3.2 BODE PLOT

The Bode plot is a frequency response plot of the sinusoidal transfer function of a system. One is a plot of the magnitude of a sinusoidal transfer function versus log ω. The other is a plot of the phase angle of a sinusoidal transfer function versus log ω. The main advantage of bode plot is that multiplication of magnitudes can be converted into addition. Also, a simple method for sketching an approximate log-magnitude curve is available. A Bode plot is a (semilog) plot of the transfer function magnitude and phase angle as a function of frequency.

The gain magnitude is many times expressed in terms of decibels (dB) = 20 $\log_{10}A$.

Semilog sheet

Two sets of axes: gain on top, phase below (identical)

Logarithmic frequency axes

Gain axis is logarithmic – either explicitly or as units of decibels(dB)

Phase axis is linear with units of degrees

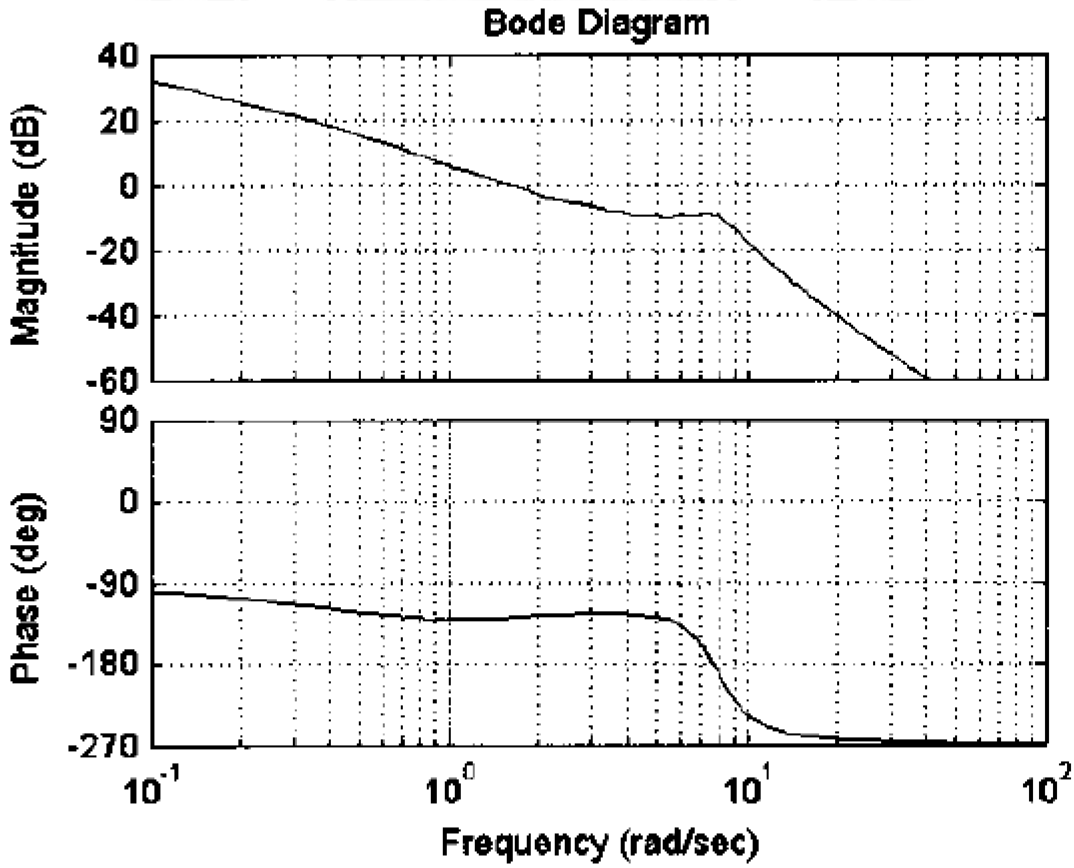

Figure 3.2.1 Magnitude and phase plots of Bode plot

[Source: "Linear Control System Analysis and Design with MATLAB" by John J D'Azzo, Constantine, Stuart, Page: 318]

BASIC FACTORS OF G(jω)

The basic factors that are very frequently occur in a typical transfer function G(jω) are,

1. Constant gain, K
2. Integral and derivative factors $(j\omega)^{\mp 1}$
3. First-order factors $(1 + j\omega T)^{\mp 1}$
4. Quadratic factors $\left(1 + 2\zeta\left(j\frac{\omega}{\omega_n}\right) + \left(j\frac{\omega}{\omega_n}\right)^2\right)^{\mp 1}$

Constant Gain, K

Let G(s)=K,

$$G(j\omega) = K = K\angle 0^o$$

$$A = |G(j\omega)|in\ db = 20\log K$$

$$\phi = \angle G(j\omega) = 0^o$$

The magnitude plot for a constant gain K is a horizontal straight line at the magnitude of 20log K db. The phase plot is a straight line at 0°.

