CAUCHY RESIDUE THEOREM

Statement:

If $f(z)$ is analytic inside and on a simple closed curve C, except at a finite number of singular points $a_1, a_2, \dots a_n$ inside C, then

$$\int_c f(z)dz = 2\pi i\ [sum\ of\ residues\ of\ f(z) at\ a_1, a_2, \dots a_n]$$

Note: Formulae for evaluation of residues

(i) If $z = a$ is a simple pole of $f(z)$ then

$$[Res\ f(z), z = a] = \lim_{z \to a} (z - a) f(z)$$

(ii) If $z = a$ is a pole of order n of $f(z)$, then

$$\left[[Res\ f(z)], z = a\right] = \frac{1}{(n-1)!} \lim_{z \to a} \left\{\frac{d^{n-1}}{dz^{n-1}} [(z-a)^n f(z)]\right\}$$

Problems based on Cauchy Residue theorem

Example: 4.46 Find the residue of $f(z) = \frac{z+2}{(z-2)(z+1)^2}$ about each singularity.

Solution:

Given $f(z) = \frac{z+2}{(z-2)((z+1)^2)}$

The poles are given by $(z-2)(z+1)^2$

$$\Rightarrow z - 2 = 0, z + 1 = 0$$

$$\Rightarrow z = 2 \text{ and } z = -1$$

∴ The Poles of $f(z)$are $z = 2$ is a simple pole and $z = -1$ is a pole of order 2

$$[Res\ f(z)]_{z=2} = \lim_{z\to 2}(z-2)\, f(z)$$

$$[Res\ f(z)]_{z=2} = \lim_{z\to 2}(z-2)\frac{\boldsymbol{z+2}}{(\boldsymbol{z-2})(z+1)^2}$$

$$= \lim_{z\to 2}\frac{\boldsymbol{z+2}}{(z+1)^2} = \frac{4}{9}$$

$$[Res\ f(z)]_{z=1} = \lim_{z\to -1}\frac{d}{dz}[(z+1)^2 f(z)]$$

$$= \lim_{z\to -1}\frac{d}{dz}\left[(z+1)^2\frac{\boldsymbol{z+2}}{(\boldsymbol{z-2})(z+1)^2}\right]$$

$$= \lim_{z\to -1}\frac{d}{dz}\left(\frac{z+2}{z-2}\right)$$

$$= \lim_{z\to -1}\left[\frac{(z-2)(1)-(z+2)(1)}{(z-2)^2}\right] = -\frac{4}{9}$$

Example: 4.47 Evaluate using Cauchy's residue theorem, $\int_c \frac{\boldsymbol{sin\pi z^2+cos\pi z^2}}{(\boldsymbol{z-1})(\boldsymbol{z-2})}\, \boldsymbol{dz}$, where C is $|z| = 3$

Solution:

Let $f(z) = \frac{\boldsymbol{sin\pi z^2+cos\pi z^2}}{(\boldsymbol{z-1})(\boldsymbol{z-2})}$

The poles are given by $(z-1)(z-2) = 0$

$\Rightarrow z = 1, 2$ are poles of order 1.

Given C is$|z| = 3$

∴ Clearly $z = 1$and $z = 2$ lies inside $|z| = 3$

To find the residues:

(i) When $z = 1$

$$[Res\, f(z)]_{z=1} = \lim_{z\to 1}(z-1)f(z)$$

$$= \lim_{z\to 1}(z-1)\frac{cos\pi z^2+sin\pi z^2}{(z-1)(z-2)}$$

$$= \lim_{z\to 1}\frac{cos\pi z^2+sin\pi z^2}{(z-2)}$$

$$= \frac{cos\pi+sin\pi}{-1}$$

$$= \frac{-1+0}{-1} = 1$$

(ii) When $z = 2$

$$[Res\, f(z)]_{z=2} = \lim_{z\to 2}(z-2)f(z)$$

$$= \lim_{z\to 2}(z-2)\frac{cos\pi z^2+sin\pi z^2}{(z-1)(z-2)}$$

$$= \lim_{z\to 2}\frac{cos\pi z^2+sin\pi z^2}{(z-1)}$$

$$= \frac{cos4\pi+sin4\pi}{1}$$

$$= \frac{1+0}{1} = 1$$

∴ By Cauchy's Residue theorem

$$\int_c f(z)dz = 2\pi i\ (sum\ of\ residues)$$

$$= 2\pi i\ (1+1) = 4\pi i$$

Example: 4.48 Evaluate $\int_c \frac{z^2}{z^2+1}\,dz$ where C is $|z| = 2$ using Cauchy's residue theorem.

