- -- -- -- --	5 5 6 -- -- --	1 1 1 1 13 --	0 0 0 0 10 10		

The minimum total transportation

cost $=11\times200+13\times50+18\times175+10\times125+13\times275+10\times125=12075$

Here, the number of allocated cells = 6 is equal to m + n - 1 = 3 + 4 - 1 = 6

∴ This solution is non-degenerate.

Optimality test using modi method

Allocation Table is

	$D1$	$D2$	$D3$	$D4$	Supply
$S1$	11 **(200)**	13 **(50)**	17	14	250
$S2$	16	18 **(175)**	14	10 **(125)**	300
$S3$	21	24	13 **(275)**	10 **(125)**	400
Demand	200	225	275	250	

Iteration-1 of optimality test

Find ui and vj for all occupied cells(i,j), where $cij=ui+vj$

Substituting, $u1=0$, we get

$c11=u1+v1\Rightarrow v1=c11-u1\Rightarrow v1=11-0\Rightarrow v1=11$

$c12=u1+v2\Rightarrow v2=c12-u1\Rightarrow v2=13-0\Rightarrow v2=13$

$c22=u2+v2\Rightarrow u2=c22-v2\Rightarrow u2=18-13\Rightarrow u2=5$

$c24=u2+v4\Rightarrow v4=c24-u2\Rightarrow v4=10-5\Rightarrow v4=5$

$c34=u3+v4\Rightarrow u3=c34-v4\Rightarrow u3=10-5\Rightarrow u3=5$

$c33=u3+v3\Rightarrow v3=c33-u3\Rightarrow v3=13-5\Rightarrow v3=8$

	$D1$	$D2$	$D3$	$D4$	Supply	ui
$S1$	11 **(200)**	13 **(50)**	17	14	250	$u1=0$
$S2$	16	18 **(175)**	14	10 **(125)**	300	$u2=5$
$S3$	21	24	13 **(275)**	10 **(125)**	400	$u3=5$
Demand	200	225	275	250		
vj	$v1=11$	$v2=13$	$v3=8$	$v4=5$		

Find d_{ij} for all unoccupied cells(i,j), where $d_{ij}=c_{ij}-(u_i+v_j)$

.$d_{13}=c_{13}-(u_1+v_3)=17-(0+8)=9$
$d_{14}=c_{14}-(u_1+v_4)=14-(0+5)=9$
$d_{21}=c_{21}-(u_2+v_1)=16-(5+11)=0$
$d_{23}=c_{23}-(u_2+v_3)=14-(5+8)=1$
$d_{31}=c_{31}-(u_3+v_1)=21-(5+11)=5$
$d_{32}=c_{32}-(u_3+v_2)=24-(5+13)=6$

	D_1	D_2	D_3	D_4	Supply	u_i
S_1	11 **(200)**	13 **(50)**	17 [9]	14 [9]	250	$u_1=0$
S_2	16 [0]	18 **(175)**	14 [1]	10 **(125)**	300	$u_2=5$
S_3	21 [5]	24 [6]	13 **(275)**	10 **(125)**	400	$u_3=5$
Demand	200	225	275	250		
v_j	$v_1=11$	$v_2=13$	$v_3=8$	$v_4=5$		

Since all $d_{ij} \geq 0$. So final optimal solution is arrived.

	D_1	D_2	D_3	D_4	Supply
S_1	11 **(200)**	13 **(50)**	17	14	250
S_2	16	18 **(175)**	14	10 **(125)**	300
S_3	21	24	13 **(275)**	10 **(125)**	400
Demand	200	225	275	250	

The minimum total transportation

cost $=11\times200+13\times50+18\times175+10\times125+13\times275+10\times125=12075$

Notice alternate solution is available with unoccupied cell S_2D_1:d_{21} = [0], but with the same optimal value.

