DIFFERENTIAL MOTIONS AND CHANGE

The differential operator Δ represents a differential operator relative to the fixed reference frame, and it is technically UΔ. However, it is possible to define another differential operator, this time relative to the current frame itself (TΔ), that will enable us to calculate the same changes in the frame. Since the differential operator relative to the frame is relative to a current frame, to find the changes in the frame we must post-multiply the frame by TΔ. The result will be the same, since both operations represent the same changes in the frame. Then:

$$[dT] = [\Delta][T] = [T][{}^T\Delta]$$

$$[T]^{-1} \times [\Delta][T] = [T]^{-1}[T][{}^T\Delta]$$

$$[{}^T\Delta] = [T]^{-1}[\Delta][T]$$

Therefore, above can be used to calculate the differential operator relative to the frame TΔ. We can multiply the matrices in above equation and simplify the result as follows. Assuming that the frame T is represented by an **n**, **o**, **a**, **p** matrix, we get:

$$T^{-1} = \begin{bmatrix} n_x & n_y & n_z & -\mathbf{p}\cdot\mathbf{n} \\ o_x & o_y & o_z & -\mathbf{p}\cdot\mathbf{o} \\ a_x & a_y & a_z & -\mathbf{p}\cdot\mathbf{a} \\ 0 & 0 & 0 & 1 \end{bmatrix} \text{ and } \Delta = \begin{bmatrix} 0 & -\delta z & \delta y & dx \\ \delta z & 0 & -\delta x & dy \\ -\delta y & \delta x & 0 & dz \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$[T^{-1}][\Delta][T] = {}^T\Delta = \begin{bmatrix} 0 & -{}^T\delta z & {}^T\delta y & {}^Tdx \\ {}^T\delta z & 0 & -{}^T\delta x & {}^Tdy \\ -{}^T\delta y & {}^T\delta x & 0 & {}^Tdz \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

As you see, TΔ is made to look exactly like the Δ matrix, but all elements are relative to the current frame, where these elements are found from the previous multiplication of matrices and are summarized as follows:

$$\begin{aligned} {}^T\delta_x &= \boldsymbol{\delta}\cdot\mathbf{n} \\ {}^T\delta_y &= \boldsymbol{\delta}\cdot\mathbf{o} \\ {}^T\delta_z &= \boldsymbol{\delta}\cdot\mathbf{a} \\ {}^Td_x &= \mathbf{n}\cdot[\boldsymbol{\delta}\times\mathbf{p}+\mathbf{d}] \\ {}^Td_y &= \mathbf{o}\cdot[\boldsymbol{\delta}\times\mathbf{p}+\mathbf{d}] \\ {}^Td_z &= \mathbf{a}\cdot[\boldsymbol{\delta}\times\mathbf{p}+\mathbf{d}] \end{aligned}$$