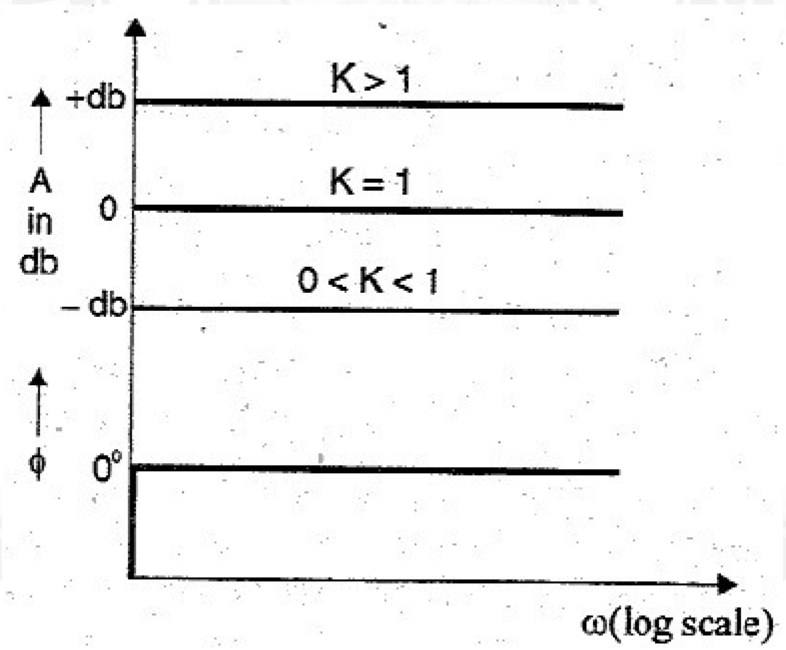

Figure 3.2.2 Bode plot of constant gain, K

[Source: "Control Systems" by A Nagoor Kani, Page: 3.10]

Integral Factor

Let G(s)=K/s,

$$G(j\omega) = \frac{K}{j\omega} = \frac{K}{\omega}\angle - 90^o$$

$$A = |G(j\omega)| in\ db = 20 \log\left(\frac{K}{\omega}\right)$$

$$\phi = \angle G(j\omega) = -90^o$$

The magnitude plot of the integral factor is a straight line with the slope of -20db/dec and passing through zero db when ω=K. The phase plot is a straight line at -90°.

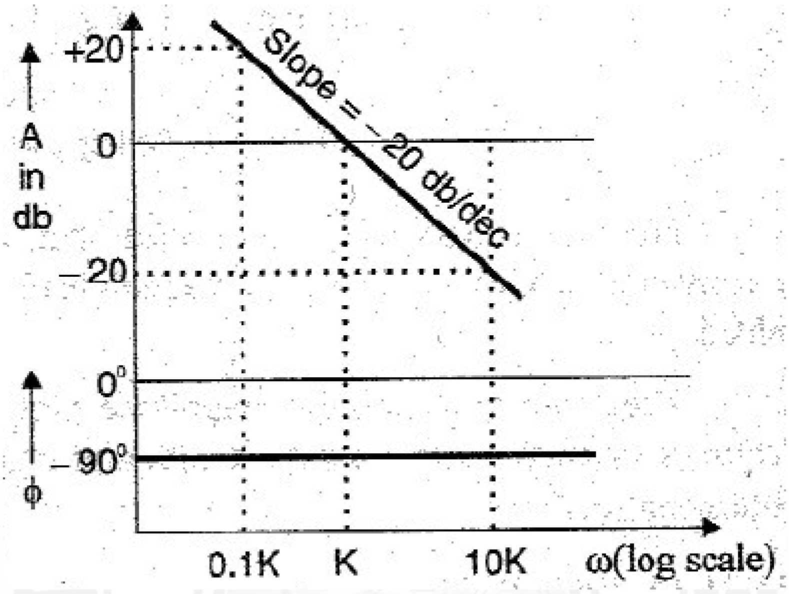

Figure 3.2.3 Bode plot of integral factor, K/jω

[Source: "Control Systems" by A Nagoor Kani, Page: 3.11]

Derivative factor

Let G(s)=Ks,

$$G(j\omega) = Kj\omega = K\omega\angle 90^o$$

$$A = |G(j\omega)| in\ db = 20 \log(K\omega)$$

$$\phi = \angle G(j\omega) = +90^o$$

The magnitude plot of the integral factor is a straight line with the slope of 20db/dec and passing through zero db when ω=K. The phase plot is a straight line at +90°.

