2.5 Maxima and Minima

One of the best application of differentiation calculus is the optimization problems, in which we find out solution to real value problem that requires minimizing or maximizing.

Definition:

Let c be a point in a domain D of a function f. Then f(c) is the

⇒ absolute maximum value of f on D if $f(c) \geq f(x)$ for all x in D.

⇒ absolute minimum value of f on D if $f(c) \leq f(x)$ for all x in D.

Definition:

Let c be a point in a domain D of a function f. Then f(c) is the

⇒ local maximum value of f if $f(c) \geq f(x)$ when x is near c.

⇒ local minimum value of f if $f(c) \leq f(x)$ when x is near c.

The extreme value theorem:

If f is continuous on a closed interval $[a, b]$, then f attains an absolute maximum value f(c) and an absolute minimum value $f(d)$ at some points c and d in $[a, b]$.

Fermat's theorem:

If f has local maximum value or minimum value at c and if f '(c) exists, then f '(c) = 0.

Critical Point:

A critical point of a function f is a point c in the domain of f such that either

f '(c) = 0

or f '(c) does not exists.

If f has local maximum value or minimum value at c, then c is a critical point of f.

Example:

Find the critical points of the following functions

(i) $f(x) = x^3 + x^2 - x$

(ii) $f(x) = x^{\frac{5}{4}} - 2x^{\frac{1}{4}}$

(iii) $f(\theta) = 4\theta - tan\theta$

(iv) $\boldsymbol{f(x) = 3x - sin^{-1}x}$

(v) $\boldsymbol{f(\theta) = 2cos\theta + sin^2\theta}$

Solutions:

(i) $\boldsymbol{f(x) = x^3 + x^2 - x}$

$$f'(x) = 3x^2 + 2x - 1$$

$$f'(x) = 0 \Rightarrow 3x^2 + 2x - 1 = 0$$

$$\Rightarrow (3x - 1)(x + 1) = 0$$

$$\Rightarrow x = \frac{1}{3}\ , -1$$

Critical points are $x = \frac{1}{3}\ , -1$.

(ii) $\boldsymbol{f(x) = x^{\frac{5}{4}} - 2x^{\frac{1}{4}}}$

$$f'(x) = \frac{5}{4}x^{\frac{1}{4}} - \frac{1}{4}2x^{-\frac{3}{4}}$$

$$f'(x) = 0 \Rightarrow \frac{1}{4}x^{\frac{1}{4}}(5 - 2x^{-1}) = 0$$

$$\Rightarrow \frac{1}{4}x^{\frac{1}{4}} = 0\ ,\ (5 - 2x^{-1}) = 0$$

$$\Rightarrow x = 0\ ,\ \frac{5x-2}{x} = 0$$

$\Rightarrow x = \frac{2}{5}$

Critical points are $x = 0, \frac{2}{5}$.

(iii) $\boldsymbol{f(\theta) = 4\theta - tan\theta}$

$$f'(x) = 4 - sec^2\theta = 4 - (1 + tan^2\theta)$$

$$= 3 - tan^2\theta$$

$$f'(x) = 0 \Rightarrow 3 - tan^2\theta$$

$$\Rightarrow tan^2\theta = 3$$

$$\Rightarrow tan\theta = \pm\sqrt{3}$$

$$tan\theta = \sqrt{3} \Rightarrow \theta = tan^{-1}\sqrt{3} = \frac{\pi}{3}$$

$$tan\theta = -\sqrt{3} \Rightarrow \theta = tan^{-1}\left(-\sqrt{3}\right) = -\frac{\pi}{3}$$

Critical points are $\theta = \frac{\pi}{3}, -\frac{\pi}{3}$

(iv) $\boldsymbol{f(x) = 3x - sin^{-1}x}$

$$f'(x) = 3 - \frac{1}{\sqrt{1-x^2}}$$

$$f'(x) = 0 \Rightarrow \frac{3\sqrt{1-x^2}-1}{\sqrt{1-x^2}} = 0$$

$$\Rightarrow 3\sqrt{1-x^2} = 1$$

$$\Rightarrow 9(1-x^2) = 1$$

$$\Rightarrow 9 - 9x^2 = 1$$

$$\Rightarrow x^2 = \frac{8}{9}$$

$$\Rightarrow x = \pm\frac{2\sqrt{2}}{3}$$

Critical points are $\theta = \pm\frac{2\sqrt{2}}{3}$

(v) $\boldsymbol{f(\theta) = 2cos\theta + sin^2\theta}$

$$f'(\theta) = -2sin\theta + 2sin\theta cos\theta$$

$$f'(\theta) = 0 \Rightarrow -2sin\theta(1 - cos\theta) = 0$$

$$\Rightarrow \theta = n\pi, n\ is\ an\ integer.$$

Critical points are $\theta = n\pi, n\ is\ an\ integer.$

The Closed Interval Method:

To find the absolute maximum and absolute minimum value of a continuous function

on the closed interval $[a, b]$

- ⇨ Find the derivatives of f in (a, b)
- ⇨ Find the critical points of f in (a, b)
- ⇨ Find the values of f at the critical points of f in (a, b)
- ⇨ Find the values of f at the end points of the interval $[a, b]$
- ⇨ The largest of the values is the absolute maximum value and smallest of the values is the absolute minimum value.

Example:

Find the absolute maximum and absolute minimum of

(i) $f(x) = 3x^4 - 4x^3 - 12x^2 + 1$on [-2 , 3]

(ii) $f(x) = (x^2 - 1)^3$on [-1 , 2]

(iii) $f(x) = x + \frac{1}{x}$on [0.2 , 4]

(iv) $f(x) = x - 2sinx$ on $[0 , 2\pi]$

(v) $f(x) = x - logx$ on [$\frac{1}{2}$, 2]

Solutions:

(i) $f(x) = 3x^4 - 4x^3 - 12x^2 + 1$

$f(x) = 3x^4 - 4x^3 - 12x^2 + 1$ is continuous on [-2 , 3]

$f'(x) = 12x^3 - 12x^2 - 24x$

$f'(x) = 0 \Rightarrow 12x^3 - 12x^2 - 24x = 0$

$\Rightarrow x(x+1)(x-2) = 0$

$\Rightarrow x = 0 , -1 , 2$ are the critical points.

The values of $f(x)$ at critical points are

$f(0) = 3(0^4) - 4(0^3) - 12(0^2) + 1 = 1$

$f(-1) = 3(-1)^4 - 4(-1)^3 - 12(-1)^2 + 1$

$= 3 + 4 - 12 + 1 = -4$

$f(2) = 3(2)^4 - 4(2)^3 - 12(2)^2 + 1$

$= 48 - 32 - 48 + 1 = -31$

The value of $f(x)$ at the end points of the interval are

$f(-2) = 3(-2)^4 - 4(-2)^3 - 12(-2)^2 + 1$

$= 48 + 32 - 48 + 1 = 33$

$f(3) = 3(3)^4 - 4(3)^3 - 12(3)^2 + 1$

$= 243 - 112 - 108 + 1 = 28$

Absolute minimum value is $f(2) = -31$

Absolute maximum value is $f(-2) = 33$

(ii) $f(x) = (x^2 - 1)^3$on [-1 , 2]

Solution:

$f(x) = (x^2 - 1)^3$ *is continuous* on [-1 , 2]

$f'(x) = 3(x^2 - 1)^2(2x) = 6x(x^2 - 1)^2$

$f'(x) = 0 \Rightarrow 6x(x^2 - 1)^2 = 0$

$\Rightarrow x = 0 , \pm 1$ are the critical points.

