UNIT – V

ORDINARY DIFFERENTIAL EQUATIONS

Introduction:

Differential equations are of fundamental importance in engineering Mathematics because many relations appear mathematically in the form of such equations.

An ordinary differential equation is an equation that contains one or several derivatives of an unknown function which we will call $y(x)$ and which we want to determine from the equation.

The study of differential equation in applied mathematics consists of three phases

(i) Formation of differential equation from the given physical situation

(ii) Solutions of differential equation, evaluating arbitrary constants from the given conditions.

(iii) Physical interpretation of the solution

5.2Higher order linear differential equations with constant coefficient

General form of a linear differential equation of the nth order with constant

coefficient is $\frac{d^n y}{dx^n} + k_1 \frac{d^{n-1} y}{dx^{n-1}} + k_2 \frac{d^{n-2} y}{dx^{n-2}} \dots k_n y = x \;\; \dots (1)$

Where $k_1, k_2, \dots, k_n$ are constants it will be convenient to denote the operation $\frac{d}{dx}$ by a single letter D.

$Dy = \frac{dy}{dx}$ similarly $D^2y = \frac{d^2y}{dx^2}, D^3y = \frac{d^3y}{dx^3}$ etc

The equation (1) above can be written as

$$(D^n + k_1 D^{n-1} + \cdots + k_n)y = x$$

$$ie\ f(D)y = x$$

Note:

1. $\frac{1}{D}x = \int x dx$
2. $\frac{1}{D-a}x = e^{ax} \int x e^{-ax} dx$
3. $\frac{1}{D+a}x = e^{-ax} \int x e^{ax} dx$

Result:

1. $\frac{1}{D-a}\varphi(x) = e^{ax} \int e^{-ax}\varphi(x) dx$
2. $\frac{1}{D+a}\varphi(x) = e^{-ax} \int e^{ax}\varphi(x) dx$

(i)**The general form of the linear differential equation of second order is**

$$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = R$$

Where P and Q are constants and R is a function of x or constant

(ii) **Differential Operators**

The symbol D stands for the operation of differential

$$Dy = \frac{dy}{dx},\ D^2y = \frac{d^2y}{dx^2}$$

$\frac{1}{D}$ stands for the operation of integration

$\frac{1}{D^2}$ stands for the operation of integration twice

$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = R$ can be written in the form

$$D^2y + PDy + Qy = R$$

$$\text{(or) } (D^2 + PD + Q)y = R$$

(iii) Complete solution is y = complementary function + Particular Integral

(iv)To find the complementary function

	Roots of A.E	**C.F**
1.	Roots are real and different m_1, m_2 ($m_1 \neq m_2$)	$Ae^{m_1x} + Be^{m_2x}$
2.	Roots are real and equal $m_1 = m_2 = m$(say)	$(Ax + B)e^{mx}$ or $(A + Bx)e^{mx}$
3.	Roots are imaginary $\propto \pm i\beta$	$e^{\propto x}[A\cos\beta x + B\sin\beta x]$
4.	Roots are $\propto \pm i\beta$ (twice)	$e^{\propto x}[(c_1 + c_2x)\cos\beta x + (c_3 + c_4x)\sin\beta x]$

(V) To find the particular integral

$$\text{P.I.} = \frac{1}{f(D)}x$$

	x	P.I
1	e^{ax}	$\text{P.I.} = \frac{1}{f(D)}e^{ax} = e^{ax}\frac{1}{f(a)}; f(a) \neq 0$ $= xe^{ax}\frac{1}{f'(a)}, f(a) = 0; f'(a) \neq 0$ $= x^2e^{ax}\frac{1}{f''(a)}; f(a) = 0, f'(a) = 0\ f''(a) \neq 0$
2	x^n	$\text{P.I.} = \frac{1}{f(D)}x^n$ $= [f(D)]^{-1}x^n$ Expand $[f(D)]^{-1}$ and then operate
3	$\sin ax (\text{or}) \cos ax$	$\text{P.I.} = \frac{1}{f(D)}[\cos ax\ (\text{or}) \sin ax]$ Replace D^2 by $-a^2$
4	$e^{ax}\varphi(x)$	$\text{P.I.} = \frac{1}{f(D)}e^{ax\varphi(x)}$ $= e^{ax}\frac{1}{f(D+a)}\varphi(x)$

Problem Based on R.H.S of the given differential equation is zero

Example:5.1

(i)Solve $(D^2 + 6D + 9)y = 0$

Solution:

Auxiliary Equation is $m^2 + 6m + 9 = 0$

$$m = -3, -3$$

$$y = C.F$$

$$= (Ax + B)e^{-3x}$$

(ii) Solve $(\mathbf{D^2 + 2D + 1})\mathbf{y = 0}$

Solution:

Auxiliary Equation is $m^2 + 2m + 1 = 0$

$$m = -1, -1$$

$$y = C.F$$

$$= (Ax + B)e^{-x}$$

Example:5.2

Solve $(\mathbf{D^2 + 1})\boldsymbol{y} = \mathbf{0}$ given $\boldsymbol{y}(\mathbf{0}) = \mathbf{0}, \boldsymbol{y}'(\mathbf{0} = \mathbf{1}$

Solution:

Auxiliary Equation is $m^2 + 1 = 0$

$$m^2 = -1$$

$$m = \pm i$$

$$y = A\cos x + B\sin x$$

$$y(x) = A\cos x + B\sin x \ldots (1)$$

$$y'(x) = -A\sin x + B\cos x \ldots (2)$$

Given $y(0) = 0$

$$(1) \Rightarrow A = 0$$

Given $y'(0) = 1$

$$(2) \Rightarrow B = 1$$

$$(1) \Rightarrow y(x) = sinx$$

Note:

The above problem is called an initial value problem because all the condition are given at a single point $i.e\ x = 0$

Example:5.3

Solve $(D^2 + 1)y = 0$ **given** $y(0) = 1, y\left(\frac{\pi}{2}\right) = 0$

Solution:

Auxiliary Equation is $m^2 + 1 = 0$

$$m^2 = -1$$

$$m = \pm i$$

$$y = A\cos x + B\sin x \ \ldots (1)$$

Given $y(0) = 0$ equation (1) $\Rightarrow$ $A = 1$

Given $y(\pi/2) = 0$ equation (1) $\Rightarrow$ $B = 0$

The solution is $y = cos\ x$

Note:

The above problem is called as boundary value problem because the conditions are given at more than one point i.e. $x = 0$ and $x = \pi/2$

Example:5.4

Solve $(D^2 + D + 1)y = 0$

Solution:

Auxiliary Equation is $m^2 + m + 1 = 0$

$$m = \frac{-1}{2} \pm \frac{\sqrt{3}}{2} i$$

$$y = e^{\frac{-x}{2}}\left[A\cos\frac{\sqrt{3}}{2}x + B\sin\frac{\sqrt{3}}{2}x\right]$$

Example:5.5

Solve $(D^3 - 3D^2 + 3D - 1)\,y = 0$

Solution:

Auxiliary Equation is $m^3 - 3m^2 + 3m - 1 = 0$

$$m = 1 \text{ (thrice)}$$

$$y = e^x[A + Bx + Cx^2]$$

Example:5.6

Solve $(D^3 + D^2 + 4D + 4)\,y = 0$

Solution:

Auxiliary Equation is $m^3 + m^2 + 4m + 4 = 0$

$$m = -1\ ,\ \pm 2\mathrm{i}$$

$$\mathrm{y} = \mathrm{Ae}^{-x} + \mathrm{e}^{0x}[\mathrm{Bcos}2x + \mathrm{Csin}2x]$$

$$= \mathrm{Ae}^{-x} + \mathrm{Bcos}2x + \mathrm{Csin}2x$$

Example:5.7

Solve $(\boldsymbol{D^4 + 4D^3 + 8D^2 + 8D + 4})\ \boldsymbol{y = 0}$

Solution:

Auxiliary Equation is $m^4 + 4m^3 + 8m^2 + 8m + 4 = 0$

$$m^4 + 4m^2 + 4 + 4m^3 + 8m + 4m^2 = 0$$

$$(m^2 + 2m + 2)^2 = 0$$

$$m = \frac{-2 \pm \sqrt{4-8}}{2}$$

$$= -1 \pm \mathrm{i}\ \text{(twice)}$$

$$\mathrm{y} = \mathrm{e}^{-x}[(\mathrm{c}_1 + \mathrm{c}_2 x)\mathrm{cos}x + (\mathrm{c}_3 + \mathrm{c}_4 x)\mathrm{sin}x]$$

Example:5.8

If $1 \pm 2\mathrm{i}$, $1 \pm 2\mathrm{i}$ are the roots of auxiliary equation corresponding to fourth order homogenous linear differential equation $\mathrm{F}(D)y = 0$.Find its solution.