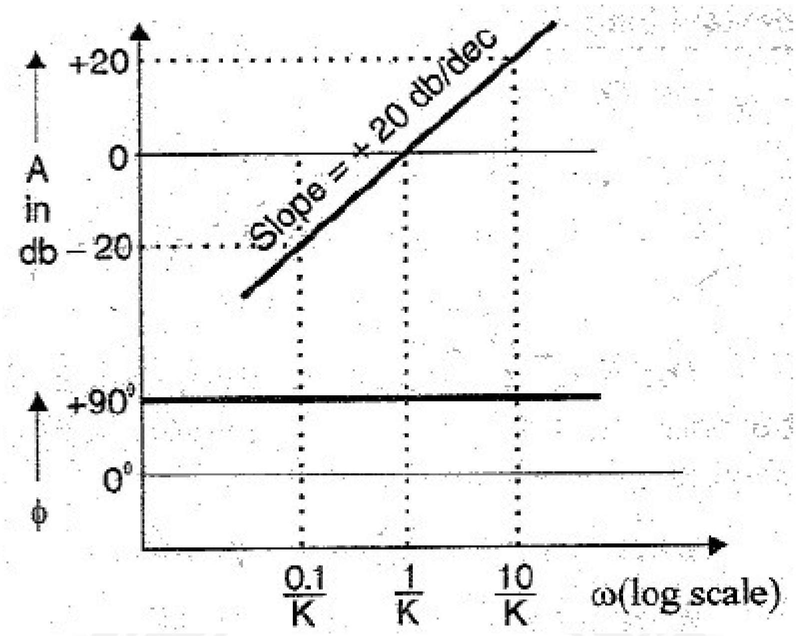

Figure 3.2.4 Bode plot of derivative factor, K x jω

[Source: "Control Systems" by A Nagoor Kani, Page: 3.11]

First order factor in denominator

Let $G(s) = \frac{1}{1+sT}$

$$G(j\omega) = \frac{1}{1 + j\omega T} = \frac{1}{\sqrt{1 + \omega^2 T^2}} \angle - \tan^{-1} \omega T$$

$$A = |G(j\omega)| in\ db = 20 \log\left(\frac{1}{\sqrt{1 + \omega^2 T^2}}\right)$$

$$\phi = \angle G(j\omega) = \angle - \tan^{-1} \omega T$$

The magnitude plot of the first order factor can be approximated by two straight lines, one is a straight line at zero db for the frequency range, $0<\omega<1/T$, and the other is a straight line with slope -20db/dec for the frequency range, $1/T<\omega<\infty$. The corner frequency is $\omega_c=1/T$ and the loss in db at the corner frequency is -3db. The phase angle of the first order factor varies from 0^o to -90^o as ω is varied from zero to infinity. The phase plot is a curve passing through -45^o at ω_c.

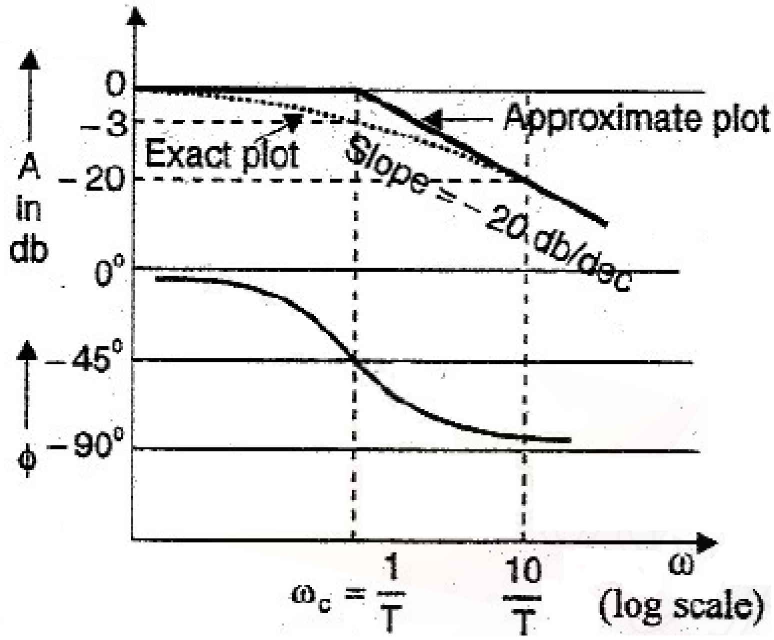

Figure 3.2.5 Bode plot of first order factor in denominator, 1/(1+jωT)

[Source: "Control Systems" by A Nagoor Kani, Page: 3.13]

First order factor in numerator

Let $G(s) = 1 + sT$

$$G(j\omega) = 1 + j\omega T = \sqrt{1 + \omega^2 T^2} \angle \tan^{-1} \omega T$$

$$A = |G(j\omega)| in\ db = 20 \log \left(\sqrt{1 + \omega^2 T^2}\right)$$

$$\phi = \angle G(j\omega) = \angle \tan^{-1} \omega T$$

The magnitude plot of the first order factor can be approximated by two straight lines, one is a straight line at zero db for the frequency range, $0<\omega<1/T$, and the other is a straight line with slope 20db/dec for the frequency range, $1/T<\omega<\infty$. The corner frequency is $\omega_c=1/T$ and the loss in db at the corner frequency is +3db. The phase angle of the first order factor varies from 0^o to $+90^o$ as ω is varied from zero to infinity. The phase plot is a curve passing through $+45^o$ at ω_c.