Problem 3 : Find Solution using Voggel's Approximation method

	D1	D2	D3	Supply
S1	4	8	8	76
S2	16	24	16	82
S3	8	16	24	77
Demand	72	102	41	

Solution:
Problem Table is

	$D1$	$D2$	$D3$	Supply
$S1$	4	8	8	76
$S2$	16	24	16	82
$S3$	8	16	24	77
Demand	72	102	41	

Here Total Demand = 215 is less than Total Supply = 235. So We add a dummy demand constraint with 0 unit cost and with allocation 20. Now, The modified table is

	$D1$	$D2$	$D3$	$D4$	Supply
$S1$	4	8	8	0	76
$S2$	16	24	16	0	82
$S3$	8	16	24	0	77
Demand	72	102	41	20	

	*D*1	*D*2	*D*3	*D4*	Supply	Row Penalty
*S*1	4	8	8	0	76	4=4-0
*S*2	16	24	16	0	82	16=16-0
*S*3	8	16	24	0	77	8=8-0
Demand	72	102	41	20		
Column Penalty	4=8-4	8=16-8	8=16-8	0=0-0		

The maximum penalty, 16, occurs in row *S*2.

The minimum *cij* in this row is *c*24 = 0.

The maximum allocation in this cell is min(82,20) = 20.

It satisfy demand of *Ddummy* and adjust the supply of *S*2 from 82 to 62 (82 - 20 = 62).

	*D*1	*D*2	*D*3	*D4*	Supply	Row Penalty
*S*1	4	8	8	0	76	4=8-4
*S*2	16	24	16	0(20)	62	0=16-16
*S*3	8	16	24	0	77	8=16-8
Demand	72	102	41	0		
Column Penalty	4=8-4	8=16-8	8=16-8	--		

The maximum penalty, 8, occurs in column *D*2.

The minimum *cij* in this column is *c*12 = 8.

The maximum allocation in this cell is min(76,102) = 76.

It satisfy supply of *S*1 and adjust the demand of *D*2 from 102 to 26 (102 - 76 = 26).

	*D*1	*D*2	*D*3	*D4*	Supply	Row Penalty
*S*1	4	8(76)	8	0	0	--
*S*2	16	24	16	0(20)	62	0=16-16
*S*3	8	16	24	0	77	8=16-8
Demand	72	26	41	0		
Column Penalty	8=16-8	8=24-16	8=24-16	--		

The maximum penalty, 8, occurs in row *S*3.

The minimum *cij* in this row is *c*31 = 8.

The maximum allocation in this cell is min(77,72) = 72.

It satisfy demand of *D*1 and adjust the supply of *S*3 from 77 to 5 (77 - 72 = 5).

	*D*1	*D*2	*D*3	*D4*	Supply	Row Penalty
*S*1	4	8(76)	8	0	0	--
*S*2	16	24	16	0(20)	62	8=24-16
*S*3	8(72)	16	24	0	5	8=24-16
Demand	0	26	41	0		
Column Penalty	--	8=24-16	8=24-16	--		

The maximum penalty, 8, occurs in row *S*2.

The minimum *cij* in this row is *c*23 = 16.

The maximum allocation in this cell is min(62,41) = 41.

It satisfy demand of *D*3 and adjust the supply of *S*2 from 62 to 21 (62 - 41 = 21).

	*D*1	*D*2	*D*3	*D*4	Supply	Row Penalty
*S*1	4	8(76)	8	0	0	--
*S*2	16	24	16(41)	0(20)	21	24
*S*3	8(72)	16	24	0	5	16
Demand	0	26	0	0		
Column Penalty	--	8=24-16	--	--		

The maximum penalty, 24, occurs in row *S*2.

The minimum *cij* in this row is *c*22 = 24.

The maximum allocation in this cell is min(21,26) = 21.

It satisfy supply of *S*2 and adjust the demand of *D*2 from 26 to 5 (26 - 21 = 5).

	*D*1	*D*2	*D*3	*D*4	Supply	Row Penalty
*S*1	4	8(76)	8	0	0	--
*S*2	16	24(21)	16(41)	0(20)	0	--
*S*3	8(72)	16	24	0	5	16
Demand	0	5	0	0		
Column Penalty	--	16	--	--		

The maximum penalty, 16, occurs in row *S*3.

The minimum *cij* in this row is *c*32 = 16.

The maximum allocation in this cell is min(5,5) = 5.

It satisfy supply of *S*3 and demand of *D*2.