The values of f(x) at critical points are

$f(0) = (0 - 1)^3 = -1$

$f(1) = (1 - 1)^3 = 0$

$f(-1) = (1 - 1)^3 = 0$

The values of f(x) at the end points of the interval are

$f(-1) = (1 - 1)^3 = 0$

$f(2) = (4 - 1)^3 = 27$

Absolute minimum value is $f(0) = -1$

Absolute maximum value is $f(2) = 27$

(iii) $\boldsymbol{f(x) = x + \frac{1}{x}}$ **on [0.2 , 4]**

Solution:

$f(x) = x + \frac{1}{x}$ is continuous on [0.2 , 4]

$f'(x) = 1 - \frac{1}{x^2}$

$f'(x) = 0 \Rightarrow \frac{x^2 - 1}{x^2} = 0$

$\Rightarrow x^2 = 1$

$\Rightarrow x = \pm 1$ are the critical points.

The values of f(x) at critical points are

$f(1) = 1 + \frac{1}{1} = 2$

$f(-1) = -1 - \frac{1}{1} = -2$

The values of f(x) at the end points of the interval are

$f(0.2) = 0.2 + \frac{1}{0.2} = 0.2 + 5 = 5.2$

$$f(4) = 4 + \frac{1}{4} = 4 + 0.25 = 4.025$$

Absolute minimum value is $f(-1) = -2$

Absolute maximum value is $f(0.2) = 5.2$

(iv) $f(x) = x - 2sinx$ on [0 , 2π]

Solution:

$f(x) = x - 2sinx$ is continuous on $[0 , 2\pi]$

$$f'(x) = 1 - 2cosx$$

$$f'(x) = 0 \Rightarrow 1 - 2cosx = 0$$

$$\Rightarrow cosx = \frac{1}{2}$$

$$\Rightarrow x = cos^{-1}\left(\frac{1}{2}\right)$$

$\Rightarrow x = \frac{\pi}{3} , \frac{5\pi}{3}$ are the critical points.

The values of f(x) at critical points are

$$f\left(\frac{\pi}{3}\right) = \frac{\pi}{3} - 2sin\frac{\pi}{3}$$

$$= \frac{\pi}{3} - 2\frac{\sqrt{3}}{2}$$

$$= \frac{\pi}{3} - \sqrt{3} \approx 0.684853$$

$$f\left(\frac{5\pi}{3}\right) = \frac{5\pi}{3} - 2sin\frac{5\pi}{3}$$

$$= \frac{5\pi}{3} - 2\left(-\frac{\sqrt{3}}{2}\right)$$

$$= \frac{5\pi}{3} + \sqrt{3} \approx 6.968039$$

The values of f(x) at the end points of the intervals are

$$f(0) = 0 - 2sin0 = 0$$

$$f(2\pi) = 2\pi - 2\sin(2\pi) = 2\pi \;\; = \;\; 6.28$$

Absolute minimum value is $f\left(\frac{\pi}{3}\right) = -0.684$

Absolute maximum value is $f\left(\frac{5\pi}{3}\right) = 6.9680$

(v) $f(x) = x - logx$ on [$\frac{1}{2}$, 2]

Solution:

$f(x) = x - logx$ is continuous on $[\frac{1}{2}, 2]$

$f'(x) = 1 - \frac{1}{x}$

$f'(x) = 0 \Rightarrow 1 - \frac{1}{x} = 0$

$\Rightarrow \frac{x-1}{x} = 0$

$\Rightarrow x = 1$ is the critical point.

The value of f(x) at critical point is

$f(1) = 1 - log1 = 1 - 0 \quad = 1$

The values of f(x) at the end points of the intervals are

$f\left(\frac{1}{2}\right) = \frac{1}{2} - log\frac{1}{2}$

$= \frac{1}{2} - (-0.6931)$

$= 1.1931$

$f(2) = 2 - \log 2$

$= 2 - 0.6931$

$= 1.3068$

Absolute maximum value is f(2) = 1.3068

Absolute minimum value is f(1) = 1

Exercise:

1. Find the critical values for the following function

(i) $f(x) = 5x^2 + 4x$ **Ans:** $-\frac{2}{5}$

(ii) $f(x) = x^{1/3} - x^{-2/3}$ **Ans:** -2

(iii) $f(x) = x^2e^{-3x}$ **Ans:** $0, 2/3$

(iv) $f(x) = x^2 - 32\sqrt{x}$ **Ans:** 4

(v) $f(x) = x^{3/4} - 2x^{1/4}$ **Ans:** $0, 4/9$

2. Find the absolute maximum and absolute minimum values for the following functions:

1. $f(x) = 8x - x^4, [-2, 1]$ **Ans**: Ab.max. is $(1) = 7$; Ab.min. is $f(-2) = -32$

2. $f(x) = x^{2/3}, [-2, 3]$ **Ans:** Ab.max. is f(3) $= 2.08$; Ab.min. is f(0) = 0

3. $f(x) = 2cosx + \sin 2x,\ [0, \frac{\pi}{2}]$ **Ans:** Ab.max. is $f\left(\frac{\pi}{2}\right) = \frac{3\sqrt{3}}{2}$; Ab.min. is $f\left(\frac{\pi}{2}\right) = 0$

4. $f(x) = xe^{-x^2/8}, [-1, 4]$ **Ans:** Ab.max. is $f(2) = 2e^{-1/2}$;
Ab.min. is $f(-1) = -e^{-1/8}$

5. $f(x) = \log(x^2 + x + 1),\ [-1, 1]$ **Ans:** Ab.max. is $f\left(-\frac{1}{2}\right) = 0.75$;
Ab.min. is $f(-1) = 0$

Rolle's Theorem:

Let f be a function that satisfies the following three conditions:

1) f is continuous on the closed interval $[a, b]$

2) f is differentiable on the open interval (a, b)

3) $f(a) = f(b)$

Then there exists a number c in (a, b) such that $f'(c) = 0$

Example:

Verify Rolle's theorem for the following functions on the given interval

a) $\boldsymbol{f(x) = x^3 - x^2 + 6x + 2, [0, 3]}$

b) $\boldsymbol{f(x) = \sqrt{x} - \frac{1}{3}x, [0, 9]}$

c) $\boldsymbol{f(x) = sin\ x, [0, \pi]}$

Solutions:

a) $\boldsymbol{f(x) = x^3 - x^2 + 6x + 2, [0, 3]}$

$f(x)$ is continuous on $[0, 3]$

$f(x)$ is differentiable on $[0, 3]$

$f(0) = 2$

$f(3) = 27 - 9 + 18 + 2 = 38$

$f(0) \neq f(3)$

Hence the Rolle's theorem is not satisfied.