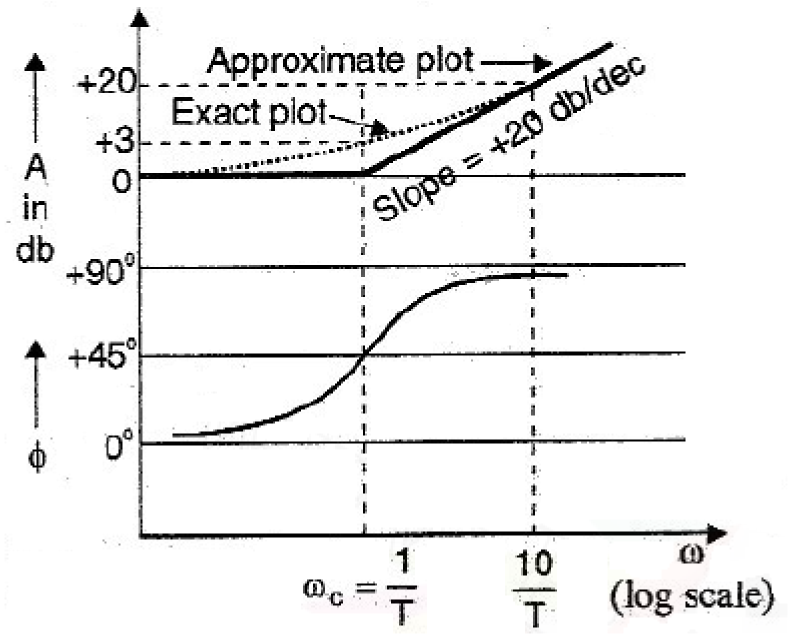

Figure 3.2.6 Bode plot of first order factor in numerator, (1+jωT)

[Source: "Control Systems" by A Nagoor Kani, Page: 3.14]

Quadratic factor in denominator

Second order closed loop transfer function is given by

$$G(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} = \frac{1}{\left(\frac{s}{\omega_n}\right)^2 + 2\zeta\frac{s}{\omega_n} + 1}$$

$$G(j\omega) = \frac{1}{\left(\frac{j\omega}{\omega_n}\right)^2 + 2\zeta\frac{j\omega}{\omega_n} + 1} = \frac{1}{-\left(\frac{\omega}{\omega_n}\right)^2 + 2\zeta\frac{j\omega}{\omega_n} + 1}$$

$$G(j\omega) = \frac{1}{\sqrt{\left(1 - \frac{\omega^2}{{\omega_n}^2}\right) + 4\zeta^2\frac{\omega^2}{{\omega_n}^2}}} \angle -\tan^{-1}\left(\frac{2\zeta\frac{\omega}{\omega_n}}{1 - \frac{\omega^2}{{\omega_n}^2}}\right)$$

At low frequencies when $\omega \ll \omega_n$, the magnitude is,

$$A = -20\log\sqrt{1 - \frac{\omega^2}{{\omega_n}^2}(2 - 4\zeta^2) + \frac{\omega^4}{{\omega_n}^4}} \cong -20\log 1 = 0$$

At high frequencies when $\omega \gg \omega_n$, the magnitude is,

$$A = -20\log\sqrt{1 - \frac{\omega^2}{{\omega_n}^2}(2 - 4\zeta^2) + \frac{\omega^4}{{\omega_n}^4}}$$

$$A \cong -20\log\sqrt{\frac{\omega^4}{{\omega_n}^4}} = -\,20\log\frac{\omega^2}{{\omega_n}^2} = -20\log\left(\frac{\omega}{\omega_n}\right)^2$$

$$\phi = \angle G(j\omega) = -\tan^{-1}\left(\frac{2\zeta\frac{\omega}{\omega_n}}{1-\frac{\omega^2}{{\omega_n}^2}}\right)$$

The magnitude plot of the quadratic factor in the denominator can be approximated by two straight lines, one is a straight line at zero db for the frequency range, $0<\omega<\omega_n$, and the other is a straight line with slope -40db/dec for the frequency range, $\omega_n<\omega<\infty$. The frequency at which the two asymptotes meet is called the corner frequency. For the quadratic factor, the frequency, ω_n is the corner frequency, ω_c. The phase angle of the quadratic factor varies from 0^o to -180^o as ω is varied from zero to infinity. The phase plot is a curve passing through -90^o at ω_c. At the corner frequency, phase angle is -90^o and independent of ζ, but at all other frequency it depends on ζ.