Solution:

Given $m = 1 \pm 2\mathrm{i}$ (twice)

$$y = \mathrm{e}^{x}[(\mathrm{c}_1 + \mathrm{c}_2 x)\mathrm{cos}2x + (\mathrm{c}_3 + \mathrm{c}_4 x)\mathrm{sin}2x]$$

Type I : Problems Based on P.I = $\frac{1}{f(D)} e^{ax} \rightarrow$ Replace D by a

Example :5.9

Solve $(D^2 - D - 6)y = 3e^{4x} + 5$

Solution:

Auxiliary Equation is $m^2 - m - 6 = 0$

$(m-3)(m+2) = 0$

$m_1 = 3,\ m_2 = -2$

C.F $= Ae^{3x} + Be^{-2x}$

P.I$_1$ $= \frac{1}{D^2-D-6} 3e^{4x}$ Replace D by 4

$= \frac{1}{16-4-6} 3e^{4x}$

$= \frac{1}{6} 3e^{4x} = \frac{1}{2} e^{4x}$

P.I$_2$ $= \frac{1}{D^2-D-6} 5$

$= \frac{1}{D^2-D-6} 5e^{0x}$ Replace D by 0

$= \frac{-1}{6} 5 = \frac{-5}{6}$

The general solution is y = C.F + P.I

$$y = Ae^{3x} + Be^{-2x} + \frac{1}{2} e^{4x} - \frac{5}{6}$$

Example:5.10

Solve $(D^2 + 6D + 8)y = e^{-2x}$

Solution:

Auxiliary Equation is $m^2 + 6m + 8 = 0$

$$(m + 2)\,(m + 4) = 0$$

$$m = -2,\ m = -4$$

$$\text{C.F} = \text{A}e^{-2x} + \text{B}e^{-4x}$$

$$\text{P.I} = \frac{1}{D^2+6D+8} e^{-2x} \qquad \text{Replace D by } -2$$

$$= \frac{1}{4-12+8} e^{-2x}$$

$$= \frac{1}{0} e^{-2x} \qquad \text{(fails)}$$

$$= x\ \frac{1}{2D+6} e^{-2x}$$

$$= x\ \frac{1}{-4+6} e^{-2x}$$

$$= \frac{x}{2} e^{-2x}$$

The general solution is y = C.F + P.I

$$y = \text{A}e^{-2x} + \text{B}e^{-4x} + \frac{x}{2}\ e^{-2x}$$

Example:5.11

Solve $(\mathbf{D^2 - 6D + 9})\boldsymbol{y} = \mathbf{e^{3x}}$

Solution:

Auxiliary Equation is $m^2 - 6m + 9 = 0$

$$(m - 3)\,(m - 3) = 0$$

$m = 3, 3$

C.F $= (Ax + B)e^{3x}$

P.I $= \frac{1}{D^2-6D+9} e^{-3x}$ Replace D by 3

$= \frac{1}{0} e^{3x}$ (fails)

$= x \frac{1}{2D-6} e^{3x}$ Replace D by 3

$= x \frac{1}{0} e^{3x}$ (fails)

$= x^2 \frac{1}{2} e^{3x}$

The general solution is y = C.F + P.I

$$y = (Ax + B)e^{3x} + \frac{x^2}{2} e^{3x}$$

Example:5.12

Solve $(D^2 - 4D + 13)y = e^{2x}$

Solution:

Auxiliary Equation is $m^2 - 4m + 13 = 0$

$$m = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$$

$$= \frac{4 \pm \sqrt{16-52}}{2}$$

$$= \frac{4 \pm \sqrt{-36}}{2}$$

$$= \frac{4 \pm 6i}{2}$$

$= 2 \pm 3i$

C.F $= e^{2x}[A\cos 3x + B\sin 3x]$

P.I $= \frac{1}{D^2-4D+13} e^{2x}$ Replace D by 2

$= \frac{1}{9} e^{2x}$

The general solution is y = C.F + P.I

$$y = e^{2x}(A\cos 3x + B\sin 3x) + \frac{1}{9} e^{2x}$$

Example:5.13

Solve $(D-2)^2 y = e^{2x}$

Solution:

Auxiliary Equation is $(m-2)^2 = 0$

$m = 2, 2$

C.F = $e^{2x}[Ax + B]$

P.I $= \frac{1}{(D-2)^2} e^{2x}$ Replace D by 2

$= \frac{1}{0} e^{2x}$ (fails)

$= x \frac{1}{2(D-2)} e^{2x}$ Replace D by 2

$= \frac{x}{2}\left[\frac{1}{2-2}\right] e^{2x}$

$= \frac{1}{0} e^{2x}$ (fails)

$= \frac{x}{2} . x \left[\frac{1}{1}\right] e^{2x} = \frac{x^2}{2} e^{2x}$

The general solution is y = C.F + P.I

$$y = e^{2x}(Ax + B) + \frac{x^2 e^{2x}}{2}$$

Example:5.14

Solve $(D^2 + 7D + 12)y = 14e^{-3x}$

Solution:

Auxiliary Equation is $m^2 + 7m + 12 = 0$

$$m = -3,\ m = -4$$

$$\text{C.F} = Ae^{-3x} + Be^{-4x}$$

$$\text{P.I} = \frac{1}{D^2+7D+12} 14e^{-3x}$$

$$= 14 \frac{1}{9-21+12} e^{-3x} \quad \text{Replace D by -3}$$

$$= \frac{1}{0} e^{-3x} \quad \text{(fails)}$$

$$= 14x \frac{1}{2D+7} e^{-3x}$$

$$= 14x \frac{1}{-6+7} e^{-3x}$$

$$= 14xe^{-3x}$$

The general solution is y = C.F + P.I

$$y = Ae^{-3x} + Be^{-4x} + 14xe^{-3x}$$

Example:5.15

Find the Particular Integral $(\mathbf{D^2-4D+4})\boldsymbol{y}=\mathbf{2}^{\boldsymbol{x}}$

Solution:

Given $(D^2-4D+4)y=2^x$

ie, $(D-2)^2y=2^x$

$$=e^{\log 2^x}$$

$$=e^{x\log 2}$$

$$=e^{(\log 2)x}$$

$$P.I=\frac{1}{(D-2)^2}e^{(\log 2)x}$$ [Replace D by log 2]

$$=\frac{1}{(\log 2-2)^2}e^{(\log 2)x}$$

Example:5.16

Solve $(\mathbf{D^2-4})\boldsymbol{y}=\mathbf{1}$

Solution:

Auxiliary Equation is $m^2-4=0$

$$m=\pm 2$$

$$C.F = Ae^{2x}+Be^{-2x}$$

$$P.I=\frac{1}{D^2-4}e^{0x}$$

$$=\frac{-1}{4}$$ [Replace D by 0]

The general solution is y = C.F+P.I

$$y = Ae^{2x} + Be^{-2x} - \frac{1}{4}$$

Example:5.17

Solve $\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 5y = -2coshx$

Solution:

Auxiliary Equation is $m^2 + 4m + 5 = 0$

$$m = \frac{-4 \pm \sqrt{16-20}}{2}$$

$$= \frac{-4 \pm 2i}{2}$$

$$= -2 \pm i$$

$$\text{C.F} = e^{-2x}[\text{A}\cos x + \text{B}\sin x]$$

$$\text{P.I} = \frac{1}{D^2+4D+5}.(-2coshx)$$

$$= \frac{1}{D^2+4D+5}.-2\left[\frac{e^x+e^{-x}}{2}\right]$$

$$= \frac{1}{D^2+4D+5}(-e^x - e^{-x})$$

$$\text{P.I}_1 = \frac{1}{D^2+4D+5}(-e^x)$$

$$= \frac{-1}{D^2+4D+5}e^x \qquad \text{[Replace D by 1]}$$

$$= \frac{-1}{10}e^x$$

$$\text{P.I}_2 = \frac{1}{D^2+4D+5}(-e^{-x})$$

$$= \frac{-1}{1-4+5}e^{-x} \qquad \text{[Replace D by } -1\text{]}$$

$$= \frac{-1}{2} e^{-x}$$

The general solution is y = C.F + P.I

$$y = e^{-2x}[Acosx + Bsinx] - \frac{e^{x}}{10} - \frac{e^{-x}}{2}$$

Example:5.18

Find the P.I of $(D^2 - 1)\, y = (e^x + 1)^2$

Solution:

Given $(D^2 - 1)y = (e^x + 1)^2$

$$= (e^x)^2 + 1 + 2e^x$$

$$= e^{2x} + e^0 + 2e^x$$

$$P.I_1 = \frac{1}{D^2-1} e^{2x} \quad \text{Replace D by 2}$$

$$= \frac{1}{4-1} e^{2x}$$

$$= \frac{1}{3} e^{2x}$$

$$P.I_2 = \frac{1}{D^2-1} e^{0x}$$

$$= \frac{1}{-1} e^{0x} \quad \text{Replace D by 0}$$

$$= -e^{0x}$$

$$= -1$$

$$P.I_3 = \frac{1}{D^2-1} 2e^{x}$$

$= 2 \frac{1}{D^2-1} e^x$ Replace D by 1

$= 2 \frac{1}{1-1} e^x$ (fails)