Solution:

Let $f(z) = \frac{z^2}{z^2+1}$

The poles are given by $z^2 + 1 = 0$

$\Rightarrow z = \pm i$ are poles of order 1.

Given C is $|z| = 2$

$\therefore$ Clearly $z = i, -i$ lies inside $|z| = 2$

To find the residue:

(i) When $z = i$

$$[Res\ f(z)]_{z=i} = \lim_{z\to i}(z - i)\frac{z^2}{(z+i)(z-i)}$$

$$= \lim_{z\to i}\frac{z^2}{(z+i)} = \frac{-1}{2i}$$

(ii) When $z = -i$

$$[Res\ f(z)]_{z=-i} = \lim_{z\to -i}(z + i)\frac{z^2}{(z+i)(z-i)}$$

$$= \lim_{z\to -i}\frac{z^2}{(z-i)}$$

$$= \frac{-1}{-2i} = \frac{1}{2i}$$

$\therefore$ By Cauchy's Residue theorem

$$\int_c f(z)dz = 2\pi i\ (sum\ of\ residues)$$

$$= 2\pi i\ (\frac{-1}{2i} + \frac{1}{2i}) = 0$$

$$\therefore \int_c \frac{z^2}{z^2+1}\, dz = 0$$

Example: 4.49 Evaluate $\int_c \frac{(z-1)}{(z+1)^2(z-2)} dz$ where C is the circle $|z-i| = 2$ using Cauchy's residue theorem.

Solution:

Let $f(z) = \frac{(z-1)}{(z+1)^2(z-2)}$

The poles are given by $(z+1)^2(z-2) = 0$

$$\Rightarrow z+1 = 0; z-2 = 0$$

$$\Rightarrow z = -1 \; is\; a\; pole\; of\; order\; 2\; and$$

$$\Rightarrow z = 2 \text{ is a pole of order 1.}$$

Given C is $|z-i| = 2$

When $z = -1, |z-i| = |-1-i| = \sqrt{2} < 2$

$\therefore z = -1$ lies inside C

When $z = 2, |z-i| = |2-i| = \sqrt{5} > 2$

$\therefore z = -1$ lies inside C

To find the residue for the inside pole:

$$[Res\, f(z)]_{z=-1} = \lim_{z\to -1} \frac{d}{dz}[(z+1)^2 f(z)]$$

$$= \lim_{z\to -1} \frac{d}{dz}\left[(z+1)^2 \frac{(z-1)}{(z+1)^2(z-2)}\right]$$

$$= \lim_{z\to -1} \frac{d}{dz}\left(\frac{z-1}{z-2}\right)$$

$$= \lim_{z \to -1} \left[\frac{(z-2)(1)-(z-1)(1)}{(z-2)} \right] = -\frac{1}{9}$$

∴ By Cauchy's Residue theorem

$$\int_c f(z)dz = 2\pi i \ (sum\ of\ residues)$$

$$= 2\pi i \left(-\frac{1}{9}\right)$$

$$\therefore \int_c \frac{(z-1)}{(z+1)^2(z-2)} dz = -2\pi i \left(\frac{1}{9}\right)$$

Example: 4.50 Evaluate $\int_c \frac{dz}{(z^2+4)^2}$ where C is the circle $|z-i| = 2$ using Cauchy's residue theorem.

Solution:

Let $f(z) = \frac{1}{(z^2+4)^2}$

The poles are given by $(z^2+4)^2$

$$\Rightarrow z^2 + 4 = 0$$

$$\Rightarrow z = \pm 2i \text{ are poles of order 2}$$

Given C is $|z - i| = 2$

When $z = 2i, |z - i| = |2i - i| = 1 < 2$

$\therefore z = 2i$ lies inside C

When $z = -2i, |z - i| = |-2i - i| = 3 > 2$

$\therefore z = -2i$ lies outside C

To find the residue for the inside pole

$$[Res\ f(z)]_{z=2i} = \lim_{z\to 2i} \frac{d}{dz}[(z-2i)^2 f(z)]$$

$$= \lim_{z\to 2i} \frac{d}{dz}\left[(z-2i)^2 \frac{1}{(z-2i)^2((z+2i)^2)}\right]$$