b) $\boldsymbol{f(x) = \sqrt{x} - \frac{1}{3}x\ , [0, 9]}$

Solution:

$f(x)$ is continuous on $[0, 9]$

$f(x)$ is differentiable on $[0, 9]$

$$f(0) = 0$$

$$f(9) = \sqrt{9} - \frac{9}{3} = 3 - 3 = 0$$

$$f(0) = 0 = f(9)$$

$$\Rightarrow f(x) = \sqrt{x} - \frac{x}{3}$$

$$\Rightarrow \ f'(x) = \frac{1}{2\sqrt{x}} - \frac{1}{3}$$

$$\Rightarrow f'(x) = 0 \Rightarrow \frac{1}{2\sqrt{x}} - \frac{1}{3} = 0$$

$$\Rightarrow \frac{1}{2\sqrt{x}} = \frac{1}{3}$$

$$\Rightarrow \sqrt{x} = \frac{3}{2}$$

Squaring, $x = \frac{9}{4} = 2.25 \in (0,9)$

Hence Rolle's theorem is verified.

c) $\boldsymbol{f(x) = sin\ x\ , [0, \pi]}$

Solution:

$f(x)$is continuous on $[0, \pi]$

$f(x)$is differentiable on $[0, \pi]$

$$f(0) = \sin 0 = 0$$

$$f(\pi) = sin\pi = 0$$

$$\Rightarrow f(0) = f(\pi) = 0$$

$$f'(x) = cosx$$

$$f'(x) = 0 \Rightarrow cosx = 0$$

$$x = \frac{\pi}{2} \in (0, \pi)$$

Hence Rolle's theorem is verified.

Example:

Prove that equation $x^3 - 15x + c = 0$ has atmost one real root in the interval [-2 , 2]

Solution:

$$Let\ f(x) = x^3 - 15x + c = 0$$

$$f(-2) = -8 + 30 + c = 22 + c$$

$$f(2) = 8 - 30 + c = -22 + c$$

$$f'(x) = 3x^2 - 15$$

Now if there were two points x = a, b such that $f(x) = 0$

∴By Rolle's theorem there exists a point $x = c$ in between them, where $f'(c) = 0$

$$\text{Now } f'(x) = 0 \Rightarrow 3x^2 - 15 = 0$$

$$\Rightarrow x^2 = 5$$

$$\Rightarrow x = \pm\sqrt{5} = \pm 2.236$$

Here both values lies outside [-2, 2]

∴ f has no more than one zero.

⇒ $f(x)$ has exactly one real root.

Example:

Let $f(x) = 1 - x^{2/3}$, Show that $f(-1) = f(1)$but there is no number c in (-1 , 1) such that $f'(x) = 0$. Why does this not contradict Rolle's theorem?

Solution:

Given $f(x) = 1 - x^{2/3}$

$$\Rightarrow f(-1) = 1 - (-1)^{\frac{2}{3}} = 0$$

$$\Rightarrow f(1) = 1 - 1^{2/3} = 0$$

$$\therefore f(-1) = f(1)$$

$$\Rightarrow f'(x) = -\frac{2}{3}x^{-1/3}$$

$$\Rightarrow f'(x) = 0 \Rightarrow -\frac{2}{3}x^{-1/3} = 0$$

$$\Rightarrow x^{-1/3} = 0$$

$$\Rightarrow \left(x^{-1/3}\right)^3 = 0^3$$

$$\Rightarrow x^{-1} = 0$$

$$\Rightarrow \frac{1}{x} = 0$$

$$\Rightarrow x = \infty$$

There is no number c in $(-1\,,1)$

f is not differentiable on $(-1\,,1)$

Mean Value Theorem:

Let f be a function that satisfies the following conditions:

1) f is continuous on the closed interval $[a, b]$

2) f is differentiable on the open interval (a, b)

Then there is a number C in (a, b) such that $f'(c) = \frac{f(b)-f(a)}{b-a}$

Example: Verify Lagrange's mean value theorem for the following functions:

a) $\boldsymbol{f(x) = x^3 + x - 1}$ **in** $[\mathbf{0}, \mathbf{2}]$

b) $\boldsymbol{f(x) = x + \frac{1}{x}}$, $\left[\frac{1}{2}, 2\right]$

c) $\boldsymbol{f(x) = e^{-2x}}$, $[\mathbf{0}, \mathbf{3}]$

d) $\boldsymbol{f(x) = 1 + x^{2/3}}$, $[-\mathbf{8}, \mathbf{1}]$

Solution:

a) $\boldsymbol{f(x) = x^3 + x - 1}$ **in** $[\mathbf{0}, \mathbf{2}]$

f is continuous on the closed interval [0,2]

f is differentiable on the open interval (0,2)

$$f'(x) = 3x^2 + 1$$

$$f'(c) = 3c^2 + 1,$$

Put $a = 0\,, b = 2$

$$\Rightarrow f(b) = f(2) = 2^3 + 2 - 1 = 9$$

$$\Rightarrow f(a) = f(0) = 0 + 0 - 1 = -1$$

$$f'(c) = \frac{f(b)-f(a)}{b-a}$$

$$\Rightarrow 3c^2 + 1 = \frac{9+1}{2-0}$$

$$\Rightarrow 2(3c^2 + 1) = 10$$

$$\Rightarrow 3c^2 + 1 = 5$$

$$\Rightarrow 3c^2 = 4$$

$$\Rightarrow c^2 = \frac{4}{3} \Rightarrow c = \pm\frac{2}{\sqrt{3}} = \pm 1.1547$$

$$c = 1.1547 \in (0,2)$$

Hence Mean value theorem is verified.

b) $\boldsymbol{f(x) = x + \frac{1}{x}}$ **,** $\boldsymbol{\left[\frac{1}{2}, 2\right]}$

Solution:

f is continuous on the closed interval $\left[\frac{1}{2}, 2\right]$

f is differentiable on the open interval $\left(\frac{1}{2}, 2\right)$

Put $a = \frac{1}{2}$, b = 2

$$\Rightarrow f(b) = f(2) = 2 + \frac{1}{2} = \frac{5}{2}$$

$$\Rightarrow f(a) = f\left(\frac{1}{2}\right) = \frac{1}{2} + 2 = \frac{5}{2}$$

$$f'(x) = 1 - \frac{1}{x^2} \Rightarrow f'(c) = 1 - \frac{1}{c^2}$$

$$f'(c) = \frac{f(b)-f(a)}{b-a}$$

$$\Rightarrow 1 - \frac{1}{c^2} = \frac{\frac{5}{2}-\frac{5}{2}}{2-\frac{1}{2}} = 0$$

$$\Rightarrow c^2 - 1 = 0$$

$$\Rightarrow c^2 = 1 \Rightarrow c = \pm 1$$

$$\Rightarrow c = 1 \in \left(\frac{1}{2}, 2\right)$$

Hence Lagrange's MVT is verified.