Initial feasible solution is

	*D*1	*D*2	*D*3	*D*4	Supply	Row Penalty
*S*1	4	8(76)	8	0	76	4 | 4 | -- | -- | -- | -- |
*S*2	16	24(21)	16(41)	0(20)	82	16 | 0 | 0 | 8 | 24 | -- |
*S*3	8(72)	16(5)	24	0	77	8 | 8 | 8 | 8 | 16 | 16 |
Demand	72	102	41	20		
Column Penalty	4 4 8 -- -- --	8 8 8 8 8 16	8 8 8 8 -- --	0 -- -- -- -- --		

The minimum total transportation cost =8×76+24×21+16×41+0×20+8×72+16×5=2424

Here, the number of allocated cells = 6 is equal to m + n - 1 = 3 + 4 - 1 = 6

∴ This solution is non-degenerate.

Optimality test using modi method

Allocation Table is

	*D*1	*D*2	*D*3	*Ddummy*	Supply
*S*1	4	8 **(76)**	8	0	76
*S*2	16	24 **(21)**	16 **(41)**	0 **(20)**	82
*S*3	8 **(72)**	16 **(5)**	24	0	77
Demand	72	102	41	20	

Iteration-1 of optimality test

Find ui and vj for all occupied cells(i,j), where $cij=ui+vj$

Substituting, $u2=0$, we get

$c22=u2+v2 \Rightarrow v2=c22-u2 \Rightarrow v2=24-0 \Rightarrow v2=24$

$c12=u1+v2 \Rightarrow u1=c12-v2 \Rightarrow u1=8-24 \Rightarrow u1=-16$

$c32=u3+v2 \Rightarrow u3=c32-v2 \Rightarrow u3=16-24 \Rightarrow u3=-8$

$c31=u3+v1 \Rightarrow v1=c31-u3 \Rightarrow v1=8+8 \Rightarrow v1=16$

$c23=u2+v3 \Rightarrow v3=c23-u2 \Rightarrow v3=16-0 \Rightarrow v3=16$

$c24=u2+v4 \Rightarrow v4=c24-u2 \Rightarrow v4=0-0 \Rightarrow v4=0$

	*D*1	*D*2	*D*3	*Ddummy*	Supply	*ui*
*S*1	4	8 **(76)**	8	0	76	$u1=-16$
*S*2	16	24 **(21)**	16 **(41)**	0 **(20)**	82	$u2=0$
*S*3	8 **(72)**	16 **(5)**	24	0	77	$u3=-8$
Demand	72	102	41	20		
vj	$v1=16$	$v2=24$	$v3=16$	$v4=0$		

Find dij for all unoccupied cells(i,j), where $dij=cij-(ui+vj)$

$d11=c11-(u1+v1)=4-(-16+16)=4$

$d13=c13-(u1+v3)=8-(-16+16)=8$

$d14=c14-(u1+v4)=0-(-16+0)=16$

$d21=c21-(u2+v1)=16-(0+16)=0$

$d33=c33-(u3+v3)=24-(-8+16)=16$

$d34=c34-(u3+v4)=0-(-8+0)=8$

	*D*1	*D*2	*D*3	*Ddummy*	Supply	*ui*
*S*1	4 [4]	8 **(76)**	8 [8]	0 [16]	76	$u1=-16$
*S*2	16 [0]	24 **(21)**	16 **(41)**	0 **(20)**	82	$u2=0$
*S*3	8 **(72)**	16 **(5)**	24 [16]	0 [8]	77	$u3=-8$
Demand	72	102	41	20		
vj	$v1=16$	$v2=24$	$v3=16$	$v4=0$		

Since all *dij*≥0. So final optimal solution is arrived.