$= 2x \frac{1}{2D} e^x$ Replace D by 1

$= 2x \frac{1}{2} e^x$

$= xe^x$

$P.I = P.I_1 + P.I_2 + P.I_3$

$= \frac{1}{3} e^{2x} - 1 + xe^x$

Type II:

Problems Based on P.I = $\frac{1}{f(D)} \sin ax$ **(or)** $\frac{1}{f(D)} \cos ax$ **Replace D^2 by - a^2**

Example : 5.19

Solve $\frac{d^3y}{dx^3} + 4\frac{dy}{dx} = \sin 2x$

Solution:

Auxiliary Equation is $m^3 + 4m = 0$

$m(m^2+4) = 0$

$m = 0, m = \pm 2i,$

$\propto = 0, \beta = 2$

$C.F = Ae^{0x} + e^{0x}[B\cos 2x + C\sin 2x]$

$P.I = \frac{1}{D^3+4D} \sin 2x$

$= \frac{1}{D(D^2+4)} \sin 2x$ Replace D^2 by -4

$= \frac{1}{D(-4+4)} \sin 2x$

$= \frac{1}{0} \sin 2x$ (fails)

$= x.\frac{1}{3D^2+4} \sin 2x$ Replace D^2 by -4

$= x.\frac{1}{3(-4)+4} \sin 2x$

$= \frac{-x}{8} \sin 2x$

The general solution is y = C.F + P.I

$$y = A + [B\cos 2x + C\sin 2x] - \frac{x}{8} \sin 2x$$

Example :5.20

Solve $(D^2 - 4D + 4)y = e^{2x} + \sin^2 x$

Solution:

Auxiliary Equation is $m^2 - 4m + 4 = 0$

$(m-2)(m-2) = 0$

$m = 2, 2$

C.F = $(Ax + B)e^{2x}$

P.I = $\frac{1}{D^2-4D+4} [e^{2x} + \sin^2 x]$

$= \frac{1}{D^2-4D+4} \left[e^{2x} + \frac{1-\cos 2x}{2}\right]$

$$\text{P.I}_1 = \frac{1}{D^2-4D+4} e^{2x} \qquad \text{Replace D by 2}$$

$$= \frac{1}{4-8+4} e^{2x}$$

$$= \frac{1}{0} e^{2x} \qquad \text{(fails)}$$

$$= x.\frac{1}{2D-4} e^{2x} \qquad \text{Replace D by 2}$$

$$= x.\frac{1}{4-4} e^{2x}$$

$$= x\frac{1}{0} e^{2x} \quad \text{(fails)}$$

$$= x^2 \frac{1}{2} e^{2x}$$

$$\text{P.I}_2 = \frac{1}{D^2-4D+4}\left(\frac{1}{2}\right) e^{0x} \qquad \text{Replace D by 0}$$

$$= \frac{1}{8}$$

$$\text{P.I}_3 = \frac{1}{D^2-4D+4}\left(\frac{-\cos 2x}{2}\right) \qquad \text{Replace } D^2 \text{by -4}$$

$$= \frac{1}{-4-4D+4}\left(\frac{-\cos 2x}{2}\right)$$

$$= \frac{1}{-4D}\left(\frac{-\cos 2x}{2}\right)$$

$$= \frac{1}{8D} \cos 2x$$

$$= \frac{1}{8}\int \cos 2x \, dx$$

$$= \frac{1}{8}\left(\frac{\sin 2x}{2}\right)$$

$$= \frac{1}{16} \sin 2x$$

The general solution y = C.F + P.I$_1$ + P.I$_2$ + P.I$_3$

$$y = (\mathrm{A}x + \mathrm{B})\mathrm{e}^{2x} + \frac{x^2}{2}\mathrm{e}^{2x} + \frac{1}{8} + \frac{1}{16}\sin 2x$$

Example :5.21

Solve $(\mathbf{D^2 - 4D + 3})y = \mathbf{\sin 3}x \mathbf{\cos 2}x$

Solution:

Auxiliary Equation is $m^2 - 4m + 3 = 0$

$$(m-1)(m-3) = 0$$

$$m = 1, m = 3$$

$$\text{C.F} = \mathrm{A}e^{x} + \mathrm{B}e^{3x}$$

$$\text{P.I} = \frac{1}{\mathrm{D}^2 - 4\mathrm{D} + 3}[\sin 3x . \cos 2x]$$

$$= \frac{1}{\mathrm{D}^2 - 4\mathrm{D} + 3}\left[\frac{\sin(3+2)x + \sin(3-2)x}{2}\right]$$

$$= \frac{1}{\mathrm{D}^2 - 4\mathrm{D} + 3}\left[\frac{\sin 5x + \sin x}{2}\right]$$

$$\text{P.I}_1 = \frac{1}{\mathrm{D}^2 - 4\mathrm{D} + 3}\left(\frac{1}{2}\sin 5x\right) \qquad \text{Replace } \mathrm{D}^2 \text{ by } -25$$

$$= \frac{1}{-25 - 4\mathrm{D} + 3}\left(\frac{1}{2}\sin 5x\right)$$

$$= \frac{1}{2}\ \frac{1}{(-22 - 4\mathrm{D})}\sin 5x$$

$$= \frac{-1}{4}\ \frac{1}{11 + 2\mathrm{D}}(\sin 5x)$$

$$= \frac{-1}{4} . \frac{1}{11 + 2\mathrm{D}}\ \frac{11 - 2\mathrm{D}}{11 - 2\mathrm{D}}(\sin 5x)$$

$$= \frac{-1}{4}\left[\frac{11\sin 5x - 2\mathrm{D}(\sin 5x)}{11^2-(2\mathrm{D})^2}\right]$$

$$= \frac{-1}{4}\left[\frac{11\sin 5x - 10\cos 5x)}{121-4\mathrm{D}^2}\right] \qquad \text{Replace } \mathrm{D}^2 \text{ by } -25$$

$$= \frac{-1}{4}\left(\frac{11\sin 5x - 10\cos 5x)}{221}\right)$$

$$\mathrm{P.I}_1 = \frac{-1}{884}(11\sin 5x - 10\cos 5x)$$

$$\mathrm{P.I}_2 = \frac{1}{\mathrm{D}^2-4\mathrm{D}+3}\left(\frac{1}{2}\sin x\right) \qquad \text{Replace } \mathrm{D}^2 \text{ by } -1$$

$$= \frac{1}{-1-4\mathrm{D}+3}\left(\frac{1}{2}\sin x\right)$$

$$= \frac{1}{2-4\mathrm{D}}\left(\frac{1}{2}\sin x\right)$$

$$= \frac{1}{4(1-2\mathrm{D})}\sin x$$

$$= \frac{1}{4}\ \frac{1}{1-2\mathrm{D}}\ \frac{1+2\mathrm{D}}{1+2\mathrm{D}}\ \sin x$$

$$= \frac{1}{4}\ \frac{1+2\mathrm{D}}{1-4\mathrm{D}^2}\sin x \qquad \text{Replace } \mathrm{D}^2 \text{ by } -1$$

$$= \frac{1}{4}\ \frac{\sin x + 2\mathrm{D}\sin x}{1-4(-1)}$$

$$= \frac{1}{20}(\sin x + 2\cos x)$$

The general solution y = C.F + P.I$_1$ + P.I$_2$

$$y = \mathrm{A}e^{x} + \mathrm{B}e^{3x} - \frac{1}{884}(11\sin 5x - 10\cos 5x) + \frac{1}{20}(\sin x + 2\cos x)$$

Example :5.22

Solve $(\mathbf{D}^2+1)y = \sin x\ \sin 2x$

Solution:

Auxiliary Equation is $m^2 + 1 = 0$

$$m^2 = -1$$

$$m = \pm\sqrt{-1} = \pm i$$

$$C.F = A\cos x + B\sin x$$

$$P.I = \frac{1}{D^2+1}(\sin x \sin 2x)$$

$$= \frac{1}{D^2+1}\left[\frac{\cos(-x) - \cos(3x)}{2}\right]$$

$$= \frac{1}{D^2+1}\left[\frac{\cos x}{2} - \frac{\cos 3x}{2}\right]$$

$$P.I_1 = \frac{1}{D^2+1}\left(\frac{\cos x}{2}\right)$$ Replace D^2 by -1

$$= \frac{1}{-1+1}\left(\frac{\cos x}{2}\right)$$

$$= \frac{1}{0}.\frac{\cos x}{2}$$ (fails)

$$= x\,\frac{1}{2D}\left(\frac{\cos x}{2}\right)$$

$$= \frac{x}{4}\,\frac{1}{D}\cos x$$

$$= \frac{x}{4}\int \cos x\, dx$$

$$= \frac{x}{4}\sin x$$

$$P.I_2 = \frac{1}{D^2+1}\left(\frac{-\cos 3x}{2}\right)$$ Replace D^2 by -9

$$= \frac{1}{-9+1}\left(\frac{-\cos 3x}{2}\right)$$

$= \frac{1}{16} \cos 3x$

The general solution is y = C.F + P.I$_1$ + P.I$_2$

$$y = \text{A}\cos x + \text{B}\sin\text{x} + \frac{x}{4}\sin x + \frac{1}{16}\cos 3x$$