c) $\boldsymbol{f(x) = e^{-2x}}$, $\boldsymbol{[0, 3]}$

Solution:

f is continuous on the closed interval$[0,3]$

f is differentiable on the open interval $(0\ ,3)$

Put a = 0 , b = 3

$\Rightarrow f(b) = f(3) = e^{-6}$

$\Rightarrow f(a) = f(0) = 1$

$\Rightarrow f'(x) = -2e^{-2x} \Rightarrow f'(c) = -2e^{-2c}$

$f'(c) = \frac{f(b)-f(a)}{b-a} \Rightarrow -2e^{-2c} = \frac{e^{-6}-1}{3}$

$\Rightarrow -6e^{-2c} = e^{-t} - 1$

$\Rightarrow e^{-2c} = -\frac{e^{-t}}{6} + \frac{1}{6} = \frac{1}{6}[1 - e^{-6}]$

Taking log on both sides,

$\Rightarrow \log e^{-2c} = \log\left[\frac{1}{6}(1 - e^{-6})\right]$

$\Rightarrow -2c = \log\left[\frac{1}{6}(1 - e^{-6})\right]$

$\Rightarrow c = -\frac{1}{2}\log\left[\frac{1}{6}(1 - e^{-6})\right]$

$\Rightarrow 0.3896 \in (0,3)$

Hence Lagrange's MVT is verified.

d) $\boldsymbol{f(x) = 1 + x^{2/3}\ ,\ [-8,1]}$

Solution:

$f(x)$ is continuous on the closed interval$[-8,1]$

$f'(x) = \frac{2}{3}x^{-1/3}$does not exists at x = 0

$\therefore f(x)$is not differentiable in (-8, 1)

Hence Lagrange's MVT is not applicable for this function.

Example:

Suppose that $f(0) = -3$ and $f'(x) \leq 5$ for all values of x. How large can $f(2)$ possible be?

Solution:

Given f is differentiable (and therefore continuous) everywhere.

In particular, we can apply the Mean Value Theorem on the interval [0,2]

There exists a number c such that $f(2) - f(0) = f'(c)(2 - 0)$

$$f(2) = f(0) + 2f'(c) = -3 + 2f'(c)$$

Given $f'(x) \leq 5$ for all x , so $f'(c) \leq 5$

we have $2f'(c) \leq 10$

$$\therefore f(2) = -3 + 2f'(c) \leq -3 + 10 = 7 \Rightarrow f(2) \leq 7$$

The largest possible value for $f(2)$ is 7

Example:

Show that the equation $x^3 + e^x = 0$ has exactly one real root.

Solution:

Let $f(x) = x^3 + e^x$, assume $f(x)$ has two roots, that is $f(a) = f(b) = 0$

The mean value theorem states, since f is continuous and differentiable

There exists $c \in (a, b)$ such that $f'(c) = \frac{f(b)-f(a)}{b-a} = \frac{0-0}{b-a} = 0$

However, $f'(x) = 3x^2 + e^x > 0$ for all x.

which is a contradiction.

$\therefore f(x)$cannot have two roots and can have atmost root.

Since, $f(0) > 0$ and $f(-10) < 0$, by the intermediate value theorem there exists $c \in (-10,0)$ such that f(c) = 0

Thus $f(x)$ has exactly one root.

Example:

Use Lagrange's MVT to prove $\frac{x}{1+x} < \log(1 + x) < x$ for all $x > 0$

Solution:

Let $f(x) = \log(1 + x)$

$f(0) = \log 1 = 0$

$f'(x) = \frac{1}{1+x}$ (or) $f'(\theta x) = \frac{1}{1+\theta x}$, $0 < \theta < 1$

Then by MVT , for the interval $[0, x]$

we have $f(x) = f(0) + xf'(\theta x)$, $\quad 0 < \theta < 1$

(or) $log(1 + x) = \frac{x}{1+\theta x}$, $\ 0 < \theta < 1$ ... (1)

$0 < \theta x < x$, since$x > 0$

$$\Rightarrow 1 < 1 + \theta x < 1 + x \qquad (\because 1 + 0 < 1 + \theta x < 1 + x)$$

$$\Rightarrow 1 > \frac{1}{1+\theta x} > \frac{1}{1+x}$$

$$\Rightarrow \frac{1}{1+x} < \frac{1}{1+\theta x} < 1$$

$$\Rightarrow \frac{x}{1+x} < \frac{x}{1+\theta x} < x \ , \ x > 0$$

$$\Rightarrow \frac{x}{1+x} < \log(1 + x) < x \text{ by (1) } \left(\because \log(1 + x) = \frac{x}{1+\theta x}\right)$$

Exercise

1. Verify Rolle's theorem for the following functions:

(i) $f(x) = x^3 + 5x^2 - 6x$, $[0,1]$

(ii) $f(x) = (x - 1)(x - 2)(x - 3)$, $[1,3]$

(iii) $f(x) = 3 + (x - 1)^{1/3}$, $[0,2]$

2. Show that the equation $x^3 + 3x + 1$ has exactly one real solution.

3. Verify Lagrange's Mean Value theorem for the following functions:

(i) $f(x) = x^2 + 3x + 2 \ in\ 1 \leq x \leq 2$

(ii) $f(x) = \frac{1}{x}$in$-1 \leq x \leq 1$

(iii) $f(x) = \frac{x}{x+2}$in $[1,4]$

(iv) $f(x) = x^{2/3}$in $[0,1]$

4. If $f(1) = 10$ and $f'(x) \geq 2$ for $1 \leq x \leq 4$ how small can f(4) possibly be? **Ans:** f(4) = 16

5. Does there exists a function f such that $f(0) = -1$, $f(2) = 4$ and $f\ '(x) \leq 2$ for all x

Ans: Does not exist.

6. Show that $\sqrt{1 + x} < 1 + \frac{1}{2}x$if $x > 0$.

Increasing/ Decreasing Test

Definition:

(a) If $f'(x) > 0$ on an interval, then f is increasing on that interval.

(b) If $f'(x) < 0$ on an interval, then f is decreasing on that interval.

The first derivative test

Definition:

Suppose that c is a critical number of a continuous function f.

(a) If f ‘ changes from positive to negative at c , then f has a local maximum at c.

(b) If f ‘ changes from negative to positive at c , then f has a local minimum at c.

(c) If f' does not change sign at c (for example if f' is positive on both sides of c or negative on both sides), then f has no local maximum or minimum at c.

Definition:

If the graph of f lies above all of its tangents on an interval I, then it is called concave upward on I. If the graph of f lies below all of its tangents on an interval I, then it is called concave downward on I.

Note:

Concave upward ≡ convex downward

Concave downward ≡ convex upward

Concavity Test

Definition:

(a) If $f''(x) > 0$ for all x in I, then the graph of f is concave upward on I.

(b) If $f''(x) < 0$ for all x in I, then the graph of f is concave downward on I.

Definition:

A point P on a curve $y = f(x)$is called an inflection point iff is continuous there and the curve changes from concave upward to concave downward or from concave downward to concave upward at P.

