5.5 **Applications : Moments and centres of mass, moment of inertia**.

For a lamina R with a density function $\rho(x,y)$ at any point (x,y) in the plane, the mass is

$$m = \iint_R \rho(x,y)dA$$

The moments about the X-axis, and Y-axis are

$$M_X = \iint_R y\,\rho(x,y)dA \qquad \text{and} \qquad M_Y = \iint_R x\,\rho(x,y)dA$$

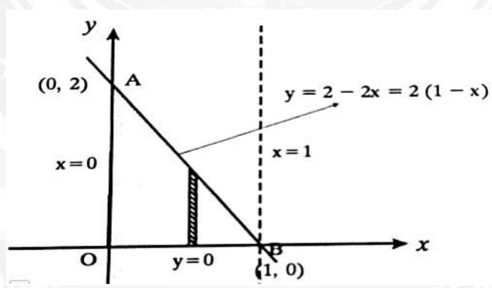
The centre of mass is given by

$$\bar{x} = \frac{M_Y}{m}, \quad \bar{y} = \frac{M_X}{m}$$

Example:

Find the mass and centre of mass of a triangular lamina with vertices$(0,0), (1,0),$ and $(0,2)$ if the density function is $\rho(x,y) = 1 + 3x + y$

Solution:



The equation the line AB is $\frac{y-y_1}{x-x_1} = \frac{y_2-y_1}{x_2-x_1}$

$$\Rightarrow \frac{y-2}{x-0} = \frac{0-2}{1-0}$$

$$\Rightarrow \frac{y-2}{x} = -2$$

$$\Rightarrow y - 2 = -2x$$

$$\Rightarrow y = 2 - 2x = 2(1-x)$$

The mass of the lamina is

$$m = \iint_R \rho(x,y)dA = \int_0^1 \int_0^{2(1-x)} (1+3x+y)dy\,dx$$

$$= \int_0^1 \left[(1+3x)y + \frac{y^2}{2}\right]_0^{2(1-X)} dx$$

$$= \int_0^1 \left[(1+3x)(2(1-x)) + \frac{4(1-x)^2}{2}\right] dx$$

$$= \int_0^1 [2(1+3x)(1-x) + 2(1-x)^2]dx$$

$$= 2\int_0^1 (1-x)[1+3x+1-x]dx$$

$$= 2\int_0^1 (1-x)[2+2x]dx$$

$$= 4\int_0^1 (1-x)[1+x]dx \;= 4\int_0^1 (1-x^2)dx$$

$$= 4\left[x - \frac{x^3}{3}\right]_0^1$$

$$= 4\left[1 - \frac{1}{3}\right] = 4\left[\frac{2}{3}\right]$$

$$m = \frac{8}{3}$$

$$\bar{x} = \frac{1}{m}\iint_R \mathrm{x}\,\rho(x,y)dA$$

$$= \frac{1}{8/3}\int_0^1 \int_0^{2(1-x)} x(1+3x+y)dy\,dx$$

$$= \frac{3}{8}\int_0^1 \int_0^{2(1-x)} [x(1+3x) + xy]dydx$$

$$= \frac{3}{8}\int_0^1 \left[x(1+3x)y + \frac{xy^2}{2}\right]_0^{2(1-x)} dx$$

$$= \frac{3}{8}\int_0^1 \left[x(1+3x)2(1-x) + x\frac{(1-x)^2}{2}\right] dx$$

$$= \frac{3}{8}\int_0^1 2(1-x)[x + 3x^2 + x - x^2]dx$$

$$= \frac{3}{8}\int_0^1 2(1-x)[2x + 2x^2]dx$$

$$= \frac{3}{2}\int_0^1 (1-x)[x + x^2]dx$$

$$= \frac{3}{2}\int_0^1 [x - x^3]dx$$

$$= \frac{3}{2}\left[\frac{x^2}{2} - \frac{x^4}{4}\right]_0^1$$

$$= \frac{3}{2}\left[\frac{1}{2} - \frac{1}{4}\right] = \frac{3}{2}\left[\frac{1}{4}\right] = \frac{3}{8}$$

$$\bar{y} = \frac{1}{m}\iint_R \mathrm{y}\, \rho(x,y)dA$$

$$= \frac{1}{8/3}\int_0^1 \int_0^{2(1-x)} y(1+3x+y)dy\, dx$$

$$= \frac{3}{8}\int_0^1 \int_0^{2(1-x)} [y(1+3x) + y^2]dydx$$

$$= \frac{3}{8}\int_0^1 \left[(1+3x)\frac{y^2}{2} + \frac{y^3}{3}\right]_0^{2(1-x)} dx$$