	*D*1	*D*2	*D*3	*Ddummy*	Supply
*S*1	4	8 **(76)**	8	0	76
*S*2	16	24 **(21)**	16 **(41)**	0 **(20)**	82
*S*3	8 **(72)**	16 **(5)**	24	0	77
Demand	72	102	41	20	

The minimum total transportation cost =8×76+24×21+16×41+0×20+8×72+16×5=2424

Problem 4: Find Solution using Voggel's Approximation method (Maximization)

	D1	D2	D3	Supply
S1	8	5	6	120
S2	15	10	12	80
S3	3	9	10	80
Demand	150	80	50	

Solution: Problem Table is

	*D*1	*D*2	*D*3	Supply
*S*1	8	5	6	120
*S*2	15	10	12	80
*S*3	3	9	10	80
Demand	150	80	50	

Problem is Maximization, so convert it to minimization by subtracting all the elements from max element 15

	*D*1	*D*2	*D*3	Supply
*S*1	7	10	9	120
*S*2	0	5	3	80
*S*3	12	6	5	80
Demand	150	80	50	

	*D*1	*D*2	*D*3	Supply	Row Penalty
*S*1	7	10	9	120	2=9-7
*S*2	0	5	3	80	3=3-0
*S*3	12	6	5	80	1=6-5
Demand	150	80	50		
Column Penalty	7=7-0	1=6-5	2=5-3		

The maximum penalty, 7, occurs in column *D*1.

The minimum *cij* in this column is *c*21=0.

The maximum allocation in this cell is min(80,150) = 80.

It satisfy supply of *S*2 and adjust the demand of *D*1 from 150 to 70 (150 - 80=70).

	*D*1	*D*2	*D*3	Supply	Row Penalty
*S*1	7	10	9	120	2=9-7
*S*2	0(80)	5	3	0	--
*S*3	12	6	5	80	1=6-5
Demand	70	80	50		
Column Penalty	5=12-7	4=10-6	4=9-5		

The maximum penalty, 5, occurs in column *D*1.

The minimum *cij* in this column is *c*11=7.

The maximum allocation in this cell is min(120,70) = 70.

It satisfy demand of *D*1 and adjust the supply of *S*1 from 120 to 50 (120 - 70=50).

	*D*1	*D*2	*D*3	Supply	Row Penalty
*S*1	7(70)	10	9	50	1=10-9
*S*2	0(80)	5	3	0	--
*S*3	12	6	5	80	1=6-5
Demand	0	80	50		
Column Penalty	--	4=10-6	4=9-5		

The maximum penalty, 4, occurs in column *D*3.

The minimum *cij* in this column is *c*33=5.

The maximum allocation in this cell is min(80,50) = 50.

It satisfy demand of *D*3 and adjust the supply of *S*3 from 80 to 30 (80 - 50=30).

	*D*1	*D*2	*D*3	Supply	Row Penalty
*S*1	7(70)	10	9	50	10
*S*2	0(80)	5	3	0	--
*S*3	12	6	5(50)	30	6
Demand	0	80	0		
Column Penalty	--	4=10-6	--		

The maximum penalty, 10, occurs in row *S*1.

The minimum *cij* in this row is *c*12=10.

The maximum allocation in this cell is min(50,80) = 50.

It satisfy supply of *S*1 and adjust the demand of *D*2 from 80 to 30 (80 - 50=30).

	D1	D2	D3	Supply	Row Penalty
S1	7(70)	10(50)	9	0	--
S2	0(80)	5	3	0	--
S3	12	6	5(50)	30	6
Demand	0	30	0		
Column Penalty	--	6	--		

The maximum penalty, 6, occurs in row *S*3.

The minimum *cij* in this row is *c*32=6.

The maximum allocation in this cell is min(30,30) = 30.

It satisfy supply of *S*3 and demand of *D*2.

Initial feasible solution is

	D1	D2	D3	Supply	Row Penalty
S1	7(70)	10(50)	9	120	2 | 2 | 1 | 10 | -- |
S2	0(80)	5	3	80	3 | -- | -- | -- | -- |
S3	12	6(30)	5(50)	80	1 | 1 | 1 | 6 | 6 |
Demand	150	80	50		
Column Penalty	7 5 -- -- --	1 4 4 4 6	2 4 4 -- --		

Allocations in the original problem

	$D1$	$D2$	$D3$	Supply
$S1$	8 (70)	5 (50)	6	120
$S2$	15 (80)	10	12	80
$S3$	3	9 (30)	10 (50)	80
Demand	150	80	50	

The maximum profit $=8\times70+5\times50+15\times80+9\times30+10\times50=2780$

Here, the number of allocated cells = 5 is equal to m + n - 1 = 3 + 3 - 1 = 5

∴ This solution is non-degenerate.