$$= \lim_{z\to 2i} \frac{d}{dz}\left(\frac{1}{z+2i}\right)^2$$

$$= \lim_{z\to 2i}\left[\frac{-2)}{(z+2i)^3)}\right]$$

$$= -\frac{2}{(4i)^3} = -\frac{2}{-64i} = \frac{1}{32i}$$

∴ By Cauchy's Residue theorem

$$\int_c f(z)dz = 2\pi i\ (sum\ of\ residues)$$

$$= 2\pi i\left(\frac{1}{32i}\right)$$

$$\therefore \frac{dz}{(z^2+4)^2} = \frac{\pi}{16}$$

Example: 4.51 Evaluate $\int_c \frac{e^z dz}{(z^2+\pi^2)^2}$ where C is the circle $|z| = 4$ using Cauchy's residue theorem.

Solution:

Let $f(z) = \frac{e^z}{(z^2+\pi^2)^2}$

The poles are given by $(z^2+\pi^2)^2 = 0$

$$\Rightarrow z^2+\pi^2 = 0$$

$\Rightarrow z = \pm\pi i$ are poles of order 2

Given C is $|z| = 4$

Clearly $z = +\pi i, z = \pi i$ lies inside $|z| = 4$

To find the residue

(i) When $z = +\pi i$

$$[Res\, f(z)]_{z=\pi i} = \lim_{z\to\pi i}\frac{d}{dz}[(z-\pi i)^2 f(z)]$$

$$= \lim_{z\to\pi i}\frac{d}{dz}\left[(z-\pi i)^2\frac{e^z}{(z-\pi i)^2(z+\pi i)^2}\right]$$

$$= \lim_{z\to\pi i}\frac{d}{dz}\left(\frac{e^z}{(z+\pi i)^2}\right)$$

$$= \lim_{z\to\pi i}\left[\frac{(z+\pi i)^2 e^z - 2(z+\pi i)e^z)}{(z+\pi i)^4)}\right]$$

$$= \lim_{z\to\pi i}\left[\frac{(z+\pi i)e^z[z+\pi i-2]}{(z+\pi i)^4}\right]$$

$$= \frac{e^{\pi i}(2\pi i-2)}{(2\pi i)^3}$$

$$= \frac{e^{\pi i}(\pi i-1)}{-4\pi^3 i}$$

$$= \frac{(cos\pi + isin\pi)(1-\pi i)}{4\pi^3 i}$$

$$= \frac{(-1+0)(1-\pi i)}{4\pi^3 i}$$

$$= \frac{(\pi i-1)}{4\pi^3 i}$$

(ii) When $z = -\pi i$

$$[Res\, f(z)]_{z=-\pi i} = \lim_{z\to-\pi i}\frac{d}{dz}[(z+\pi i)^2 f(z)]$$

$$= \lim_{z\to-\pi i}\frac{d}{dz}\left[(z+\pi i)^2\frac{e^z}{(z-\pi i)^2(z+\pi i)^2}\right]$$

$$= \lim_{z \to -\pi i} \frac{d}{dz}\left(\frac{e^z}{(z-\pi i)^2}\right)$$

$$= \lim_{z \to -\pi i} \left[\frac{(z-\pi i)^2 e^z - 2(z-\pi i)e^z)}{(z-\pi i)^4}\right]$$

$$= \lim_{z \to -\pi i} \left[\frac{(z-\pi i)e^z[z-\pi i-2]}{(z-\pi i)^4}\right]$$

$$= \frac{e^{-\pi i}(-2\pi i-2)}{(-2\pi i)^3}$$

$$= \frac{(-2)(cos\pi - isin\pi)(\pi i+1)}{8\pi^3 i}$$

$$= \frac{-(-1-0)(\pi i+1)}{4\pi^3 i}$$

$$= \frac{(1+\pi i)}{4\pi^3 i}$$

∴ By Cauchy's Residue theorem

$$\int_c f(z)dz = 2\pi i\,(sum\ of\ residues)$$

$$= 2\pi i\left[\frac{(\pi i-1)}{4\pi^3 i} + \frac{(\pi i+1)}{4\pi^3 i}\right]$$

$$= \frac{2\pi i}{4\pi^3 i}[2\pi i] = \frac{i}{\pi}$$

$$\therefore \int_c \frac{e^z dz}{(z^2+\pi^2)^2} = \frac{i}{\pi}$$

Example: 4.52 Evaluate $\int_c \frac{dz}{zsinz}$ where C is the circle $|z| = 1$ using Cauchy's residue theorem.