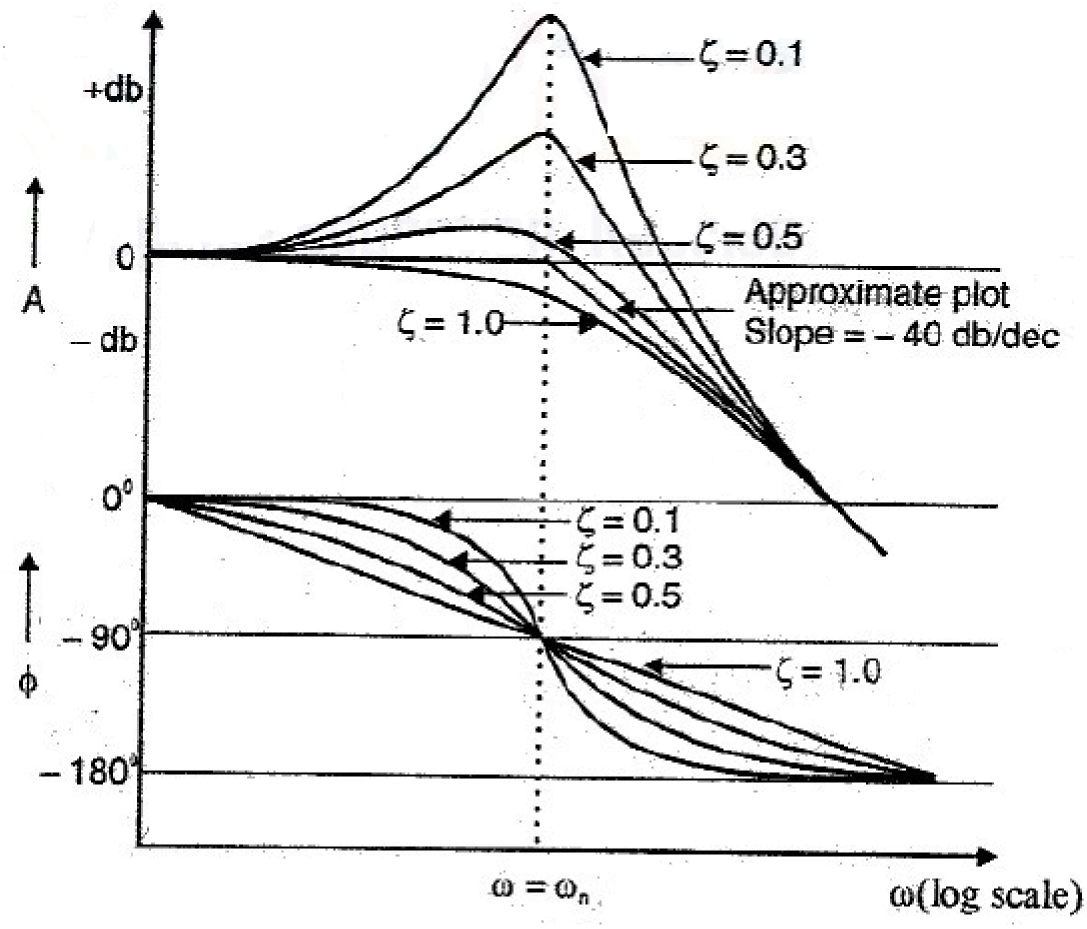

Figure 3.2.7 Bode plot of quadratic factor in denominator

[Source: "Control Systems" by A Nagoor Kani, Page: 3.15]

Quadratic factor in the numerator

$$G(s) = \frac{s^2 + 2\zeta\omega_n s + \omega_n^2}{\omega_n^2} = \left(\frac{s}{\omega_n}\right)^2 + 2\zeta\frac{s}{\omega_n} + 1$$

$$G(j\omega) = \left(\frac{j\omega}{\omega_n}\right)^2 + 2\zeta\frac{j\omega}{\omega_n} + 1 = -\left(\frac{\omega}{\omega_n}\right)^2 + 2\zeta\frac{j\omega}{\omega_n} + 1$$

$$G(j\omega) = \sqrt{\left(1 - \frac{\omega^2}{{\omega_n}^2}\right) + 4\zeta^2\frac{\omega^2}{{\omega_n}^2}} \angle \tan^{-1}\left(\frac{2\zeta\frac{\omega}{\omega_n}}{1 - \frac{\omega^2}{{\omega_n}^2}}\right)$$

At low frequencies when $\omega << \omega_n$, the magnitude is,

$$A = 20\log\sqrt{1 - \frac{\omega^2}{{\omega_n}^2}(2 - 4\zeta^2) + \frac{\omega^4}{{\omega_n}^4}} \cong 20\log 1 = 0$$

At high frequencies when $\omega >> \omega_n$, the magnitude is,

$$A = 20\log\sqrt{1 - \frac{\omega^2}{{\omega_n}^2}(2 - 4\zeta^2) + \frac{\omega^4}{{\omega_n}^4}}$$

$$A \cong 20\log\sqrt{\frac{\omega^4}{{\omega_n}^4}} = 20\log\frac{\omega^2}{{\omega_n}^2} = 20\log\left(\frac{\omega}{\omega_n}\right)^2$$

$$\phi = \angle G(j\omega) = \tan^{-1}\left(\frac{2\zeta\frac{\omega}{\omega_n}}{1 - \frac{\omega^2}{{\omega_n}^2}}\right)$$

The magnitude plot of the quadratic factor in the denominator can be approximated by two straight lines, one is a straight line at zero db for the frequency range, $0 < \omega < \omega_n$, and the other is a straight line with slope +40db/dec for the frequency range, $\omega_n < \omega < \infty$. The frequency at which the two asymptotes meet is called the corner frequency. For the quadratic factor, the frequency, ω_n is the corner frequency, ω_c. The phase angle of the quadratic factor varies from 0^o to $+180^o$ as ω is varied from zero to infinity. The phase plot is a curve passing through $+90^o$ at ω_c. At the corner frequency, phase angle is $+90^o$ and independent of ζ, but at all other frequency it depends on ζ.