The Second Derivative Test

Definition:

Suppose f'' is continuous near c,

(a) If $f'(c) = 0$ and $f''(c) > 0$, then f has a local minimum at c.

(b) If $f'(c) = 0$ and $f''(c) < 0$, then f has a local maximum at c.

Example:

Find where the function $f(x) = 3x^4 - 4x^3 - 12x^2 + 5$ is increasing and where it is decreasing.

Solution:

Given $f(x) = 3x^4 - 4x^3 - 12x^2 + 5$

$$f'(x) = 12x^3 - 12x^2 - 24x$$

$$= 12x(x^2 - x - 2)$$

$$= 12x(x - 2)(x + 1)$$

$$f'(x) = 0 \Rightarrow 12x(x - 2)(x + 1) = 0$$

$$\Rightarrow x(x - 2)(x + 1) = 0$$

$\Rightarrow x = 0, 2, -1$are the critical values.

We divide the real line into intervals whose end points are the critical points. $x = 0, 2, -1$ and list them in a table

Interval	$12x$	$x - 2$	$x + 1$	$f'(x)$	$f(x)$
$x < -1$	-	-	-	-	decreasing
$-1 < x < 0$	-	-	+	+	increasing
$0 < x < 2$	+	-	+	-	decreasing
$x > 2$	+	+	+	+	increasing

∴The function is increasing in $-1 < x < 0$ and $x > 2$ and it is decreasing in $x < -1$ and $0 < x < 2$

Example:

Find the local maximum and minimum values of $y = x^5 - 5x + 3$ using both the first and second derivative tests.

Solution:

Given $y = f(x) = x^5 - 5x + 3$

$f'(x) = 5x^4 - 5$

$f'(x) = 0 \Rightarrow 5x^4 - 5 = 0$

$\Rightarrow x^4 - 1 = 0 \Rightarrow x^4 = 1 \Rightarrow x^2 = \pm 1$

$\Rightarrow x = 1, -1$are the critical points.

Interval	Sign of f'	Behaviour of f
$-\infty < x < 1$	+	increasing
$-1 < x < 1$	-	decreasing
$1 < x < \infty$	+	increasing

First derivative test tells us that

(i) Local maximum at $x = -1$

$f(-1) = -1 + 5 + 3 = 7$

Second derivative test tells us that

(ii) Local minimum at $x = 1$

$f(1) = 1 - 5 + 3 = -1$

$f''(x) = 20x^3$

$f''(x) = 0 \Rightarrow 20x^3 = 0 \Rightarrow x = 0$

Interval	$f''(x)$	Behaviour of f
$(-\infty,0)$	-	Concave down
$(0,\infty)$	+	Concave up

$f'(1) = 0\,, f''(1) = 20\,, f(1) = -1$ is a local minimum

$f'(-1) = 0\,, f''(-1) = -20\,, f(-1) = 7$ is a local maximum

Example:

If $f(x) = 2x^3 + 3x^2 - 36x$ find the intervals on which is increasing or decreasing, the local maximum and local minimum values of f , the intervals of concavity and the inflection points.

Solution:

Given $f(x) = 2x^3 + 3x^2 - 36x$

$$f'(x) = 6x^2 + 6x - 36$$

$$f'(x) = 0 \Rightarrow 6(x^2 + x - 6) = 0$$

$$\Rightarrow 6(x+3)(x-2) = 0$$

$\Rightarrow x = -3$, 2are the critical points.

$$f''(x) = 12x + 6$$

We divide the real line into intervals whose end points are the critical points $x = 2$, -3 and list them in a table.

Interval	$6(x+3)$	$x-2$	$f'(x)$	$f(x)$
$x < -3$	$-$	$-$	$+$	increasing
$-3 < x < 2$	$+$	$-$	$-$	decreasing
$x > 2$	$+$	$+$	$+$	increasing

Now we apply the first derivative test to find the local extremum values.

$f(x)$changes from increasing to decreasing at $x = -3$. Thus the function has a local maximum $x = -3$ and local maximum value is $f(-3) = 2(-3)^3 + 3(-3)^2 - 36(-3)$

$$= 2(-27) + 3(9) + 108$$

$$= -54 + 27 + 108 \quad = 81$$

$f(x)$ changes from decreasing to increasing at $x = 2$. Thus the function has a local minimum $x = 2$ and local minimum value is $f(2) = 2(2)^3 + 3(2)^2 - 36(2)$

$$= 2(8) + 3(4) - 72$$

$$= 16 + 12 - 7 \ = -44$$

For concavity test ,$f''(x) = 0$

$$\Rightarrow 12x + 6 = 0$$

$$\Rightarrow x = -\frac{1}{2}$$

We divide the real line into intervals whose end points are the critical points $x = -\frac{1}{2}$ and list them in a table.

Interval	$f''(x)$	concavity
$x < -1/2$	$-$	downward
$x > -1/2$	$+$	upward

Since the curve changes from concave downward to concave upward at $x = -\frac{1}{2}$

The point of inflection is $\left[-\frac{1}{2}, f\left(-\frac{1}{2}\right)\right]$

$$f\left(-\frac{1}{2}\right) = 2\left(-\frac{1}{2}\right)^3 + 3\left(-\frac{1}{2}\right)^2 - 36\left(-\frac{1}{2}\right)$$

$$= 2\left(-\frac{1}{8}\right) + 3\left(\frac{1}{4}\right) + 18$$

$$= -\frac{1}{4} + \frac{3}{4} + 18$$

$$= \frac{-1+3+72}{4}$$

$$= \frac{74}{4} = \frac{37}{2}$$

Hence the point of inflection are $\left(-\frac{1}{2}, \frac{37}{2}\right)$

Example:

Find the interval of concavity and the inflection points. Also find the extreme values on what interval is f increasing or decreasing.

a) $(x) = \boldsymbol{sinx + cosx}$ **,** $\boldsymbol{0 \le x \le 2\pi}$

b) $\boldsymbol{f(x) = e^{2x} + e^{-x}}$

c) $\boldsymbol{f(x) = x + 2sinx}$ **,** $\boldsymbol{0 \le x \le 2\pi}$

Solution:

a) $(x) = \boldsymbol{sinx + cosx}$ **,** $\boldsymbol{0 \le x \le 2\pi}$

$f'(x) = cosx - sinx$

$f'(x) = 0 \Rightarrow cosx = sinx$

$\Rightarrow x = \frac{\pi}{4}, \frac{5\pi}{4}$ are the critical points.