Example :5.23

Find the P.I of (D^2+4) y = cos2x

Solution:

P.I $= \frac{1}{D^2+4}\cos 2x$ Replace D^2 by -4

$= \frac{1}{-4+4}\cos 2x$

$= \frac{1}{0}\cos 2x$ (fails)

$= x\frac{1}{2D}\cos 2x$

$= \frac{x}{2D}\cos 2x$

$= \frac{x}{2}\int \cos 2x \, dx$

$= \frac{x}{2}\frac{\sin 2x}{2} = \frac{x}{4} sin2x$

$P.I = \frac{x}{4} sin2x$

Example :5.24

Find the P.I of $\frac{d^3y}{dx^3} + 4\frac{dx}{dx} = sin2x$

Solution:

$$\text{P.I} = \frac{1}{D^3+4D}\sin 2x$$

$$= \frac{1}{D(D^2+4)}\sin 2x \qquad \text{Replace } D^2 \text{ by } -4$$

$$= \frac{1}{D(-4+4)}\sin 2x \qquad \text{(fails)}$$

$$= x\frac{1}{3D^2+4}\sin 2x \qquad \text{Replace } D^2 \text{ by } -4$$

$$= x\frac{1}{3(-4)+4}\sin 2x$$

$$= x\frac{1}{-12+4}\sin 2x$$

$$= \frac{-x}{8}\sin 2x$$

$$P.I = \frac{-x}{8}\sin 2x$$

Example :5.25

Find the P.I of (D^3+1) $y = cos(2x-1)$

Solution:

$$\text{P.I} = \frac{1}{D^3+1}\cos(2x-1)$$

$$= \frac{1}{D.D^2+1}\cos(2x-1) \qquad \text{Replace } D^2 \text{ by } -4$$

$$= \frac{1}{1-4D}\cos(2x-1)$$

$$= \frac{1}{1-4D}\ \frac{1+4D}{1+4D}\cos(2x-1)$$

$$= \frac{1+4D}{1-16D^2}\cos(2x-1) \qquad \text{Replace } D^2 \text{ by } -4$$

$$= \frac{1+4D}{1-16(-4)}\cos(2x-1)$$

$$= \frac{1+4D}{1+64} \cos(2x-1)$$

$$= \frac{1}{65}(1+4D)\cos(2x-1)$$

$$= \frac{1}{65}[\cos(2x-1) + 4D\,(\cos(2x-1)]$$

$$= \frac{1}{65}[\cos(2x-1) + 4\,(-\sin(2x-1)(2))]$$

$$P.I = \frac{1}{65}[\cos(2x-1) - 8\sin(2x-1)]$$

Example :5.26

Solve (D^2-D+1) $y = \sin^3 x$

Solution:

Auxiliary Equation is $m^2 - m + 1 = 0$

$$m = \frac{1 \pm \sqrt{1-4}}{2}$$

$$= \frac{1 \pm i\sqrt{3}}{2}$$

$$= \frac{1}{2} \pm \frac{i\sqrt{3}}{2}$$

$$C.F = e^{\frac{1}{2}x}\left[A\cos\frac{\sqrt{3}}{2}x + B\cos\frac{\sqrt{3}}{2}x\right]$$

$$\text{P.I} = \frac{1}{D^2-D+1}\sin^3 x$$

$$= \frac{1}{D^2-D+1}\left[\frac{3\sin x - \sin 3x}{4}\right]$$

$$P.I_1 = \frac{1}{D^2-D+1}\left[\frac{3}{4}\sin x\right] \qquad \text{Replace } D^2 \text{ by } -1$$

$$= \frac{1}{-1-D+1}\left[\frac{3}{4}\sin x\right]$$

$$= \frac{1}{-D}.\frac{3}{4}\sin x$$

$$= \frac{-3}{4} \ . \ \frac{1}{D}\sin x$$

$$= \frac{-3}{4}\int \sin x \, dx$$

$$P.I_1 = \frac{-3}{4}(-cosx)$$

$$P.I_2 = \frac{1}{D^2-D+1}\left[\frac{-sin3x}{4}\right] \qquad \text{Replace } D^2 \text{by} -9$$

$$= \frac{1}{-9-D+1}\left[\frac{-sin3x}{4}\right]$$

$$= \frac{1}{-8-D}\left[\frac{-sin3x}{4}\right]$$

$$= \frac{1}{4}\frac{1}{8+D}\frac{8-D}{8-D}(\sin 3x)$$

$$= \frac{1}{4}\left[\frac{8\sin 3x - D(\sin 3x)}{64-D^2}\right]$$

$$= \frac{1}{4}\left[\frac{8\sin 3x - 3\cos 3x}{64-(-9)}\right]$$

$$= \frac{1}{4}\left[\frac{8\sin 3x - 3\cos 3x}{73}\right]$$

$$= \frac{1}{292}(8\sin 3x - 3\cos 3x)$$

The general solution y = C.F + P.I$_1$ + P.I$_2$

$$y = e^{\frac{1}{2}x}\left[A\ \cos\frac{\sqrt{3}}{2}x + B\ \cos\frac{\sqrt{3}}{2}x\right] + \frac{3}{4}\cos x + \frac{1}{292}[8\sin 3x - 3\cos 3x]$$

Type III: Problems Based on R.H.S= $e^{ax} + \sin ax\ (or)\ e^{ax} + \cos ax$

Example :5.27

Find the particular integral of $(D^2 + 2D + 2)\, y = e^{-2x} + \cos 2x$

Solution:

$$P.I_1 = \frac{1}{D^2+2D+2} e^{-2x}$$

$$= \frac{1}{4-4+2} e^{-2x} \qquad \text{(Replace D by } -2)$$

$$= \frac{1}{2} e^{-2x}$$

$$P.I_2 = \frac{1}{D^2+2D+2} \cos 2x$$

$$= R.P \frac{1}{D^2+2D+2} e^{i2x}$$

$$= R.P \frac{1}{(i2)^2+2(i2)+2} e^{i2x} \qquad \text{Replace D by } i2$$

$$= R.P \frac{1}{-4+4i+2} e^{i2x}$$

$$= R.P \left[\frac{-1}{10} + \frac{-1}{5} i\right] (cos2x + isin\, 2x)$$

$$= \frac{-1}{10} \cos 2x + \frac{1}{5} \sin 2x$$

The general solution y = C.F + P.I$_1$ + P.I$_2$

$$y = e^{-x}[Acos\, x + B \sin x] + \frac{1}{2} e^{-2x} + \frac{1}{5} sin2x - \frac{1}{10} cos2x$$

Example :5.28

Solve $(D^2 - 3D + 2)y = 2\cos(2x + 3) + 2e^x$

Solution:

Auxiliary Equation is $m^2 - 3m + 2 = 0$

$m = 1\ , m = 2$

C.F = $Ae^x + Be^{2x}$

P.I$_1$ = $\frac{1}{D^2-3D+2} 2e^x$

$= 2\ \frac{1}{1-3+2} e^x$ Replace D by 1

$= 2\ \frac{1}{0} e^x$ (fails)

$= 2x\ \frac{1}{2D-3} e^x$

$= 2x\ \frac{1}{2-3} e^x$ Replace D by 1

$= -2xe^x$

P.I$_2$ = $\frac{1}{D^2-3D+2} 2\cos(2x+3)$

$= 2\frac{1}{-4-3D+2}\cos(2x+3)$ Replace D^2 by -4

$= 2\frac{1}{-3D-2}\cos(2x+3)$

$= 2\frac{1}{-3D-2}\ \frac{-3D+2}{-3D+2}\cos(2x+3)$

$= 2\ \frac{-3D+2}{9D^2-4}\cos(2x+3)$ Replace D^2 by -4

$= 2\ \frac{-3D+2}{-40}\cos(2x+3)$

$= \frac{-3D+2}{-20}\cos(2x+3)$

$= 6\sin(2x+3) + 2\cos(2x+3)\ /-20$

$$= -\frac{1}{10}\cos(2x+3) - \frac{3}{10}\sin(2x+3)$$

The general solution is y = C.F + P.I$_1$ + P.I$_2$

$$y = Ae^{x} + Be^{2x} - 2xe^{x} - \frac{1}{10}\cos(2x+3) - \frac{3}{10}\sin(2x+3)$$