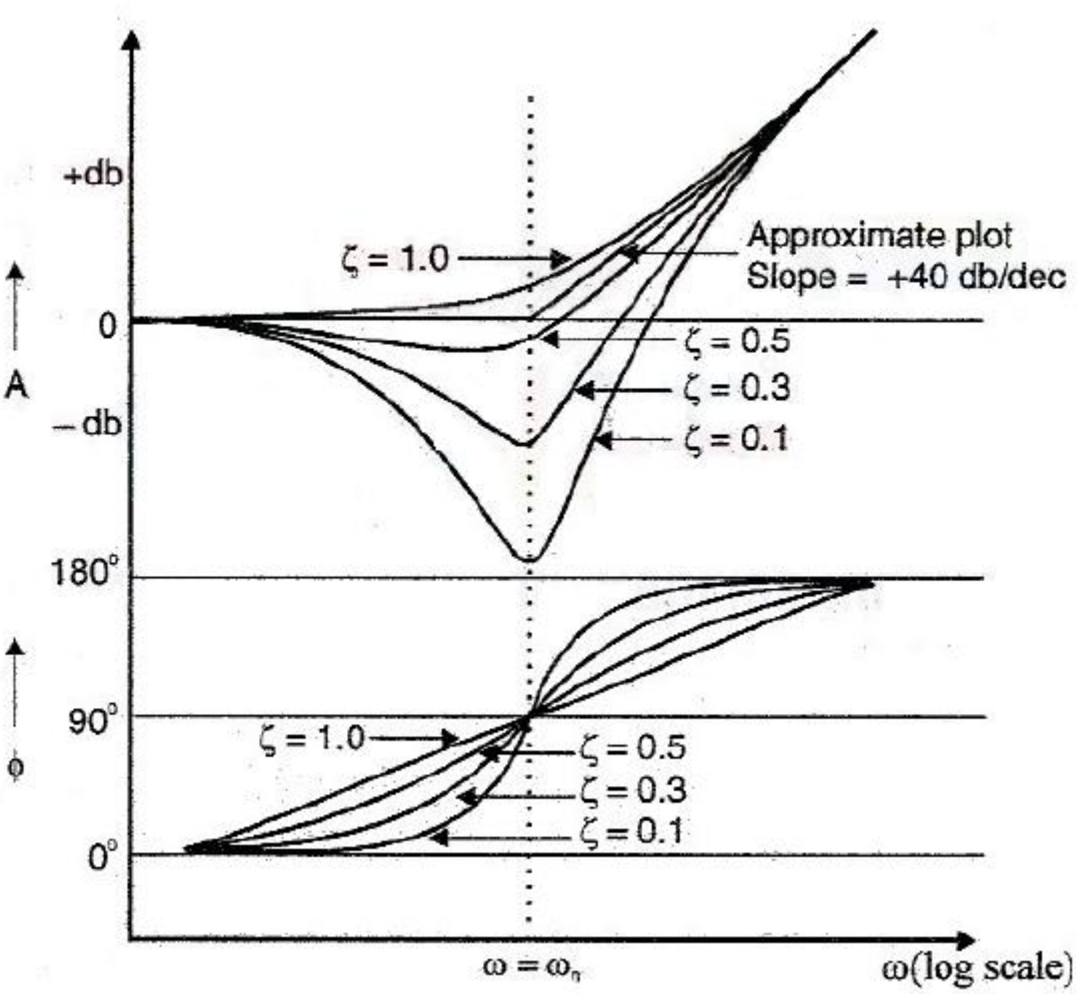

Figure 3.2.8 Bode plot of quadratic factor in numerator

[Source: "Control Systems" by A Nagoor Kani, Page: 3.16]

Derivative factor: magnitude

$$M = 20\log\left|j\omega\right| = 20\log\omega \text{ dB}$$

$$\angle j\omega = 90^0$$

$$\Delta M = 20\log\omega_2 - 20\log\omega_1 = 20\log\frac{\omega_2}{\omega_1}\text{ dB/decade}$$

$$\Delta M = 20\log 10 = 20 \text{ dB/decade}$$

$$\Delta M = 20\log 2 \approx 6 \text{ dB/octave}$$

Integral factor: magnitude

$$M = 20\log\left|\frac{1}{j\omega}\right| = -20\log\omega \text{ dB}$$

$$\angle j\omega = 270^0$$

$$\Delta M = -20\log\omega_2 + 20\log\omega_1 = -20\log\frac{\omega_2}{\omega_1}\text{ dB/decade}$$

