2.5 PROPERTIES OF LAPLACE TRANSFORM

1. **Linearity:**

Statement:

If $x_1(t) \overset{\mathcal{L}}{\leftrightarrow} X_1(s)$ with a region of convergence denoted as R_1

and $x_2(t) \overset{\mathcal{L}}{\leftrightarrow} X_2(s)$ with a region of convergence denoted as R_2

then $ax_1(t) + bx_2(t) \overset{\mathcal{L}}{\leftrightarrow} aX_1(s) + bX_2(s)$, with ROC containing $R_1 \cap R_2$

Proof:

$$\mathcal{L}\{z(t)\} = \mathcal{L}\{ax_1(t) + bx_2(t)\} = \int_{-\infty}^{\infty} \{ax_1(t) + bx_2(t)\}e^{-st}\,dt$$

$$= a\int_{-\infty}^{\infty} x_1(t)e^{-st}\,dt + b\int_{-\infty}^{\infty} x_2(t)e^{-st}\,dt$$

$$= aX_1(s) + bX_2(s)$$

2. **Time Shifting:**

Statement:

If $x(t) \overset{\mathcal{L}}{\leftrightarrow} X(s)$ with ROC= R

then $x(t-\tau) \overset{\mathcal{L}}{\leftrightarrow} e^{-s\tau}X(s)$ with ROC= R

Proof:

$$\mathcal{L}\{x(t-\tau)\} = \int_{-\infty}^{\infty} x(t-\tau)e^{-st}\,dt$$

Let t-τ=p

$$= \int_{-\infty}^{\infty} x(p)e^{-s(p+\tau)}\,dt$$

$$= e^{-s\tau}\int_{-\infty}^{\infty} x(p)e^{-sp}\,dt$$

$$= e^{-s\tau}X(s)$$

3. Shifting in s-Domain:

Statement:

If $x(t) \overset{\mathcal{L}}{\leftrightarrow} X(s)$ with ROC= R

then $e^{s_o t}x(t) \overset{\mathcal{L}}{\leftrightarrow} X(s - s_o)$ with ROC= R+Re{s_o}

Proof:

$$\mathcal{L}\{e^{s_o t}x(t)\} = \int_{-\infty}^{\infty} e^{s_o t}x(t)e^{-st}dt$$

$$= \int_{-\infty}^{\infty} x(t)e^{-(s-s_o)t}dt$$

$$= X(s - s_o)$$

4. Time Scaling:

Statement:

If $x(t) \overset{\mathcal{L}}{\leftrightarrow} X(s)$ with ROC= R

then $x(at) \overset{\mathcal{L}}{\leftrightarrow} \frac{1}{|a|}X\left(\frac{s}{a}\right)$ with ROC= R_1=aR

Proof:

Case 1: For a>0:

$$\mathcal{L}\{x(at)\} = \int_{-\infty}^{\infty} x(at)e^{-st}dt$$

Using the substitution of $\lambda=at$; $dt=ad\lambda$

$$= \frac{1}{a}\int_{-\infty}^{\infty} x(\lambda)e^{-\left(\frac{s}{a}\right)\lambda}d\lambda$$

$$= \frac{1}{a}X\left(\frac{s}{a}\right)$$

Case 2: For a< 0 :

$$\mathcal{L}\{x(at)\} = \int_{-\infty}^{\infty} x(at)e^{-st}dt$$

Using the substitution of $\lambda=at$; $dt=ad\lambda$

$$= -\frac{1}{a}\int_{-\infty}^{\infty} x(\lambda)e^{-\left(\frac{s}{a}\right)\lambda}\, d\lambda$$

$$= -\frac{1}{a}X\left(\frac{s}{a}\right)$$

Combining the two cases, we get $x(at) \overset{\mathcal{L}}{\leftrightarrow} \frac{1}{|a|}X\left(\frac{s}{a}\right)$ with ROC= $R_1 = aR$

5. **Conjugation:**

Statement:

If $x(t) \overset{\mathcal{L}}{\leftrightarrow} X(s)$ with ROC= R

then $x^*(t) \overset{\mathcal{L}}{\leftrightarrow} X^*(s^*)$ with ROC= R

Proof:

$$\mathcal{L}\{x^*(t)\} = \int_{-\infty}^{\infty} x^*(t)e^{-st}\, dt$$

$s=\sigma+j\omega$

$$= \int_{-\infty}^{\infty} x^*(t)e^{-\sigma t}e^{-j\omega t}\, dt$$

$$= \left(\int_{-\infty}^{\infty} x(t)e^{-\sigma t}e^{j\omega t}\, dt\right)^*$$

$$= \left(\int_{-\infty}^{\infty} x(t)e^{-(\sigma-j\omega)t}\, dt\right)^*$$

$$= \left(\int_{-\infty}^{\infty} x(t)e^{-(s^*)t}\, dt\right)^*$$

$$= \left(X(s^*)\right)^* = X^*(s^*)$$

6. Convolution Property:

Statement:

If $x_1(t) \overset{\mathcal{L}}{\leftrightarrow} X_1(s)$ with ROC = R_1

and $x_2(t) \overset{\mathcal{L}}{\leftrightarrow} X_2(s)$ with ROC = R_2

then $x_1(t) * x_2(t) \overset{\mathcal{L}}{\leftrightarrow} X_1(s).X_2(s)$, with ROC containing $R_1 \cap R_2$

Proof:

$$\mathcal{L}\{z(t)\} = \mathcal{L}\{x_1(t) * x_2(t)\} = \int_{-\infty}^{\infty} \{x_1(t) * x_2(t)\} e^{-st} dt$$

$$= \int_{-\infty}^{\infty} \left\{ \int_{-\infty}^{\infty} x_1(\tau) x_2(t-\tau) d\tau \right\} e^{-st} dt$$

$$\mathcal{L}\{x_1(t) * x_2(t)\} = \int_{-\infty}^{\infty} x_1(\tau) \left\{ \int_{-\infty}^{\infty} x_2(t-\tau) e^{-st} dt \right\} d\tau$$

$$= \int_{-\infty}^{\infty} x_1(\tau) \{e^{-s\tau} X_2(s)\} d\tau$$

$$= X_2(s) \int_{-\infty}^{\infty} x_1(\tau) e^{-s\tau} d\tau$$

$$= X_1(s).X_2(s)$$

7. Differentiation in the Time Domain:

Statement:

If $x(t) \overset{\mathcal{L}}{\leftrightarrow} X(s)$ with ROC= R

then $\frac{dx(t)}{dt} \overset{\mathcal{L}}{\leftrightarrow} s\,X(s)$ with ROC containing R

Proof:

Inverse Laplace transform is given by

$$x(t) = \frac{1}{2\pi j} \int_{\sigma - j\infty}^{\sigma + j\infty} X(s) e^{st} ds$$

Differentiating above on both sides with respect to 't'

$$\frac{dx(t)}{dt} = \frac{1}{2\pi j}\int_{\sigma-j\infty}^{\sigma+j\infty} \{sX(s)\}e^{st}\,ds$$

Comparing both equations $s\,X(s)$ is the Laplace transform of $\frac{dx(t)}{dt}$.

8. **Differentiation in the s-Domain:**

Statement:

If $x(t) \overset{\mathcal{L}}{\leftrightarrow} X(s)$ with ROC= R

then $-tx(t) \overset{\mathcal{L}}{\leftrightarrow} \frac{dX(s)}{ds}$ with ROC = R

Proof:

Laplace transform is given by

$$X(s) = \int_{-\infty}^{\infty} x(t)e^{-st}\,dt$$

Differentiating above on both sides with respect to 's'

$$\frac{dX(s)}{ds} = \int_{-\infty}^{\infty} \{-tx(t)\}e^{-st}\,dt$$

Comparing both equations $\frac{dX(s)}{ds}$ is the Laplace transform of $-tx(t)$.

9. **Integration in the Time Domain:**

Statement:

If $x(t) \overset{\mathcal{L}}{\leftrightarrow} X(s)$ with ROC= R

then $\int_{-\infty}^{t} x(\tau)d\tau \overset{\mathcal{L}}{\leftrightarrow} \frac{1}{s}X(s)$ with ROC containing $R \cap \{Re\{s\} > 0\}$

Proof:

$$\int_{-\infty}^{t} x(\tau)d\tau = x(t) * u(t)$$

$$\mathcal{L}\left\{\int_{-\infty}^{t} x(\tau)d\tau\right\} = \mathcal{L}\{x(t) * u(t)\} = X(s).\,\mathcal{L}\{u(t)\} = X(s)\frac{1}{s}$$

10.The Initial and Final Value Theorems:

Statement:

If x(t) and $\frac{dx(t)}{dt}$ are Laplace transformable, and under the specific constraints that x(t)=0 for t<0 containing no impulses at the origin,one can directly calculate, from the Laplace transform, the initial value $x(0^+)$, i.e., x(t) as t approaches zero from positive values of t. Specifically the ***initial -value theorem*** states that

$$x(0^+) = \lim_{s\to\infty} sX(s)$$

Also, if x(t)=0 for t<0 and, in addition, x(t) has a finite limit as t→∞, then the ***final-value theorem*** says that

$$\lim_{t\to\infty} x(t) = \lim_{s\to 0} sX(s)$$