Example :5.29

Solve $(D^3 - 2D^2 + 4D - 8)y = e^{2x} + \frac{1}{2}sin2x$

Solution:

Auxiliary Equation is $m^3 - 2m^2 + 4m - 8 = 0$

$$m^2(m-2) + 4\,(m-2) = 0$$

$$(m-2)(m^2+4) = 0$$

$$m = 2, m = \pm 2i$$

C.F $= Ae^{2x} + e^{0x}[Bcos2x + Csin2x]$

P.I$_1$ = $\frac{1}{D^3-2D^2+4D-8}e^{2x}$ Replace D by 2

$$= \frac{1}{8-8+8-8}e^{2x}$$

$$= \frac{1}{0}e^{2x}$$ (fails)

$$= x\ \frac{1}{3D^2-4D+4}e^{2x}$$ Replace D by 2

$$= x\ \frac{1}{3(4)-4(2)+4}e^{2x}$$

$$= \frac{x}{8}e^{2x}$$

$$\text{P.I}_2 = \frac{1}{2} \cdot \frac{1}{D^3-2D^2+4D-8} sin2x$$

$$= \frac{1}{2} \text{ I.P } \frac{1}{D^3-2D^2+4D-8} e^{i2x}$$

$$= \frac{1}{2} \text{ I.P } \frac{1}{(i2)^3-2(i2)^2+4(i2)-8} e^{i2x}$$ Replace D by i2

$$= \frac{1}{2} \text{ I.P } \frac{1}{-8i+8+8i-8} e^{i2x}$$

$$= \frac{1}{2} \text{ I.P } \frac{1}{0} e^{i2x}$$ (fails)

$$= \frac{x}{2} \text{ I.P } \frac{1}{3D^2-4D+4} e^{i2x}$$

$$= \frac{x}{2} \text{ I.P } \frac{1}{3(2i)^2-4(2i)+4} e^{i2x}$$ Replace D by 2i

$$= \frac{x}{2} \text{ I.P } \left[\frac{-1}{16} + \frac{1}{16} i\right] (cos2x + isin2x)$$

$$= \frac{x}{2} \left[\frac{-1}{16} sin2x + \frac{1}{16} cos2x\right]$$

$$= \frac{x}{32} (cos2x - sin2x)$$

The general solution is $y = \text{C.F} + \text{P.I}_1 + \text{P.I}_2$

$$y = Ae^{2x} + e^{0x}[Bcos2x + Csin2x] + \frac{x}{8} e^{2x} + \frac{x}{32} (cos2x - sin2x)$$

Type IV : Problems Based on R.H.S = Polynomial in x

Binomial expression

$$(1+x)^{-1} = 1 - x + x^2 - x^3 + \cdots \ldots \ldots \ldots ....$$

$$(1-x)^{-1} = 1 + x + x^2 + x^3 + \cdots \ldots \ldots \ldots ....$$

$$(1+x)^{-2} = 1 - 2x + 3x^2 - 4x^3 + \cdots \ldots \ldots \ldots$$

$$(1-x)^{-2} = 1 + 2x + 3x^2 + 4x^3 + \cdots \ldots\ldots\ldots..$$

Example:5.30

Solve $y'' + 4y' + 5y = 3x - 2$

Solution:

Auxiliary Equation is $m^2 + 4m + 5 = 0$

$$m = \frac{-4 \pm \sqrt{16-20}}{2}$$

$$= \frac{-4 \pm \sqrt{-4}}{2}$$

$$= \frac{-4 \pm 2i}{2} = -2 \pm i$$

$$\propto = -2,\ \beta = 1$$

$$\text{C.F} = e^{-2x}[A\cos x + B\sin x]$$

$$\text{P.I} = \frac{1}{D^2+4D+5}(3x-2)$$

$$= \frac{1}{5\left[1+\frac{D^2+4D}{5}\right]}(3x-2)$$

$$= \frac{1}{5}\left[1+\frac{D^2+4D}{5}\right]^{-1}(3x-2)$$

$$= \frac{1}{5}\left[1-\frac{D^2+4D}{5}\right](3x-2)$$

$$= \frac{1}{5}\left[1-\frac{D^2}{5}-\frac{4D}{5}\right](3x-2)$$

$$= \frac{1}{5}\left[(3x-2)-\frac{D^2}{5}(3x-2)-\frac{4D}{5}(3x-2)\right]$$

$$= \frac{1}{5}\left[3x-2-0-\frac{4}{5}(3)\right]$$

$$= \frac{1}{5}\left[\frac{15x-10-12}{5}\right]$$

$$= \frac{1}{25}[15x - 22]$$

The general solution is y = C.F +P.F

$$y = e^{-2x}[Acosx + Bsinx] + \frac{1}{25}[15x - 22]$$

Example:5.31

Solve $\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = x^2 + 3$

Solution:

$$(D^2 - 5D + 6)y = x^2 + 3$$

Auxiliary Equation is $m^2 - 5m + 6 = 0$

m = 3, 2

C.F = Ae^{3x} + Be^{2x}

P.I$_1$ = $\frac{1}{D^2-5D+6}x^2$

$$= \frac{1}{6\left[1+\frac{D^2-5D}{6}\right]}x^2$$

$$= \frac{1}{6}\left[1 + \frac{D^2-5D}{6}\right]^{-1} x^2$$

$$= \frac{1}{6}\left[1 - \left(\frac{D^2-5D}{6}\right) + \left(\frac{D^2-5D}{6}\right)^2 \ldots.\right] x^2$$

$$= \frac{1}{6}\left[1 - \frac{D^2}{6} + \frac{5D}{6} + \frac{25}{36}D^2\right] x^2$$

$$= \frac{1}{6}\left[x^2 - \frac{D^2(x^2)}{6} + \frac{5D(x^2)}{6} + \frac{25}{36}D^2(x^2)\right]$$

$$= \frac{1}{6}\left[x^2 - \frac{2}{6} + \frac{5\times 2x}{6} + \frac{25}{36}(2)\right]$$

$$= \frac{1}{6}\left[x^2 + \frac{5}{3}x + \frac{19}{18}\right]$$

$$\text{P.I}_2 = \frac{1}{D^2-5D+6}3e^{0x}$$

$$= \frac{1}{2}$$

The general solution is y =C.F + P.I$_1$ +P.I$_2$

$$y = Ae^{3x} + Be^{2x} + \frac{1}{6}\left[x^2 + \frac{5}{3}x + \frac{19}{18}\right] + \frac{1}{2}$$

Example:5.32

Solve $(D^3 + 8)\, y = x^4 + 2x + 1$

Solution :

Auxiliary Equation is $m^3 + 8 = 0$

$$m = -2, m^2 - 2m + 4 = 0$$

$$m = \frac{1 \pm i\sqrt{3}}{2}$$

$$= \frac{1}{2} \pm i\frac{\sqrt{3}}{2}$$

$$\alpha = \frac{1}{2} \qquad \beta = \frac{\sqrt{3}}{2}$$

$$\text{C.F} = Ae^{-2x} + Be^{\frac{1}{2}x}\left[Bcos\frac{\sqrt{3}}{2}x + Ccos\frac{\sqrt{3}x}{2}\right]$$

$$\text{P.I} = \frac{1}{D^3+8}(x^4 + 2x + 1)$$

$$= \frac{1}{8\left[1+\frac{D^3}{8}\right]}(x^4+2x+1)$$

$$= \frac{1}{8}\left[1+\frac{D^3}{8}\right]^{-1}(x^4+2x+1)$$

$$= \frac{1}{8}\left[1-\frac{D^3}{8}+\left(\frac{D^3}{8}\right)^2 \dots\right](x^4+2x+1)$$

$$= \frac{1}{8}\left[1-\frac{D^3}{8}\right](x^4+2x+1)$$

$$= \frac{1}{8}\left[(x^4+2x+1)-\frac{D^3}{8}(x^4+2x+1)\right]$$

$$= \frac{1}{8}\left[x^4+2x+1-\frac{24x}{8}\right]$$

$$= \frac{1}{8}[x^4+2x+1-3x]$$

$$= \frac{1}{8}[x^4-x+1]$$

The general solution is $y = C.F + P.I$

$$y = Ae^{-2x} + B^{\frac{1}{2}x}\left[Bcos\frac{\sqrt{3}}{2}x + Ccos\frac{\sqrt{3}}{2}x\right] + \frac{1}{8}[x^4-x+1]$$

Example:5.33

Solve $(D^2 - D)y = x$

Solution:

Auxiliary Equation is $m^2 - m = 0$

$$m(m-1) = 0$$

$$m = 0, 1$$

C.F = $Ae^{0x} + Be^{x}$ = $A + Be^{x}$

$$\text{P.I} = \frac{1}{D^2-D}x$$

$$= \frac{1}{D(D-1)}x$$

$$= \frac{1}{-D(1-D)}x$$

$$= \frac{1}{-D}(1-D)^{-1}$$

$$= \frac{1}{-D}[1+D+D^2+\cdots]x$$

$$= \frac{1}{-D}[1+D(x)]$$

$$= \frac{-1}{D}[x+1]$$

$$= \frac{-1}{D}(x+1)$$

$$= -\left[\frac{x^2}{2}+x\right]$$

The general solution is y = C.F +P.I

$$y = A + Be^x - \frac{x^2}{2} - x$$

Example:5.34

Solve $(D^4 - 2D^3 + D^2)y = x^3$

Solution:

Auxiliary Equation is $m^4 - 2m^3 + m^2 = 0$

$$m^2(m^2 - 2m + 1) = 0$$

$$m^2(m-1)^2 = 0$$

$$m = 0,1 \quad \text{(twice)}$$

$$\text{C.F} = (A + Bx)e^{0x} + (C + Dx)e^{x}$$

$$= A + Bx + (C + Dx)e^{x}$$

$$\text{P.I} = \frac{1}{D^4 - 2D^3 + D^2} x^3$$

$$= \frac{1}{D^2(D^2 - 2D + 1)} x^3$$

$$= \frac{1}{D^2[1 + (D^2 - 2D)]} x^3$$

$$= \frac{1}{D^2}[1 + (D^2 - 2D)]^{-1} x^3$$

$$= \frac{1}{D^2}[1 - (D^2 - 2D) + (D^2 - 2D)^2 - (D^2 - 2D)^3 + - - -]x^3$$

$$= \frac{1}{D^2}[1 - D^2 + 2D + D^4 - 4D^3 + 4D^2 + 8D^3]x^3$$

$$= \frac{1}{D^2}[x^3 - 6x + 6x^2 - 24 + 24x + 48]$$

$$= \frac{1}{D^2}[x^3 + 18x + 6x^2 + 24]$$

$$= \frac{1}{D}\left[\frac{x^4}{4} + \frac{18x^2}{2} + \frac{6x^3}{3} + 24x\right]$$

$$= \frac{x^5}{20} + 3x^3 + \frac{x^4}{2} + 12x^2$$

The general solution is y = C.F + P.I

$$y = A + Bx + (c + Dx)e^{x} + \frac{x^5}{20} + 3x^3 + \frac{x^4}{2} + 12x^2$$

Example:5.35

Solve $(D^3 - 3D^2 - 6D + 8)y = x$

Solution:

Auxiliary Equation is $m^3 - 3m^3 - 6m + 8 = 0$

$$m = 1,\ m = -2,\ m = 4$$

$$\text{C.F} = Ae^{x} + Be^{-2x} + ce^{4x}$$

$$\text{P.I} = \frac{1}{D^3-3D^2-6D+8}x$$

$$= \frac{1}{8\left[1+\frac{D^3-3D^2-6D}{8}\right]}x$$

$$= \frac{1}{8}\left[1+\frac{D^3-3D^2-6D}{8}\right]^{-1}x$$

$$= \frac{1}{8}\left[1-\left(\frac{D^3-3D^2-6D}{8}\right)+\cdots\right]x$$

$$= \frac{1}{8}\left[1+\frac{6D}{8}\right]x$$

$$= \frac{1}{8}\left[x+\frac{3}{4}\right]$$

The general solution y = C.F + P.I

$$y = Ae^{x} + Be^{-2x} + Ce^{4x} + \frac{1}{8}\left[x+\frac{3}{4}\right]$$

Example:5.36

Solve $[D^2 - 4D + 3]y = cos2x + 2x^2$

Solution:

Auxiliary Equation is $m^2 - 4m + 3 = 0$

$$m = 1, 3$$

$$\text{C.F} = Ae^{x} + Be^{3x}$$

$$\text{P.I} = \frac{1}{D^2-4D+3} cos2x$$

$$= \frac{1}{-4-4D+3} cos2x \qquad Replace\ D^2 \text{ by} -4$$

$$= \frac{1}{-1-4D} cos2x$$

$$= \frac{1}{-(1+4D)} cos2x$$

$$= -\frac{(1-4D)}{(1+4D)(1-4D)} cos2x$$

$$= -\frac{(1-4D)}{1-16D^2} cos2x$$

$$= -\frac{(cos2x+8sin2x)}{1-16(-4)} \qquad \text{Replace } D^2 \text{ by} -4$$

$$\text{P.I}_1 = \frac{-1}{65}(cos2x + 8sin2x)$$

$$\text{P.I}_2 = \frac{1}{D^2-4D+3}(2x^2)$$

$$= \frac{1}{3\left(1+\frac{D^2-4D}{3}\right)}(2x^2)$$

$$= \frac{1}{3}\left[1 + \frac{D^2-4D}{3}\right]^{-1} (2x^2)$$

$$= \frac{1}{3}\left[1 - \frac{D^2-4D}{3} + \left(\frac{D^2-4D}{3}\right)^2\right](2x^2)$$

$$= \frac{1}{3}\left[1 - \frac{D^2}{3} + \frac{4D}{3} + \left(\frac{4D}{3}\right)^2\right](2x^2)$$

$$= \frac{1}{3}\left[2x^2 - \frac{1}{3}D^2(2x^2) + \frac{4}{3}D(2x^2) + \frac{16}{9}D^2(2x^2)\right]$$

$$= \frac{1}{3}\left[2x^2 - \frac{1}{3}(4) + \frac{4}{3}(4x) + \frac{16}{9}(4)\right]$$

$$= \frac{1}{3}\left[\frac{18x^2-12+48x+64}{9}\right]$$

$$= \frac{1}{27}[18x^2 + 48x + 52]$$

The general solution is y = C.F + P.I$_1$ + P.I$_2$

$$y = Ae^x + Be^{3x} - \frac{1}{65}(cos2x + 8sin2x) + \frac{1}{27}[18x^2 + 48x + 52]$$

Example:5.37

Solve $(D^2 + 4)y = x^4 + cos^2x$

Solution:

Auxiliary Equation is $m^2 + 4 = 0$

$$m = \pm\sqrt{-4}$$

$$= \pm 2i$$

C.F $= Acos2x + Bsin2x$

P.I$_1$$= \frac{1}{D^2+4}x^4$

$$= \frac{1}{4\left(1+\frac{D^2}{4}\right)}x^4$$

$$= \frac{1}{4}\left(1 + \frac{D^2}{4}\right)^{-1} x^4$$

$$= \frac{1}{4}\left[1 - \frac{D^2}{4}\left(\frac{D^2}{4}\right)^2\right]x^4$$

$$= \frac{1}{4}\left[x^4 - \frac{D^2(x^4)}{4} + \frac{D^4(x^4)}{16}\right]$$

$$= \frac{1}{4}\left[x^4 - \frac{12x^2}{4} + \frac{24}{16}\right]$$

$$= \frac{1}{4}\left[x^4 - 3x^2 + \frac{3}{2}\right]$$

$$= \frac{1}{8}[2x^4 - 6x^2 + 3]$$

$$\text{P.I}_2 = \frac{1}{D^2+4} cos^2 x$$

$$= \frac{1}{D^2+4}\left(\frac{1+cos2x}{2}\right)$$

$$= \frac{1}{D^2+4}\ \frac{1}{2} e^{0x} + \frac{1}{D^2+4}\ \frac{cos2x}{2}$$

Replace D by 0 ; Replace D^2 by -4

$$= \frac{1}{8} + \frac{1}{-4+4}\ \frac{cos2x}{2}$$

$$= \frac{1}{8} + \frac{1}{0}\ \frac{cos2x}{2} \qquad \text{(fails)}$$

$$= \frac{1}{8} + \frac{x}{2D}\left(\frac{cos2x}{2}\right)$$

$$= \frac{1}{8} + \frac{x}{2}\left(\frac{sin2x}{4}\right)$$

$$= \frac{1}{8} + \frac{x}{8} sin2x$$

The general solution is y = C.F + P.I$_1$ + P.I$_2$

$$y = Acos2x + Bsin2x + \frac{1}{8}(2x^4 - 6x^2 + 3) + \frac{1}{8} + \frac{x}{8} sin2x$$

Type V: Problems based on R.H.S $e^{ax}F(x)$

$$\text{P.I} = \frac{1}{f(D+a)} e^{ax} F(x) \quad \text{Replace } x \text{ by D+a}$$

$$= e^{ax} \frac{1}{f(D+a)} F(x)$$

Example:5.38

Solve $(D^2+6D+9)y = e^{-2x}x^3$

Solution:

Auxiliary Equation is $m^2+6m+9=0$

$$(m+3)\quad(m+3)=0$$

$$m=-3,-3$$

C.F = $(A+Bx)e^{3x}$

P.I $= \frac{1}{D^2+6D+9}e^{-2x}x^3$ Replace D by $D-2$

$$= e^{-2x}\frac{1}{(D-2)^2+6(D-2)+9}x^3$$

$$= e^{-2x}\frac{1}{D^2-4D+4+6D-12+9}x^3$$

$$= e^{-2x}\frac{1}{D^2+2D+1}x^3$$

$$= e^{-2x}\frac{1}{1+(D^2+2D)}x^3$$

$$= e^{-2x}[1+(D^2+2D]^{-1}x^3$$

$$= e^{-2x}[1-(D^2+2D)+(D^2+2D)^2-(D^2+2D)^3]x^3$$

$$= e^{-2x}[1-D^2-2D+D^4+4D^3+4D^2-8D^3]x^3$$

$$= e^{-2x}[x^3-D^2(x^3)-2D(x^3)+4D^3(x^3)-4D^2(x^3)-\ 8D^3(x^3)]$$

$$= e^{-2x}[x^3-6x-2(3x^2)+4(6)+4(6x)-8(6)]$$

$$= e^{-2x}[x^3-6x-6x^2+24+24x-48]$$

$$= e^{-2x}[x^3-6x^2+18x-24]$$

The general solution is y = C.F + P.I

$$y = (A + Bx)e^{3x} + e^{-2x}(x^3 - 6x^2 + 18x - 24)$$

Example :5.39

Solve $(D^2 + 1)y = xsinhx$

Solution:

Auxiliary Equation is $m^2 + 1 = 0$

$m^2 = -1$

$m^2 = \pm i$

C.F $= Acosx + Bsinx$

P.I $= \frac{1}{D^2+1} xsinhx$

$= \frac{1}{D^2+1}\left[x\left(\frac{e^x - e^{-x}}{2}\right)\right]$

$= \frac{1}{2}\left[\frac{1}{D^2+1} xe^x - \frac{1}{D^2+1} xe^{-x}\right]$

Replace $by\ D + 1$; Replace D by $D - 1$

$= \frac{1}{2}\left[e^x \frac{1}{(D+1)^2+1} x - e^x \frac{1}{(D-1)^2+1}\right]$

$= \frac{1}{2}\left[e^x \frac{1}{D^2+2D+2} x - e^{-x} \frac{1}{D^2-2D+2} x\right]$

$= \frac{1}{2}\left[e^x \frac{1}{2\left[1+\left(\frac{D^2+2D}{2}\right)\right]} x - e^{-x} \frac{1}{2\left[1+\left(\frac{D^2-2D}{2}\right)\right]} x\right]$

$= \frac{1}{2}\left[\frac{e^x}{2}\left[1 + \left(\frac{D^2+2D}{2}\right)\right]^{-1} x - \frac{e^{-x}}{2}\left[1 + \left(\frac{D^2-2D}{2}\right)\right]^{-1} x\right]$

$$= \frac{1}{2}\left[\frac{e^x}{2}\left(1-\frac{D^2}{2}-\frac{2D}{2}\right)x-\frac{e^{-x}}{2}\left(1-\frac{D^2}{2}+\frac{2D}{2}\right)x\right]$$

$$= \frac{1}{2}\left[\frac{e^x}{2}(x-1)-\frac{e^{-x}}{2}(x+1)\right]$$

$$= \frac{1}{2}\left[\frac{e^x x}{2}-\frac{e^x}{2}-\frac{xe^{-x}}{2}-\frac{e^{-x}}{2}\right]$$

$$= \frac{1}{2}\left[x\left(\frac{e^x-e^{-x}}{2}\right)-\left(\frac{e^x+e^{-x}}{2}\right)\right]$$

$$= \frac{1}{2}[xsinhx-coshx]$$

The general solution is y = C.F + P.I

$$y = Acosx + Bsinx + \frac{1}{2}(xsinhx - coshx)$$

TYPE VI:

Problems based on f(x) = $x^n \sin ax$ or $x^n \cos ax$ $P.I = \frac{1}{f(x)} x^n \sin ax$ or $x^n \cos ax$

Example 5.40

Solve $(D^2+1)y = xsin\ x$

Solution:

Auxiliary Equation is $m^2+1=0$

$$m^2+1=-1$$

$$m=\pm i$$

C.F $= Acos\ x + Bsin\ x$

P.I $= \frac{1}{D^2+1}\ xsin\ x$

$= \frac{1}{D^2+1}\ x\ I.P\ of\ e^{ix}$ Replace D $by D + i$

$= I.P\ of e^{ix}\ \frac{1}{(D+i)^2+1}\ x$

$= I.P\ of e^{ix}\ \frac{1}{D^2+2Di+i^2+1}\ x$

$= I.P\ of e^{ix}\ \frac{1}{D^2+2Di+i^2+1}\ x$

$= I.P\ of e^{ix}\ \frac{1}{D^2+2Di}\ x$

$= I.P\ of e^{ix}\ \frac{1}{2Di}\left(1+\frac{D}{2i}\right)^{-1} x$

$= I.P\ of e^{ix}\ \frac{1}{2Di}\left(x-\frac{D(x)}{2i}\right)$

$= I.P\ of (\cos x + i\sin x)\ \left(\frac{x^2}{4i}+\frac{x}{4}\right)$

$= I.P\ of (\cos x + i\sin x)\ \left(\frac{-x^2 i}{4}+\frac{x}{4}\right)$

$= I.P\ of \left(\frac{-ix^2}{4}\cos x + \frac{x\cos x}{4} - \frac{x^2 sinx}{4} + \frac{ix\ sinx}{4}\right)$

$= \frac{-x^2}{4}\cos x + \frac{xsin\ x}{4}$

The general solution is y = C.F + P.I

$$y = Acosx + Bsinx - \frac{x^2}{4}cosx + \frac{xsin\ x}{4}$$

Example:5.41

Solve $(D^2 - 4D + 4)y = 3x^2e^{2x}sin\ 2x$

Solution:

Auxiliary Equation is $m^2 - 4m + 4 = 0$

$$m = 2, 2$$

C.F = $(A + Bx)e^{2x}$

P.I $= \frac{1}{D^2-4D+4} \, 3x^2 e^{2x} sin\, 2x$ Replace D by $D + 2$

$$= 3e^{2x} \frac{1}{(D+2)^2-4(D+2)+4} \, x^2 sin2x$$

$$= 3e^{2x} \frac{1}{D^2} \, x^2 sin2x$$

$$= 3e^{2x} \frac{1}{D^2} \, x^2 I.P \, of e^{i2x}$$ Replace D by $D + 2i$

$$= 3e^{2x} I.P \, of e^{i2x} \frac{1}{(D+2i)^2} \, x^2$$

$$= 3e^{2x} I.P \, of e^{i2x} \frac{1}{-4\left[1-\frac{D^2+4Di}{4}\right]} \, x^2$$

$$= 3e^{2x} I.P \, of e^{i2x} \frac{1}{-4}\left[1-\frac{D^2+4Di}{4}\right]^{-1} x^2$$

$$= 3e^{2x} I.P \, of e^{i2x} \frac{1}{-4}\left[1+\left(\frac{D^2+4Di}{4}\right)+\left(\frac{D^2+4Di}{4}\right)^2+\cdots\right] x^2$$

$$= 3e^{2x} I.P \, of e^{i2x} \frac{1}{-4}\left[1+\frac{D^2}{4}+Di+D^2 i\right](x^2)$$

$$= \frac{3e^{2x}}{-4} I.P \, of (\cos 2x + isin\, 2x)\left(x^2 + \frac{1}{2} + i2x - 2\right)$$

$$= \frac{-3}{4} e^{2x} I.P \, of (\cos 2x + isin\, 2x)\left(x^2 + 2xi - \frac{3}{2}\right)$$

$$= \frac{-3}{4} e^{2x} I.P \, of (x^2 \cos 2x + i2x \, cos\, 2x - \frac{3}{2}\cos 2x + i\, x^2 \sin 2x -$$

$$2x \sin 2x - i\frac{3}{2} \sin 2x\ )$$

$$= \frac{-3e^{2x}}{4}\left[2\,x\cos 2x + x^2 sin\,2x - \frac{3}{2}\sin 2x\right]$$

The general solution y = C.F + P.I

$$y = (A + Bx)e^{2x} - \frac{3}{4}e^{2x}\ \left[2x\cos 2x + x^2 sin\,2x - \frac{3}{2}\sin 2x\right]$$

Example:5.42

Solve $(\boldsymbol{D^2 - 2D})\boldsymbol{y = e^x x^2 cos\ x}$

Solution:

Auxiliary Equation is $m^2 - 2m = 0$

$$m(m-2) = 0$$

$$m = 0, m = 2$$

C.F = $(Ae^{0x} + Be^{2x})$

$= A + Be^{2x}$

P.I $= \frac{1}{D^2-2D}\ e^x x^2 cos\ x$ Replace D by $D + 1$

$$= e^x \frac{1}{(D+1)^2-2(D+1)}\ x^2\cos x$$

$$= e^x \frac{1}{D^2+2D+1-2D-2}\ x^2\cos x$$

$$= e^x \frac{1}{D^2-1}\ x^2\ R.P\ of\ e^{ix}$$

$$= e^x R.P\ of e^{ix}\frac{1}{(D+i)^2-1}\ x^2$$ Replace D by $D + i$

$$= e^x R.P\ of e^{ix}\frac{1}{D^2+2Di-1-1}\ x^2$$

$$= e^x R.P\ of e^{ix} \frac{1}{-2\left[1-\frac{D^2+2Di}{2}\right]}\ x^2$$

$$= e^x R.P\ of e^{ix} \frac{1}{-2}\left[1-\ \frac{D^2+2Di}{2}\right]^{-1}\ x^2$$

$$= \frac{e^x}{-2} R.P\ of e^{ix}\ \left[1+\frac{D^2}{2}+iD-D^2\right]x^2$$

$$= \frac{e^x}{-2} R.Pof(\cos x + i sin\ x)\ (x^2+2xi-1)$$

$$= \frac{e^x}{4} R.P\ of(x^2\cos x + i2x\ cos\ x - \cos x + ix^2\sin x -$$

$$2x\sin x - i\sin x\ )$$

$$= \frac{e^x}{-2}[\ x^2\cos x - \cos x - 2x\sin x]$$

The general solution is y = C.F+P.I

$$y = (A+Be^{2x}) - \frac{e^x}{2}\ [x^2\ \cos x + \cos x\ - 2x\ \sin x]$$

Example:5.43

Solve $(D^2-2D+1)y = xe^x\sin x$

Solution:

Auxiliary Equation is $m^2-2m+1=0$

$$m = 1{,}1$$

C.F = $(A+Bx)e^x$

P.I = $\frac{1}{D^2-2D+1}\ x\ e^x sin\ x$

$$= \ I.P\ of \frac{1}{D^2-2D+1}\ x\ e^x e^{ix}$$

$$= I.P\ of \frac{1}{D^2-2D+1}\ x\, e^{(1+i)x}$$

$$= I.P\ of \frac{1}{(D-1)^2}\ xe^{(1+i)x}$$ Replace D by $D+1+i$

$$= I.P\ of e^{(1+i)x} \frac{1}{(D+1+i-1)^2}$$

$$= I.P\ of e^{(1+i)x} \frac{1}{(D+i)^2}$$

$$= I.P\ of e^{(1+i)x} \frac{1}{i^2}\left[1+\frac{D}{i}\right]^{-2}\ x$$

$$= I.P\ of e^{(1+i)x}(-1)\left(x-\frac{2}{i}\right)$$

$$= I.P\ of e^x e^{ix}\ (-1)(x+2i)$$

$$= -e^x(\text{x} \sin x + 2 \cos x)$$

The general solution is y = C.F + P.I

$$y = (A+Bx)e^x - e^x\ [x \sin x + 2 \cos x]$$

Example: 5.44

Solve $(D^2+4)y = x^2 \cos 2x$

Solution:

Auxiliary Equation is $m^2+4=0$

$$m = \pm 2i$$

C.F = $(A \cos 2x + B \sin 2x)$

P.I = $\frac{1}{D^2+4}\ R.P\ of\ x^2 e^{i2x}$

$$= R.P\ of e^{i2x} \frac{1}{(D+2i)^2+4}\ x^2$$

$$= R.P\ of e^{i2x} \frac{1}{D^2+4Di}\ x^2$$

$$= R.P\ of e^{i2x} \frac{1}{4i}\frac{1}{D}\left[1+\ \frac{D}{4i}\right]^{-1}\ x^2$$

$$= R.P\ of e^{i2x}\left(\frac{-i}{4}\right)\frac{1}{D}\left[1-\frac{D}{4i}+\left(\frac{D}{4i}\right)^2\right]x^2$$

$$= R.P\ of e^{i2x}\left(\frac{-i}{4}\right)\frac{1}{D}\left[x^2-\frac{2x}{4i}+\frac{2}{16}\right]$$

$$= R.Pof e^{i2x}\ \left(\frac{-i}{4}\right)\left[\frac{x^3}{3}-\frac{1}{2i}\frac{x^2}{2}-\frac{1}{8}x\right]$$

$$= R.Pof\ \left(\frac{-i}{4}\right)\left[\frac{-x^3 i}{12}+\frac{1}{16}x^2+\frac{x}{32}i\right]e^{i2x}$$

$$= R.Pof\left[\left(\frac{-x^3}{12}+\frac{x}{32}\right)i+\frac{1}{16}x^2\right][\cos 2x+i\sin 2x]$$

$$= -\left[\left(\frac{-x^3}{12}+\frac{x}{32}\right)\sin 2x-\frac{1}{16}x^2\cos 2x\right]$$

The general solution is y = C.F+P.I

$$y = A\cos 2x + B\sin 2x + \frac{x^2}{16}\cos 2x + \frac{x^3}{12}\sin 2x - \frac{x}{32}\sin 2x$$

Type VII: Problems based on $\frac{1}{D-a}f(x) = e^{ax}\int e^{-ax} f(x)dx$

Example:5.45

Solve $(D^2+a^2)y = \sec ax$

Solution:

Auxiliary Equation is $m^2+a^2=0$

$$m = \pm ia$$

C.F = $(A\cos ax + B\sin ax)$

$$\text{P.I} = \frac{1}{D^2+a^2}\sec ax$$

$$= \frac{1}{(D-ia)(D+ia)}\sec ax$$

$$= \left(\frac{\frac{1}{2ia}}{D-ia} - \frac{\frac{1}{2ia}}{D+ia}\right)\sec ax$$

$$= \frac{1}{2ia}e^{iax}\int e^{-iax}\sec ax\ dx - \frac{1}{2ia}e^{-iax}\int e^{iax}\sec ax\ dx$$

$$\left[\because \frac{1}{D-m}\ x = e^{mx}\int xe^{-mx}\ dx\right]$$

$$= \frac{1}{2ia}e^{iax}\int(1 - i\tan ax)\ dx - \frac{1}{2ia}e^{-iax}\int(1 + i\tan ax)\ dx$$

$$= \frac{1}{2ia}e^{iax}\int(x - \frac{i}{a}\log\sec ax) - \frac{1}{2ia}e^{-iax}\int(x + \frac{i}{a}\log\sec ax)$$

$$= \frac{x}{a}\left(\frac{e^{iax}-e^{-iax}}{2i}\right) - \frac{1}{a^2}\log\sec ax\left(\frac{e^{iax}+e^{-iax}}{2}\right)$$

$$= \frac{x}{a}\sin ax - \frac{1}{a^2}\cos ax\log\sec ax$$

The general solution is y = C.F + P.I

$$y = A\cos ax + B\sin ax + \frac{x}{a}\sin ax - \frac{1}{a^2}\cos ax\log\sec ax$$

Example:5.46

Solve $\boldsymbol{\frac{d^2y}{dx^2} + \frac{3dy}{dx} + 2y = e^{e^x}}$

Solution:

Auxiliary Equation is $m^2 + 3m + 2 = 0$

$$m = -1, -2$$

$$\text{C.F} = (A\,e^{-x} + Be^{-2x})$$

$$\text{P.I} = \frac{1}{D^2+3D+2} e^{e^x}$$

$$= \left[\frac{1}{(D+1)(D+2)}\right] e^{e^x}$$

$$= \left[\frac{1}{D+1} - \frac{1}{D+2}\right] e^{e^x}$$

$$= \frac{1}{D+1} e^{e^x} - \frac{1}{D+2} e^{e^x}$$

$$= e^{-x} \int e^x e^{e^x} dx - e^{-2x} \int e^{2x} e^{e^x} \, dx$$

Take $z = e^x$ $\qquad dz = e^x dx$

$$= e^{-x} \int e^z \, dz - e^{-2x} \int z \, e^z \, dz$$

$$= e^{-x} e^z - e^{-2x} [z e^z - e^z]$$

$$= e^{x-} e^{e^z} - e^x e^{e^x} e^{-2x} + e^{-2x} e^{e^x}$$

$$= e^{-x} e^{e^z} - e^x e^{e^x} + e^{-2x} e^{e^x}$$

$$= e^{-2x} e^{e^x}$$

The general solution is y = C.F + P.I

$$y = A e^{-x} + B e^{-2} + e^{-2x} e^{e^x}$$