$$\Delta M = -20\log 10 = -20 \text{ dB/decade}$$

$$\Delta M = 20\log 2 \approx -6 \text{ dB/octave}$$

First-order derivative factor: magnitude

$$M = 20\log|1 + j\omega\tau| = 20\log(\sqrt{1+[\omega\tau]}^2 \text{ dB}$$

For $\omega \ll \omega_c$, $M \approx 0$ dB

For $\omega \gg \omega_{c'}$

$$M \approx 20\log\frac{\omega}{\omega_c}\text{ dB}$$

Here, $\omega_c = 1/\tau$ = corner frequency

For $\omega > \omega_c$

$$\Delta M = 20\log\omega_2 - 20\log\omega_1 = 20\log\frac{\omega_2}{\omega_1}$$

$$\Delta M = 20\log 10 = 20 \text{ dB/decade}$$

$$\Delta M = 20\log 2 \approx 6 \text{ dB/octave}$$

First-order derivative factor: phase

$$\theta = \angle 1 + j\omega\tau = \arctan(\omega\tau)$$

$$\theta \approx 0 \qquad ; w < \frac{w_c}{10}$$

$$\theta = 45^0\left(1 + \log\frac{\omega}{\omega_c}\right) \qquad ; \frac{w_c}{10} < w < 10w_c$$

$$\theta \approx 90 \qquad ; w > 10w_c$$

First-order integral factor: magnitude

$$M = 20\log\left|\frac{1}{1+j\omega\tau}\right| = 20\log\left(\frac{1}{\sqrt{1+[\omega\tau]^2}}\right)\text{dB}$$

$$M \approx 0, \qquad w << w_c$$

$$M \approx -20\log\frac{\omega}{\omega_c}dB, \quad w >> w_c$$

$$\Delta M = -20\log\omega_2 + 20\log\omega_1 = -20\log\frac{\omega_2}{\omega_1}\text{dB/decade}$$

$$\Delta M = -20\log 2 \approx -6 \text{ dB/octave}$$

First-order integral factor: phase

$$\theta=360, \ \omega<\omega_c/10$$

$$\theta = 360 - 45^0\left(1+\log\frac{\omega}{\omega_c}\right), \ \omega_c/10<\omega<10\ \omega_c$$

$$\theta = 360 - 45^0\left(1+\log\frac{\omega}{\omega_c}\right)$$

$$\theta=270, \ \omega>10\ \omega_c$$

Second-order derivative factor: magnitude

$$M = 20\log|\omega_n^2 - \omega^2 + j2\zeta\omega\omega_n|$$

$$= 20\log\left(\omega_n^2\sqrt{\left(1-\frac{\omega^2}{\omega_n^2}\right)^2 + \left(2\zeta\frac{\omega}{\omega_n}\right)^2}\right)$$

$$\mathbf{M \approx 40\log w_n, w << w_n}$$

$$\mathbf{M = 20\log\ (2zw_n^{\ 2}), w = w_n}$$

$$\mathbf{M = 40\ \log w, w >> w_n}$$

For $w >> w_n$

$$\Delta M = 40\log\omega_2 - 40\log\omega_1 = 40\log\frac{\omega_2}{\omega_1}\ \text{dB/decade}$$

$$\Delta M = 40\log 10 = 40\ \text{dB/decade}$$

$$\Delta M = 40\log 2 \approx 12\ \text{dB/octave}$$

Second-order derivative factor: phase

$$\theta = \angle|\,\omega_n^2 - \omega^2 + j2\zeta\omega\omega_n\,| = \arctan\left(\frac{2\zeta\dfrac{\omega}{\omega_n}}{1-\dfrac{\omega^2}{\omega_n^2}}\right)$$

$$\theta = 0^0,\quad w < \frac{w_n}{10}$$

$$\theta = 90^0,\quad w = w_n$$

$$\theta = 180^0,\quad w > 10w_n$$

Second-order integral factor

$$M = 20\log\left|\frac{1}{\omega_n^2 - \omega^2 + j2\zeta\omega\omega_n}\right| dB = 20\log\left(\frac{1}{\omega_n^2\sqrt{\left(1-\dfrac{\omega^2}{\omega_n^2}\right)^2 + \left(2\zeta\dfrac{\omega}{\omega_n}\right)^2}}\right) dB$$

$M \approx -40\log\omega_n,\ \omega \ll \omega_n$

$M = -20\log(2\zeta\omega_n^2),\ \omega = \omega_n$

$M = -40\log\omega,\ \omega \gg \omega_n$

$$\Delta M = -40\log\omega_2 + 40\log\omega_1 = -40\log\frac{\omega_2}{\omega_1}\ dB/decade$$

PROCEDURE FOR MAGNITUDE PLOT OF BODE PLOT

Step 1: Convert the transfer function into Bode form or time constant form.

Step 2: List the corner frequencies in the increasing order and prepare a table as shown

Term	Corner frequency rad/sec	Slope db/dec	Change in Slope db/dec

In the above table, enter K or $K/(j\omega)^n$ or $K(j\omega)^n$ as the first term and the other terms in the increasing order of corner frequencies. Then enter the corner frequency, slope contributed by each term and change in slope at every corner frequency.

Step 3: Choose an arbitrary frequency ω_l which is lesser than the lowest corner frequency. Calculate the db magnitude of K or $K/(j\omega)^n$ or $K(j\omega)^n$ at ω_l and at the lowest corner frequency.