Problem 5: Find Solution using Voggel's Approximation method, also find optimal solution using modi method

	D1	D2	D3	D4	D5	Supply
S1	5	8	6	6	3	8
S2	4	7	7	6	5	5
S3	8	4	6	6	4	9
Demand	4	4	5	4	8	

Solution:

Problem Table is

	$D1$	$D2$	$D3$	$D4$	$D5$	Supply
$S1$	5	8	6	6	3	8
$S2$	4	7	7	6	5	5
$S3$	8	4	6	6	4	9
Demand	4	4	5	4	8	

Here Total Demand = 25 is greater than Total Supply = 22. So We add a dummy supply constraint with 0 unit cost and with allocation 3.

Now, The modified table is

	$D1$	$D2$	$D3$	$D4$	$D5$	Supply
$S1$	5	8	6	6	3	8
$S2$	4	7	7	6	5	5
$S3$	8	4	6	6	4	9
$S4$	0	0	0	0	0	3
Demand	4	4	5	4	8	

	$D1$	$D2$	$D3$	$D4$	$D5$	Supply	Row Penalty
$S1$	5	8	6	6	3	8	2=5-3
$S2$	4	7	7	6	5	5	1=5-4
$S3$	8	4	6	6	4	9	0=4-4
$S4$	0	0	0	0	0	3	0=0-0
Demand	4	4	5	4	8		
Column Penalty	4=4-0	4=4-0	6=6-0	6=6-0	3=3-0		

The maximum penalty, 6, occurs in column $D3$.

The minimum cij in this column is $c43 = 0$.

The maximum allocation in this cell is min(3,5) = 3.

It satisfy supply of *Sdummy* and adjust the demand of $D3$ from 5 to 2 (5 - 3 = 2).

	$D1$	$D2$	$D3$	$D4$	$D5$	Supply	Row Penalty
$S1$	5	8	6	6	3	8	2=5-3
$S2$	4	7	7	6	5	5	1=5-4
$S3$	8	4	6	6	4	9	0=4-4
$S4$	0	0	0(3)	0	0	0	--
Demand	4	4	2	4	8		
Column Penalty	1=5-4	3=7-4	0=6-6	0=6-6	1=4-3		

The maximum penalty, 3, occurs in column $D2$.

The minimum c_{ij} in this column is $c_{32} = 4$.

The maximum allocation in this cell is min(9,4) = 4.

It satisfy demand of $D2$ and adjust the supply of $S3$ from 9 to 5 (9 - 4 = 5).

	$D1$	$D2$	$D3$	$D4$	$D5$	Supply	Row Penalty
$S1$	5	8	6	6	3	8	2=5-3
$S2$	4	7	7	6	5	5	1=5-4
$S3$	8	4(4)	6	6	4	5	2=6-4
$S4$	0	0	0(3)	0	0	0	--
Demand	4	0	2	4	8		
Column Penalty	1=5-4	--	0=6-6	0=6-6	1=4-3		

The maximum penalty, 2, occurs in row $S1$.

The minimum c_{ij} in this row is $c_{15} = 3$.

The maximum allocation in this cell is min(8,8) = 8.

It satisfy supply of $S1$ and demand of $D5$.

	$D1$	$D2$	$D3$	$D4$	$D5$	Supply	Row Penalty
$S1$	5	8	6	6	3(8)	0	--
$S2$	4	7	7	6	5	5	2=6-4
$S3$	8	4(4)	6	6	4	5	0=6-6
$S4$	0	0	0(3)	0	0	0	--
Demand	4	0	2	4	0		
Column Penalty	4=8-4	--	1=7-6	0=6-6	--		

The maximum penalty, 4, occurs in column $D1$.

The minimum c_{ij} in this column is $c_{21} = 4$.

The maximum allocation in this cell is min(5,4) = 4.

It satisfy demand of $D1$ and adjust the supply of $S2$ from 5 to 1 (5 - 4 = 1).