Solution:

Let $f(z) = \frac{1}{zsinz}$

The poles are given by $zsinz = 0$

$$\Rightarrow z\left[z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots\right] = 0$$

$$\Rightarrow z^2\left[1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \cdots\right] = 0$$

$\Rightarrow z = 0$ is a pole of order 2

Given C is $|z| = 1$

$\therefore z = 0$ lies inside C

To find the residue for the inside pole

$$[Res\, f(z)]_{z=0} = \lim_{z\to 0} \frac{d}{dz}[(z-0)^2 f(z)]$$

$$= \lim_{z\to 0} \frac{d}{dz}\left[(z)^2 \frac{1}{zsinz}\right]$$

$$= \lim_{z\to 0} \frac{d}{dz}\left(\frac{z}{sinz}\right)$$

$$= \lim_{z\to 0}\left[\frac{sinz(1) - z(cosz)}{(sinz)^2}\right]$$

$$= \frac{0-0}{0} = \left[\frac{0}{0}\right] \text{form}$$

$$= \lim_{z\to 0} \frac{cosz - [z\,(-sinz) + cosz(1)]}{2sinzcosz} \text{(by L' Hospital rule)}$$

$$= \lim_{z\to 0} \frac{cosz + zsinz - cosz}{2sinzcosz}$$

$$= \lim_{z\to 0} \frac{zsinz}{2sinzcosz}$$

$$= \lim_{z\to 0} \frac{z}{2cosz}$$

$$= \frac{0}{2} = 0$$

∴ By Cauchy's Residue theorem

$$\int_c f(z)dz = 2\pi i \; (sum\ of\ residues)$$

$$= 2\pi i \; [0]$$

$$\therefore \int_c \frac{dz}{zsinz} = 0$$

Example: 4.53 Evaluate $\int_c \frac{tan^z/_2}{(z-a)^2}$, where $-2 < a < 2$ and C is the boundary of the square whose sides lie along $x = \pm 2$ and $y = \pm 2$

Solution:

Let $f(z) = \frac{tan^z/_2}{(z-a)^2}$

The poles are given by $(z-a)^2 = 0$

$\Rightarrow z = a$ is a pole of order 2

C is the square with vertices $(-2,2), (2,-2)(2,2) and\ (-2,-2)$

Clearly $z = a$ lies inside C

To find the residue for the inside pole

$$[Res\ f(z)]_{z=a} = \lim_{z \to a} \frac{d}{dz}[(z-a)^2 f(z)]$$

$$= \lim_{z \to a} \frac{d}{dz}\left[(z-a)^2 \frac{tan^z/_2}{(z-a)^2}\right]$$

$$= \lim_{z \to a} \frac{d}{dz}\left(tan\, ^z/_2\right)$$

$$= \lim_{z \to a} \left[sec^2 \frac{z}{2} \left({}^1/_2 \right) \right]$$

$$= \frac{1}{2} sec^2 \left(\frac{a}{2} \right)$$

∴ By Cauchy's Residue theorem

$$\int_c f(z)dz = 2\pi i \ (sum\ of\ residues)$$

$$= 2\pi i \left[\frac{1}{2} sec^2 \left(\frac{a}{2} \right) \right]$$

$$\int_c \frac{tan^z/_2}{(z-a)^2} = \pi i \left[sec^2 \left(\frac{a}{2} \right) \right]$$

Example: 4.54 Evaluate $\int_c \frac{dz}{z^2 sinhz}$ where C is the circle $|z-1| = 2$ using Cauchy's residue theorem.

Solution:

Let $f(z) = \frac{1}{z^2 sinhz}$

The poles are given by $z^2 sinhz = 0$

$\Rightarrow z^2 = 0\ (or) sinhz = 0$

$\Rightarrow z = 0\ or)\ z = \ sinh^{-1}(0) = 0$is a pole of order 1.