Step 4: Then calculate the gain (db magnitude) at every corner frequency one by one by using the formula,

$$\text{Gain at } \omega_y = \text{change in gain from } \omega_x \text{ to } \omega_y + \text{Gain at } \omega_x$$

$$A_y = (\text{Slope from } \omega_x \text{ to } \omega_y \times \log(\omega_y/\ \omega_x) + \text{Gain at } \omega_x$$

Step 5: Choose an arbitrary frequency ω_h which is greater than the highest corner frequency. Calculate the gain at ω_h by using the formula in step 4.

Step 6: In a semilog graph sheet mark the required range of frequency on x-axis (log scale) and the range of db magnitude on y-axis (ordinary scale) after choosing proper units.

Step 7: Mark all the points obtained in steps 3, 4, 5 on the graph and join the points by straight lines. Mark the slope at every part of the graph.

PROCEDURE FOR PHASE PLOT OF BODE PLOT

The phase plot is an exact plot obtained with exact phase angles of $G(j\omega)$ computed for various values of ω and is then tabulated. The choice of frequencies are preferably the frequencies chosen for magnitude plot. Usually the magnitude plot and phase plot are drawn in a single semilog sheet on a common frequency scale. Take another y-axis in the graph where the magnitude plot is drawn and, in this y-axis, mark the desired range of phase angles after choosing proper units. From the tabulated values of ω and phase angles, mark all the points on the graph. Join the points by a smooth curve.

DETERMINATION OF GAIN MARGIN AND PHASE MARGIN FROM BODE PLOT

The gain margin in db is given by the negative of db magnitude of $G(j\omega)$ at the phase crossover frequency, ω_{pc}. The ω_{pc} is the frequency at which phase of $G(j\omega)$ is 180^o. if the db magnitude of $G(j\omega)$ at ω_{pc} is negative then gain margin is positive and vice versa.

Let Φ_{gc} be the phase angle of $G(j\omega)$ at gain cross over frequency, ω_{gc}. The ω_{gc} is the frequency at which the db magnitude of $G(j\omega)$ is zero. Now the phase margin, γ is given by, $\gamma = 180^o + \Phi_{gc}$. If Φ_{gc} is less negative than -180^o, then phase margin is positive and vice versa. The positive and negative gain margins and phase margins are illustrated in figure 3.2.9.

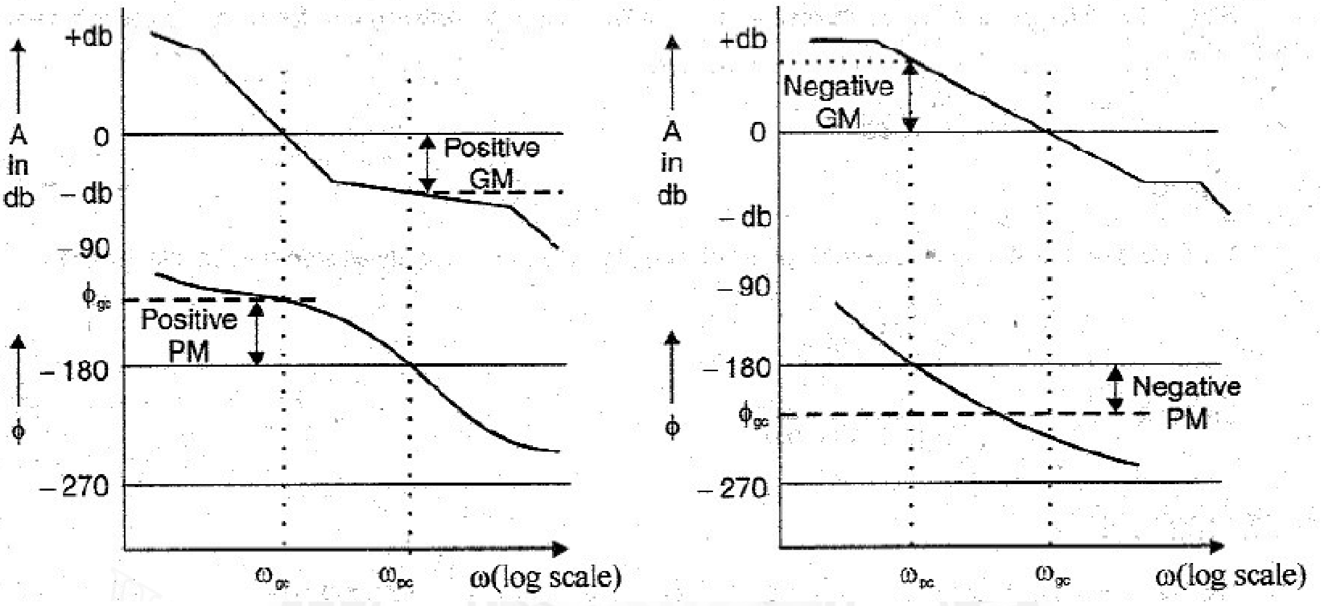

Figure 3.2.9 Gain margin and Phase margin in Bode plot

[Source: "Control Systems" by A Nagoor Kani, Page: 3.20]