	$D1$	$D2$	$D3$	$D4$	$D5$	Supply	Row Penalty
$S1$	5	8	6	6	3(8)	0	--
$S2$	4(4)	7	7	6	5	1	1=7-6
$S3$	8	4(4)	6	6	4	5	0=6-6
$S4$	0	0	0(3)	0	0	0	--
Demand	0	0	2	4	0		
Column Penalty	--	--	1=7-6	0=6-6	--		

The maximum penalty, 1, occurs in column $D3$.

The minimum c_{ij} in this column is $c_{33} = 6$.

The maximum allocation in this cell is min(5,2) = 2.

It satisfy demand of $D3$ and adjust the supply of $S3$ from 5 to 3 (5 - 2 = 3).

	$D1$	$D2$	$D3$	$D4$	$D5$	Supply	Row Penalty
$S1$	5	8	6	6	3(8)	0	--
$S2$	4(4)	7	7	6	5	1	6
$S3$	8	4(4)	6(2)	6	4	3	6
$S4$	0	0	0(3)	0	0	0	--
Demand	0	0	0	4	0		
Column Penalty	--	--	--	0=6-6	--		

The maximum penalty, 6, occurs in row $S3$.

The minimum cij in this row is $c34 = 6$.

The maximum allocation in this cell is min(3,4) = 3.

It satisfy supply of $S3$ and adjust the demand of $D4$ from 4 to 1 (4 - 3 = 1).

	$D1$	$D2$	$D3$	$D4$	$D5$	Supply	Row Penalty
$S1$	5	8	6	6	3(8)	0	--
$S2$	4(4)	7	7	6	5	1	6
$S3$	8	4(4)	6(2)	6(3)	4	0	--
$S4$	0	0	0(3)	0	0	0	--
Demand	0	0	0	1	0		
Column Penalty	--	--	--	6	--		

The maximum penalty, 6, occurs in row $S2$.

The minimum cij in this row is $c24 = 6$.

The maximum allocation in this cell is min(1,1) = 1.

It satisfy supply of $S2$ and demand of $D4$.

Initial feasible solution is

	$D1$	$D2$	$D3$	$D4$	$D5$	Supply	Row Penalty
$S1$	5	8	6	6	3(8)	8	2 | 2 | 2 | -- | -- | -- | -- |
$S2$	4(4)	7	7	6(1)	5	5	1 | 1 | 1 | 2 | 1 | 6 | 6 |
$S3$	8	4(4)	6(2)	6(3)	4	9	0 | 0 | 2 | 0 | 0 | 6 | -- |
$S4$	0	0	0(3)	0	0	3	0 | -- | -- | -- | -- | -- | -- |
Demand	4	4	5	4	8		
Column Penalty	4 1 1 4 -- -- --	4 3 -- -- -- -- --	6 0 0 1 1 -- --	6 0 0 0 0 0 6	3 1 1 -- -- -- --		

The minimum total transportation cost $=3 \times 8+4 \times 4+6 \times 1+4 \times 4+6 \times 2+6 \times 3+0 \times 3=92$

Here, the number of allocated cells = 7, which is one less than to m + n - 1 = 4 + 5 - 1 = 8

∴ This solution is degenerate

To resolve degeneracy, we make use of an artificial quantity(d).

The quantity d is assigned to that unoccupied cell, which has the minimum transportation cost.

The quantity d is assigned to $S3D5$, which has the minimum transportation cost = 4.

	$D1$	$D2$	$D3$	$D4$	$D5$	Supply
$S1$	5	8	6	6	3 (8)	8
$S2$	4 (4)	7	7	6 (1)	5	5
$S3$	8	4 (4)	6 (2)	6 (3)	4 (d)	9
$S4$	0	0	0 (3)	0	0	3
Demand	4	4	5	4	8	

Optimality test using modi method, Allocation Table is

	$D1$	$D2$	$D3$	$D4$	$D5$	Supply
$S1$	5	8	6	6	3 (8)	8
$S2$	4 (4)	7	7	6 (1)	5	5
$S3$	8	4 (4)	6 (2)	6 (3)	4 (d)	9
$Sdummy$	0	0	0 (3)	0	0	3
Demand	4	4	5	4	8	

