3.2 Centroids by Integration

In this section we will use the integral form to find the centroids of non-homogenous objects or shapes with curved boundaries.

$$\bar{x} = \frac{\int \bar{x}_{\text{el}}\, dA}{\int dA} \qquad \bar{y} = \frac{\int \bar{y}_{\text{el}}\, dA}{\int dA} \qquad \bar{z} = \frac{\int \bar{z}_{\text{el}}\, dA}{\int dA}$$

With the integral equations we are mathematically breaking up a shape into an infinite number of infinitesimally small pieces and adding them together by integrating. This powerful method is conceptually identical to the discrete sums we introduced first.

Integration Process

Determining the centroid of a area using integration involves finding weighted average values $\bar{x}$and $\bar{y}$, by evaluating these three integrals,

$$A = \int dA, \qquad Q_x = \int \bar{y}_{\text{el}}\, dA \qquad Q_y = \int \bar{x}_{\text{el}}\, dA,$$

where

- d**A** is a differential bit of area called the *element*.
- **A** is the *total area* enclosed by the shape, and is found by evaluating the first integral.
- $\bar{x}$el and $\bar{y}$e1are the coordinates of the *centroid of the element*. These are frequently functions of *x* or, y, not constant values.
- $Q_x$and Q_y are the *First moments of Area* with respect to the *x* and *y* axis.

General procedure to find centroid by integration

1) Choose appropriate co-ordinate system depending upon symmetry of the problem. e.g. for circular arcs, the polar co-ordinate system will be more convenient to use than the rectangular cartesian co-ordinate system.

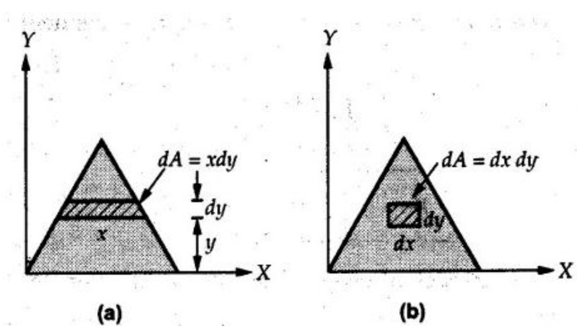


Figure 3.2.1

2) Choose a suitable differential element. For areas choose differential element of area dA. Choose first order differential element so that only one integration will be required to cover the complete area. Second order differential elements will require double integration. e.g., for a triangle, it will be more convenient to take a differential element shown in Fig. 3.2.1 (a) than the element shown in Fig. 3.2.1(b).

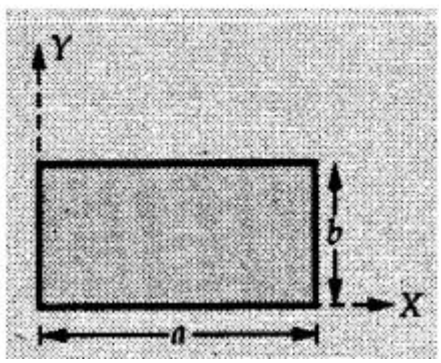
3) Determine the co-ordinates of the centroid of the differential element.

4) Evaluate the integrals in equations to find co-ordinates of centroid.

Solved Examples for Understanding

Example 3.2.1

Determine the x and y co-ordinates of centroid of the rectangle shown in Fig. 5.2.4 from first principles.



Solution :

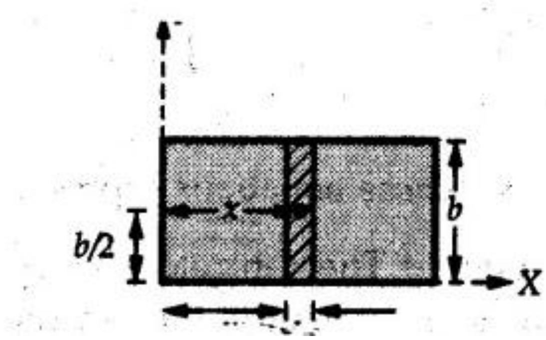
Consider the differential element to be a vertical strip of width *dx* as shown in Fig. The area of rectangle is