Interval	Sign of f '	Behaviour of f
$0 < x < \frac{\pi}{4}$	+	increasing
$\frac{\pi}{4} < x < \frac{5\pi}{4}$	+	increasing
$\frac{5\pi}{4} < x < 2\pi$	-	decreasing

(i) Maximum at $\frac{\pi}{4}$, $f'\left(\frac{\pi}{4}\right) = sin\frac{\pi}{4} + cos\frac{\pi}{4}$

$$= \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} = \frac{2}{\sqrt{2}} = \sqrt{2}$$

(ii) Minimum at $\frac{5\pi}{4}$, $f'\left(\frac{5\pi}{4}\right) = sin\frac{5\pi}{4} + cos\frac{5\pi}{4}$

$$= -\sqrt{2}$$

$$f''(x) = -sinx - cosx = -(sinx + cosx)$$

$f''(x) = 0 \Rightarrow -(sinx + cosx) = 0$

$\Rightarrow sinx = -cosx$

$\Rightarrow x = \frac{3\pi}{4}, \frac{7\pi}{4}$

Interval	Sign of f''	Behaviour of f
$0 < x < \frac{3\pi}{4}$	$-$	Concave down
$\frac{3\pi}{4} < x < \frac{7\pi}{4}$	$+$	Concave up

$\frac{3\pi}{4} < x < 2\pi$	$-$	Concave down

Inflection points are $\left(\frac{3\pi}{4}, 0\right), \left(\frac{7\pi}{4}, 0\right)$

Since $f\left(\frac{3\pi}{4}\right) = 0, f\left(\frac{7\pi}{4}\right) = 0$

b) $f(x) = e^{2x} + e^{-x}$

$$f'(x) = 2e^{2x} - e^{-x}$$

$$f'(x) = 0 \Rightarrow 2e^{2x} - e^{-x} = 0$$

$$\Rightarrow 2e^{2x} = e^{-x}$$

$$\Rightarrow e^{3x} = \frac{1}{2}$$

$$\Rightarrow 3x = \log\left(\frac{1}{2}\right)$$

$$\Rightarrow x = \frac{1}{3}[\log 1 - log2]$$

$$\Rightarrow x = \frac{1}{3}[0 - 0.693]$$

$\Rightarrow -0.23$are the critical points.

Interval	Sign of f ‘	Behaviour of f
$-\infty < x < -0.23$	$-$	decreasing
$-0.23 < x < \infty$	$+$	increasing

The first derivative test tells us that there is a local minimum at $x = -0.23$

$$f(-0.23) = f\left(-\frac{1}{3}log2\right) = f\left(log2^{-\frac{1}{3}}\right)$$

$$= e^{2log2^{-1/3}} + e^{-log2^{-1/3}}$$

$$= e^{log\left(2^{-1/3}\right)^2} + e^{log\left(2^{-1/3}\right)^{-1}}$$

$$= \left(2^{-1/3}\right)^2 + \left(2^{-1/3}\right)^{-1}$$

$$= (2)^{-2/3} + (2)^{1/3}$$

$$f''(x) = 4e^{2x} + e^{-x}$$

$$f''(x) = 0 \Rightarrow 4e^{2x} + e^{-x} = 0$$

$$\Rightarrow 4e^{2x} = -e^{-x}$$

$$\Rightarrow e^{3x} = -\frac{1}{4}$$

$$\Rightarrow 3x = log\left(-\frac{1}{4}\right)$$

$$\Rightarrow x = \frac{1}{3} log\left(-\frac{1}{4}\right)$$

$$\Rightarrow x = \frac{1}{3}(-log\,4\,)$$

$$= -\frac{1}{3}(log\,4\,) = -0.46$$

Interval	Sign of f''	Behaviour of f
$-\infty < x < -0.46$	+	Concave up
$-0.46 < x < \infty$	+	Concave up

No inflection points.

c) $f(x) = x + 2sinx\,,\ \ 0 \le x \le 2\pi$

$$f'(x) = 1 + 2cosx$$

$$f'(x) = 0 \Rightarrow 2cosx = -1$$

$$\Rightarrow cosx = -\frac{1}{2}$$

$\Rightarrow x = \frac{2\pi}{3}, \frac{4\pi}{3}$ are the critical points.

Interval	Sign of f'	Behaviour of f
$0 < x < \frac{2\pi}{3}$	+	increasing
$\frac{2\pi}{3} < x < \frac{4\pi}{3}$	−	decreasing
$\frac{4\pi}{3} < x < 2\pi$	+	increasing

The first derivatives test tells us that there is a

(i) Local maximum at $\frac{2\pi}{3}$

$$f\left(\frac{2\pi}{3}\right) = \frac{2\pi}{3} + 2\sin\left(\frac{2\pi}{3}\right) = 3.83$$

(ii)Local minimum at $\frac{4\pi}{3}$

$$f\left(\frac{4\pi}{3}\right) = \frac{4\pi}{3} + 2\sin\left(\frac{4\pi}{3}\right) = 2.46$$

$$f''(x) = -2sinx$$

$$f''(x) = 0 \Rightarrow -2sinx = 0$$

$$\Rightarrow sinx = 0 \Rightarrow x = 0, \pi, 2\pi$$

Interval	Sign of f''	Behaviour of f
$0 < x < \pi$	+	Concave up
$\pi < x < 2\pi$	−	Concave down

Inflection points are (π, π)

Example:

Find a cubic function $f(x) = ax^3 + bx^2 + cx + d$ that has a local maximum value of 3 at $x = -2$ and a local minimum value of 0 at $x = 1$

Solution:

Given $f(x) = ax^3 + bx^2 + cx + d$

$$f'(x) = 3ax^2 + 2bx + c$$

$$f'^{(x)} = 0 \Rightarrow 3ax^2 + 2bx + c = 0$$

Given the critical points are $x = -2$, $x = 1$

$$\Rightarrow 3ax^2 + 2bx + c = (x+2)(x-1)$$

$$\Rightarrow 3ax^2 + 2bx + c = x^2 + x - 2$$

Equating the like terms we get

$3a = 1,\qquad 2b = 1,\qquad c = -2$

$a = \frac{1}{3}\qquad b = \frac{1}{2}$

Given $f(-2) = 3$ and $f(1) = 0$

$$f(1) = 0 \Rightarrow a + b + c + d = 0$$

$$\Rightarrow \frac{1}{3} + \frac{1}{2} - 2 + d = 0$$

$$\Rightarrow d = 2 - \frac{1}{3} - \frac{1}{2} = \frac{7}{6}$$

$$\therefore f(x) = \frac{1}{3}x^3 + \frac{1}{2}x^2 - 2x + \frac{7}{6}$$

Exercise:

1. Find the interval on which f is increasing or decreasing:

(i) $f(x) = x^4 - 2x^2 + 3$

Ans: Decreasing on $(-\infty, -1) \cup (0,1)$and increasing on$(-1,0)$and $(1, \infty)$

(ii) $f(x) = \sqrt{3}x - 2cosx,\ \ 0 \le x \le 2\pi$

Ans: Increasing on $\left(0, \frac{4\pi}{3}\right)$and decreasing on$\left(\frac{4\pi}{3}, 2\pi\right)$

c) $f(x) = \frac{8x}{x^2+4}$ **Ans:** Decreasing on $(-\infty, -2) \cup (2, \infty)$and increasing on$(-2,2)$

2. Find the local maximum and minimum values of f

(i) $f(x) = 4x^3 + 3x^2 - 6x + 1$

Ans: Local minimum is$f\left(\frac{1}{2}\right) = -\frac{3}{4}$ Local maximum is $f(-1) = 6$