Iteration-1 of optimality test

Find ui and vj for all occupied cells(i,j), where $cij=ui+vj$

Substituting, $u3=0$, we get

$c32=u3+v2 \Rightarrow v2=c32-u3 \Rightarrow v2=4-0 \Rightarrow v2=4$

$c33=u3+v3 \Rightarrow v3=c33-u3 \Rightarrow v3=6-0 \Rightarrow v3=6$

$c43=u4+v3 \Rightarrow u4=c43-v3 \Rightarrow u4=0-6 \Rightarrow u4=-6$

$c34=u3+v4 \Rightarrow v4=c34-u3 \Rightarrow v4=6-0 \Rightarrow v4=6$

$c24=u2+v4 \Rightarrow u2=c24-v4 \Rightarrow u2=6-6 \Rightarrow u2=0$

$c21=u2+v1 \Rightarrow v1=c21-u2 \Rightarrow v1=4-0 \Rightarrow v1=4$

$c35=u3+v5 \Rightarrow v5=c35-u3 \Rightarrow v5=4-0 \Rightarrow v5=4$

$c15=u1+v5 \Rightarrow u1=c15-v5 \Rightarrow u1=3-4 \Rightarrow u1=-1$

	$D1$	$D2$	$D3$	$D4$	$D5$	Supply	ui
$S1$	5	8	6	6	3 (8)	8	$u1=-1$
$S2$	4 (4)	7	7	6 (1)	5	5	$u2=0$
$S3$	8	4 (4)	6 (2)	6 (3)	4 (d)	9	$u3=0$
$Sdummy$	0	0	0 (3)	0	0	3	$u4=-6$
Demand	4	4	5	4	8		
vj	$v1=4$	$v2=4$	$v3=6$	$v4=6$	$v5=4$		

Find d_{ij} for all unoccupied cells(i,j), where $d_{ij}=c_{ij}-(u_i+v_j)$

$d_{11}=c_{11}-(u_1+v_1)=5-(-1+4)=2$
$d_{12}=c_{12}-(u_1+v_2)=8-(-1+4)=5$
$d_{13}=c_{13}-(u_1+v_3)=6-(-1+6)=1$
$d_{14}=c_{14}-(u_1+v_4)=6-(-1+6)=1$
$d_{22}=c_{22}-(u_2+v_2)=7-(0+4)=3$
$d_{23}=c_{23}-(u_2+v_3)=7-(0+6)=1$
$d_{25}=c_{25}-(u_2+v_5)=5-(0+4)=1$
$d_{31}=c_{31}-(u_3+v_1)=8-(0+4)=4$
$d_{41}=c_{41}-(u_4+v_1)=0-(-6+4)=2$
$d_{42}=c_{42}-(u_4+v_2)=0-(-6+4)=2$
$d_{44}=c_{44}-(u_4+v_4)=0-(-6+6)=0$
$d_{45}=c_{45}-(u_4+v_5)=0-(-6+4)=2$

	*D*1	*D*2	*D*3	*D*4	*D*5	Supply	*ui*
*S*1	5 [2]	8 [5]	6 [1]	6 [1]	3 (8)	8	$u_1=-1$
*S*2	4 (4)	7 [3]	7 [1]	6 (1)	5 [1]	5	$u_2=0$
*S*3	8 [4]	4 (4)	6 (2)	6 (3)	4 (d)	9	$u_3=0$
Sdummy	0 [2]	0 [2]	0 (3)	0 [0]	0 [2]	3	$u_4=-6$
Demand	4	4	5	4	8		
vj	$v_1=4$	$v_2=4$	$v_3=6$	$v_4=6$	$v_5=4$		

Since all $d_{ij} \geq 0$. So final optimal solution is arrived.

	*D*1	*D*2	*D*3	*D*4	*D*5	Supply
*S*1	5	8	6	6	3 (8)	8
*S*2	4 (4)	7	7	6 (1)	5	5
*S*3	8	4 (4)	6 (2)	6 (3)	4 (d)	9
Sdummy	0	0	0 (3)	0	0	3
Demand	4	4	5	4	8	

The minimum total transportation cost $=3\times8+4\times4+6\times1+4\times4+6\times2+6\times3+0\times3=92$