A = *a b*

The area of differential element is

dA = b dx

$$\therefore \quad A = \int_0^a b\,dx = b\,[x]_0^a$$



 $A = ab$

The distance of centroid of differential element from Y-axis is *x*.

$$\therefore \quad \overline{X} = \frac{\int x\,dA}{A} = \frac{\int_0^a (x)(b\,dx)}{ab} = \frac{b\left[\frac{x^2}{2}\right]_0^a}{ab} = \frac{1}{a}\cdot\frac{a^2}{2}$$

$$\therefore \quad \boxed{\overline{X} = \frac{a}{2}}$$

The distance of centroid of differential element from X-axis is b/2.

$$\therefore \quad \overline{Y} = \frac{\int y\,dA}{A} = \frac{\int_0^a \left(\frac{b}{2}\right)(b\,dx)}{ab} = \frac{\frac{b^2}{2}[x]_0^a}{ab} = \frac{\frac{b^2}{2}\cdot a}{ab}$$

$$\therefore \quad \boxed{\overline{Y} = \frac{b}{2}}$$

Example 3.2.2

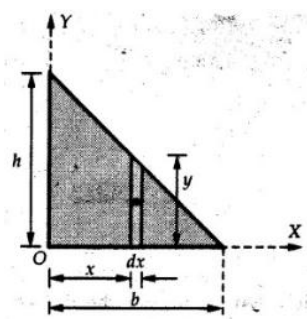
Find the centroid of the area enclosed by a right angled triangle from first principles.

Solution:

Consider a right-angle triangle of base b and height h as shown in Fig.

Consider an elementary strip of width *dx* and height *y* parallel to Y-axis as shown. As *dx* is very small, it is approximately a rectangular area. Its area is

dA = y dx



By similarity of triangles,

$$\frac{y}{b-x} = \frac{h}{b}$$

$$y = \frac{h}{b}(b-x)$$

$$y = h-\frac{h}{b}x$$

$$dA = \left(h-\frac{h}{b}x\right)dx$$

$$A = \int dA = \int_0^b \left(h-\frac{h}{b}x\right)dx = \left[hx-\frac{h}{b}\cdot\frac{x^2}{2}\right]_0^b = hb-\frac{h}{b}\cdot\frac{b^2}{2} = bh-\frac{bh}{2}$$

$$\therefore \quad A = \frac{bh}{2}$$

$$\overline{X} = \frac{\int x\,dA}{A} = \frac{\int_0^b x\left(h-\frac{h}{b}x\right)dx}{\left(\frac{bh}{2}\right)} = \frac{2}{b}\int_0^b \left(x-\frac{x^2}{b}\right)dx = \frac{2}{b}\left[\frac{x^2}{2}-\frac{x^3}{3b}\right]_0^b$$

$$= \frac{2}{b}\left[\frac{b^2}{2}-\frac{b^3}{3b}\right] = 2b\left[\frac{1}{2}-\frac{1}{3}\right] = 2b\left[\frac{1}{6}\right]$$

$$\boxed{\overline{X} = \frac{b}{3}}$$

$$\overline{Y} = \frac{\int y\,dA}{A}$$

where *y* is the centroidal distance of area *dA* from x-axis.

$$\therefore \quad y = \frac{y}{2} = \frac{1}{2}\left(h-\frac{h}{b}x\right)$$

$$\therefore \quad \overline{Y} = \frac{\int_0^b \frac{1}{2}\left(h-\frac{h}{b}x\right)\left(h-\frac{h}{b}x\right)dx}{\left(\frac{bh}{2}\right)} = \frac{h}{b}\int_0^b \left(1-\frac{x}{b}\right)^2 dx = \frac{h}{b}\left[\frac{\left[1-\frac{x}{b}\right]^3}{3(-1/b)}\right]_0^b$$

$$= -\frac{h}{3}\left[\left(1-\frac{b}{b}\right)^2-(1-0)^3\right] = -\frac{h}{3}(-1)$$

$$\therefore \quad \boxed{\overline{Y} = \frac{h}{3}}$$