$$= \frac{3}{8}\int_0^1 \left[(1+3x)\frac{4(1-x)^2}{2} + \frac{8(1-x)^3}{3}\right] dx$$

$$= \frac{3}{8}\int_0^1 2(1-x)^2 \left[1 + 3x + \frac{4}{3}(1-x)\right] dx$$

$$= \frac{3}{8}\int_0^1 2(1-x)^2\left[\frac{3+9x+4-4x}{3}\right]dx$$

$$= \frac{1}{4}\int_0^1 (1-x)^2[7+5x]dx$$

$$= \frac{1}{4}\int_0^1 (1-x)^2[7+5x]dx$$

$$= \frac{1}{4}\int_0^1 (1+x^2-2x)[7+5x]dx$$

$$= \frac{1}{4}\int_0^1 (7+5x+7x^2+5x^3-14x-10x^2)dx$$

$$= \frac{1}{4}\int_0^1 (7-9x-3x^2+5x^3)dx$$

$$= \frac{1}{4}\left[7x-\frac{9x^2}{2}-\frac{3x^3}{3}+\frac{5x^4}{4}\right]_0^1$$

$$= \frac{1}{4}\left[7-\frac{9}{2}-1+\frac{5}{4}\right]$$

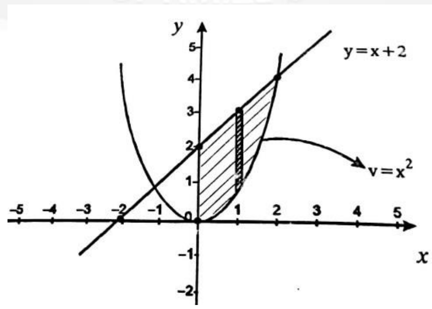
$$= \frac{1}{4}\left[\frac{11}{4}\right] = \frac{11}{16}$$

The centre of mass is $\left(\frac{3}{8},\frac{11}{16}\right)$.

2. find the mass and centre of mass of the lamina that occupies the region D and has the given density function ρ. D is bounded by $y = x^2$ and $y = x + 2$; $\rho(x,y) = kx$

Solution:

The mass of a homogeneous lamina is given by,



$$m = \iint_R \rho(x,y)dA = \int_{-1}^{2}\int_{x^2}^{(x+2)} kx\,dy\,dx$$

$$= \int_{-1}^{2} kx(x - x^2 + 2)\, dx$$

$$= k\left[\frac{x^3}{3} - \frac{x^4}{4} + 2\frac{x^2}{2}\right]_{-1}^{2}$$

$$= k\left[\left(\frac{8}{3} - \frac{16}{4} + 4\right) - \left(-\frac{1}{3} - \frac{1}{4} + 1\right)\right]$$

$$= k\left[\frac{32}{12} - \frac{15}{12}\right] = k\frac{27}{12} = \frac{9}{4}k$$

$$\bar{x} = \frac{1}{m}\iint_R \mathrm{x}\,\rho(x,y)dA$$

$$= \frac{1}{\left(\frac{9}{4}k\right)} \int_{-1}^{2}\int_{x^2}^{x+2} kx^2\, dy\, dx$$

$$= \frac{1}{\left(\frac{9}{4}k\right)} \int_{-1}^{2} kx^2(x - x^2 + 2)dx$$

$$= \frac{4}{9k}k \int_{-1}^{2} (x^3 - x^4 + 2x^2)dx = \frac{4}{9}\left[\frac{x^4}{4} - \frac{x^5}{5} + 2\frac{x^3}{3}\right]_{-1}^{2}$$

$$= \frac{4}{9}\left[\left(\frac{176}{60} + \frac{13}{60}\right)\right] = \frac{7}{5}$$

$$\bar{y} = \frac{1}{m}\iint_R \mathrm{y}\,\rho(x,y)dA$$

$$= \frac{1}{\left(\frac{9}{4}k\right)} \int_{-1}^{2}\int_{x^2}^{x+2} kxy\, dy\, dx$$

$$= \frac{1}{\left(\frac{9}{4}k\right)} \int_{-1}^{2} \frac{1}{2}kx\,[(x+2)^2 - x^4]dx$$

$$= \frac{2}{9}\int_{-1}^{2}[x^3 + 4x + 4x^2 - x^5]dx$$

$$= \frac{2}{9}\left[\frac{x^4}{4} + 2x^2 + \frac{4x^3}{3} - \frac{x^6}{6}\right]_{-1}^{2}$$

$$= \frac{2}{9}\left[12 - \frac{6}{8}\right] = \frac{5}{2}$$

The centre of the gravity of the lamina is $\left(\frac{7}{5}, \frac{5}{2}\right)$