(ii) $f(x) = \frac{x^2}{x^2+3}$ **Ans:** Local minimum is $f(0) = 0$

(iii) $f(x) = x - sinx\ ,\ \ 0 \le x \le 2\pi$ **Ans:** neither maximum nor minimum

3. Find the interval of concavity and the inflection points:

(i) $f(x) = x^4 - 8x^2 + 16$

Ans: concave up on $\left(-\infty, -\frac{2}{\sqrt{3}}\right) \cup \left(\frac{2}{\sqrt{3}}, \infty\right)$and concave down on$\left(-\frac{2}{\sqrt{3}}, \frac{2}{\sqrt{3}}\right)$

(ii) $f(x) = e^{1/x}$ **Ans:** concave down on$\left(-\infty, -\frac{1}{2}\right)$ and concave up on $\left(-\frac{1}{2}, 0\right) \cup (0, \infty)$

(iii) $f(x) = x + 2sinx\ ,\ \ 0 \le x \le 2\pi$

Ans: concave up on $(0, \pi)$and concave down on$(\pi, 2\pi)$

4. Suppose f'' is continuous on$(-\infty, \infty)$

(i) If $f'(2) = 0$ and$f''(2) = -5$, what can you say about ?

(ii) If $f'(6) = 0$ and$f''(6) = 0$, what can you say about ?

Ans: f has a local maximum at 2 and f has a horizontal tangent at 6.

5. Show that the curve $y = e^{-x}$ and $y = -e^{-x}$ touch the curve $y = e^{-x}\, sinx$ at its inflection points.

Indeterminate Form and L' Hospital Rule

The indeterminate forms are $\frac{0}{0}$, $0 \times \infty$, $\frac{\infty}{\infty}$, $\infty - \infty$, 0^0 , ∞^0 , 1^∞.

Type: I (For Indeterminate form of $\frac{0}{0}$)

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f'(x)}{g'(x)}$$

Example:

Evaluate the following

1. $\lim_{x\to -1}\frac{x^2-1}{x+1}$ 2. $\lim_{x\to 0}\frac{\sqrt{1+2x}-\sqrt{1-4x}}{x}$

3. $\lim_{x\to 1}\frac{x-(n+1)x^{n+1}+nx^{n+2}}{(1-x)^2}$ 4. $\lim_{x\to 1}\frac{1-x}{\log x}$

5. $\lim_{x\to a}\frac{a^x-x^a}{x^x-b^b}$ 6. $\lim_{x\to 0}\frac{e^x-1-x}{x^2}$

7. $\lim_{x\to \pi/2}\left(\frac{1+cos2x}{(\pi-2x)^2}\right)$ 8. $\lim_{x\to 0}\frac{\tan hx}{tanx}$

9. $\lim_{x\to 0}\frac{sin^{-1}x}{x}$ 10. $\lim_{x\to 0}\frac{e^{2x}-1}{sinx}$

Solution:

1. $\lim_{x\to -1}\frac{x^2-1}{x+1} = \frac{(-1)^2-1}{-1+1} = \frac{0}{0}$

Applying L'Hospital's Rule:

$$\lim_{x\to -1}\frac{x^2-1}{x+1} = \lim_{x\to -1}\frac{2x}{1} = -2$$

2. $\lim_{x\to 0}\frac{\sqrt{1+2x}-\sqrt{1-4x}}{x} = \frac{1-1}{0} = \frac{0}{0}$

Applying L'Hospital's Rule:

$$\lim_{x\to 0}\frac{\sqrt{1+2x}-\sqrt{1-4x}}{x}=\lim_{x\to 0}\frac{\frac{2}{2\sqrt{1+2x}}-\frac{(-4)}{2\sqrt{1-4x}}}{1}$$

$$=1+2=3$$

3. $\lim_{x\to 1}\frac{x-(n+1)x^{n+1}+nx^{n+2}}{(1-x)^2}=\frac{1-(n+1)+n}{0}=\frac{0}{0}$

Applying L'Hospital's Rule:

$$=\lim_{x\to 1}\frac{1-(n+1)^2x^2+n(n+2)x^{n+1}}{2(1-x)(-1)}=\frac{1-(n+1)^2+n(n+2)}{0}$$

$$=\frac{0}{0}$$

$$=\lim_{x\to 1}\frac{-n(n+1)^2x^{n-1}+n(n+1)(n+2)x^n}{2}$$

$$=\frac{-n(n+1)^2+n(n+1)(n+2)}{2}$$

$$=\frac{n(n+1)(-n-1+n+2)}{2}$$

$$=\frac{n(n+1)}{2}$$

4. $\lim_{x\to 1}\frac{1-x}{\log x}=\frac{0}{0}$

Applying L'Hospital's Rule:

$$\lim_{x\to 1}\frac{1-x}{\log x}=\frac{-1}{1/x}=-1$$

5. $\lim_{x\to a}\frac{a^x-x^a}{x^x-b^b}=\frac{0}{0}$

Let $u=a^x$ Let $u=x^x$

$log\,u = x\,log\,a$ $log\,u = x\,log\,x$

$$\frac{1}{u}\frac{du}{dx}=\log a\;\frac{1}{u}\frac{du}{dx}=x\frac{1}{x}+\log x$$

$$\frac{du}{dx}=u\log a\;\frac{du}{dx}=u[1+\log x]$$

$$=a^x loga = x^x(1+logx)$$

Applying L'Hospital's Rule

$$\lim_{x\to a}\frac{a^x-x^a}{x^x-b^b}=\lim_{x\to a}\frac{a^x logx-ax^{a-1}}{x^x(1+logx)}=\frac{a^a loga-aa^a a^{-1}}{a^a(1+loga)}$$

$$= \frac{a^a(loga-1)}{a^a(1+loga)}$$

$$= \frac{loga-loge}{loge+loga}$$

$$= \frac{\log\left(\frac{a}{e}\right)}{\log(ae)} (\because \log e = 1$$

6. $\lim_{x\to0} \frac{e^x-1-x}{x^2} = \frac{e^0-1-0}{0} = \frac{0}{0}$

Applying L'Hospital's Rule

$$\lim_{x\to0} \frac{e^x-1-x}{x^2} = \lim_{x\to0} \frac{e^x-1}{2x} = \frac{0}{0}$$

$$= \lim_{x\to0} \frac{e^x}{2}$$

$$= \frac{1}{2}$$

7. $\lim_{x\to\pi/2} \frac{1+\cos 2x}{(\pi-2x)^2} = \frac{1+cos\pi}{(\pi-\pi)^2} = \frac{0}{0} [\because cos\pi = -1]$

Applying L'Hospital's Rule

$$\lim_{x\to\pi/2} \frac{1+\cos 2x}{(\pi-2x)^2} = \lim_{x\to\pi/2} \frac{-2sin2x}{2(\pi-2x)(-2)}$$

$$= \frac{-2sin2\pi/2}{2(\pi-2\pi/2)(-2)} = \frac{0}{0}$$

Again Applying L'Hospital's Rule

$$= \lim_{x\to\frac{\pi}{2}} \frac{2cos2x}{2(-2)} = \frac{-2}{-4} = \frac{1}{2}$$

8. $\lim_{x\to0} \frac{\tan hx}{tanx} = \frac{0}{0}$

Applying L'Hospital's Rule

$$\lim_{x\to0} \frac{\tan hx}{tanx} = \lim_{x\to0} \frac{sech^2x}{sec^2x} = \frac{1}{1} = 1$$

9. $\lim_{x\to0} \frac{sin^{-1}x}{x} = \frac{0}{0}$

Applying L'Hospital's Rule

$$\lim_{x\to0} \frac{sin^{-1}x}{x} = \lim_{x\to0} \frac{\frac{1}{\sqrt{1-x^2}}}{1} = 1$$

10. $\lim_{x\to 0} \frac{e^{2x}-1}{sinx} = \frac{1-1}{0} = \frac{0}{0}$

Applying L'Hospital's Rule

$$\lim_{x\to 0} \frac{e^{2x}-1}{sinx} = \lim_{x\to 0} \frac{2e^{2x}}{cosx} = \frac{2}{1} = 2$$

Type: II (For Indeterminate form of $\frac{\infty}{\infty}$)

Example:

Evaluate the following

1. $\lim_{x\to\infty} \frac{e^x}{x^2}$

2. $\lim_{x\to 0} \frac{\log\ sin2x}{\log\ \ sinx}$

3. $\lim_{x\to 0} \frac{\log x}{cosecx}$

4. $\lim_{x\to 0} xlogx$

5. $\lim_{x\to 0} \left(\frac{1}{x} - \frac{1}{e^x-1}\right)$

Solutions:

1. $\lim_{x\to\infty} \frac{e^x}{x^2} = \frac{e^\infty}{\infty^2} = \frac{\infty}{\infty}$

Applying L'Hospital's Rule

$$\lim_{x\to\infty} \frac{e^x}{x^2} = \frac{e^x}{2x} = \frac{\infty}{\infty}$$

$$= \lim_{x\to\infty} \frac{e^x}{2} = \frac{\infty}{2} = \infty$$

2. $\lim_{x\to 0} \frac{\log\ sin2x}{\log sinx} = \frac{\infty}{\infty}$

Applying L'Hospital's Rule

$$\lim_{x\to 0} \frac{\log\ sin2x}{\log sinx} = \lim_{x\to 0} \frac{\frac{1}{\sin 2x}2cos2x}{\frac{1}{sinx}cosx}$$

$$= \lim_{x\to 0} \frac{2cot2x}{cotx} = \frac{\infty}{\infty}$$

$$= \lim_{x\to 0} \frac{2tanx}{tan2x} = \frac{0}{0}$$

Again Applying L'Hospital's Rule

$$= \lim_{x\to 0} \frac{2sec^2x}{2sec^2 2x} = \frac{2}{2} = 1$$

3. $\lim_{x\to 0} \frac{\log x}{cosecx}$

Applying L'Hospital's Rule

$$\lim_{x\to 0} \frac{\log x}{cosecx} = \lim_{x\to 0} \frac{1/x}{-cosecxcotx} = \frac{\infty}{\infty}$$

$$= \lim_{x\to 0} \frac{-sinx\ sinx}{xcosx} = \lim_{x\to 0} \frac{-sin^2x}{xcosx} = \frac{0}{0}$$

Again Applying L'Hospital's Rule

$$= \lim_{x\to 0} \frac{2sinxcosx}{-sinx+cosx} = \frac{0}{1} = 0$$

Type: III (Indeterminate form are $\mathbf{0 \times \infty\ ,\ \infty - \infty\ , \infty^0}$ *and* $\mathbf{1^\infty}$)

Example:

Evaluate

1. $\lim_{x\to 0} xlogx$

2. $\lim_{x\to 0} \left(\frac{1}{x} - \frac{1}{e^x-1}\right)$

Solution:

$$\lim_{x\to 0} xlogx\ = o \times \infty$$

$$= \lim_{x\to 0} \frac{logx}{1/x} = \frac{\infty}{\infty}$$

Applying L'Hospital's Rule

$$\lim_{x\to 0} xlogx\ = \lim_{x\to 0} \frac{1/x}{(-1/x^2)} = \lim_{x\to 0}(-x)\ = 0$$

2. $\lim_{x\to 0} \left(\frac{1}{x} - \frac{1}{e^x-1}\right) = \infty - \infty$

Consider $\frac{1}{x} - \frac{1}{e^x-1} = \frac{e^x-1-x}{x(e^x-1)} = \frac{0}{0}$

Applying L'Hospital's Rule

$$\lim_{x\to 0} \left(\frac{1}{x} - \frac{1}{e^x-1}\right) = \lim_{x\to 0} \frac{e^x-1}{xe^x+(e^x-1)} = \frac{0}{0}$$

Again Applying L'Hospital's Rule

$$= \lim_{x\to 0} \frac{e^x}{xe^x+e^x+e^x} = \frac{1}{0+1+1}$$

$$= \frac{1}{2}$$

Exercise:

1. **Evaluate**

(i) $\lim_{x\to 1} \frac{x^3-2x^2+1}{x^3-1}$ **Ans:** $-\frac{1}{3}$

(ii) $\lim_{t\to 1} \frac{t^8-1}{t^5-1}$ **Ans:** $\frac{8}{5}$

(iii) $\lim_{x\to 0} \frac{\sqrt{1+x^2}-\sqrt{1-x^2}}{x^2}$ **Ans:** 1

(iv) $\lim_{x\to 1} \frac{\log x}{x-1}$ **Ans :** 1

(v) $\lim_{x\to 0} \frac{xe^x-\log(1+x)}{x^2}$ **Ans:** 3/2

(vi) $\lim_{x\to 0} \frac{\tan x - x}{x^3}$ **Ans:** 1/3

(vii) $\lim_{x\to 0} \frac{\sin\log(1+x)}{\log(1+sinx)}$ **Ans:** 1

(viii) $\lim_{x\to 0} \frac{e^{3x}+e^{-3x}-2}{5x^2}$ **Ans:** 9/5

(ix) $\lim_{x\to 0} \frac{e^x+sinx-1}{\log(1+x)}$ **Ans**: 2

(x) $\lim_{x\to 0} \frac{cosecx-cotx}{x}$ **Ans:** ½

2. **Evaluate**

(i) $\lim_{x\to\infty} \frac{\log x}{\sqrt{x}}$ **Ans:** 0

(ii) $\lim_{x\to\infty} \frac{\log x}{\sqrt[3]{x}}$ **Ans:** 0

(iii) $\lim_{x\to\infty} \frac{\log x}{\cot x}$ **Ans:** 0

3. **Evaluate**

(i) $\lim_{x\to\infty} x^3 e^{-x^2}$ **Ans:** 0

(ii) $\lim_{x\to\infty} x^{1/x}$ **Ans:** 1

(iii) $\lim_{x\to 0^+} x^x$ **Ans:** -1