Given C is $|z-1| = 2$

∴Clearly $z = 0$ lies inside C.

To find residue for the inside pole at $z = 0$

$$f(z) = \frac{1}{z^2 sinhz}$$

$$= \frac{1}{z^2\left[z+\frac{z^3}{3!}+\frac{z^5}{5!}+\cdots\right]}$$

$$= \frac{1}{z^3\left[1+\frac{z^2}{6}+\frac{z^4}{120}+\cdots\right]}$$

$$= \frac{1}{z^3}[1+u]^{-1} \qquad \text{where} \quad u = 1+\frac{z^2}{6}+\cdots$$

$$= \frac{1}{z^3}[1-u+u^2-u^3 \ldots]$$

$$= \frac{1}{z^3}\left[1-\left(\frac{z^2}{6}+\frac{z^4}{120}+\cdots\right)+\left(\frac{z^2}{6}+\frac{z^4}{120}+\cdots\right)^2 \ldots\right]$$

$$= \frac{1}{z^3} - \frac{1}{6z} - \frac{z}{120} + \cdots$$

$[Res\ f(z)]_{z=0}$ = Coefficient of $\frac{1}{z}$ in the Laurent's expansion of $f(z)$

$$\therefore [Res\ f(z)]_{z=0} = -\frac{1}{6}$$

∴ By Cauchy's Residue theorem

$$\int_c f(z)dz = 2\pi i\ (sum\ of\ residues)$$

$$= 2\pi i\left[-\frac{1}{6}\right]$$

$$\therefore \int_c \frac{dz}{z^2 sinhz} = \frac{-\pi i}{3}$$

Example: 4.55 Evaluate $\int_c \frac{z}{cosz}\,dz$ where C is the circle $\left|z-\frac{\pi}{2}\right| = \frac{\pi}{2}$

Solution:

Let $f(z) = \frac{z}{cosz}$

The poles are given by $cosz = 0$

$\Rightarrow z = (2n+1)\frac{\pi}{2}, n = 0, \pm 1, \pm 2, \ldots$ are poles of order 1

Given C is $\left|z - \frac{\pi}{2}\right| = \frac{\pi}{2}$

Here $z = \frac{\pi}{2}$ lies inside the circle and others lies outside.

$$[Res\, f(z)]_{z=\frac{\pi}{2}} = \lim_{z \to \frac{\pi}{2}} \left(z - \frac{\pi}{2}\right) f(z)$$

$$[Res\, f(z)]_{z=\frac{\pi}{2}} = \lim_{z \to \frac{\pi}{2}} \left(z - \frac{\pi}{2}\right) \frac{z}{cosz}$$

$$= \frac{0}{0} \text{ (form)}$$

Using L ‘ Hospital’s rule

$$= \lim_{z \to \frac{\pi}{2}} \frac{\left(z-\frac{\pi}{2}\right)(1)+z(1)}{-sinz}$$

$$= \lim_{z \to \frac{\pi}{2}} \frac{\left(z-\frac{\pi}{2}\right)+z}{-sinz}$$

$$= -\frac{\pi}{2}$$

∴ By Cauchy’s Residue theorem

$$\int_c f(z)dz = 2\pi i\ (sum\ of\ residues)$$

$$= 2\pi i \left[-\frac{\pi}{2}\right]$$

$$\therefore \int_c \frac{z}{cosz}\, dz = -\pi^2 i$$

Example: 4.56 Evaluate $\int_c \; z^2 e^{1/z}\, dz$ where C is the unit circle using Cauchy's residue theorem.

Solution:

Let $f(z) = z^2 e^{1/z}$

Here $z = 0$ is the only singular point.

Given C is $|z| = 1$

∴Clearly $z = 0$ lies inside C.

To find residue of $f(z)$ $at\ z = 0$

We find the Laurent's series of $f(z)$ $about\ \ z = 0$

$$\Rightarrow f(z) = z^2 e^{1/z}$$

$$\Rightarrow z^2 \left[1 + \frac{1}{z} + \frac{1}{2!z^2} + \cdots\right]$$

$[Res\ f(z)]_{z=0}$ = C0efficient of $\frac{1}{z}$ in the Laurent's expansion of $f(z)$

$$\therefore [Res\ f(z)]_{z=0} = \frac{1}{6}$$

∴ By Cauchy's Residue theorem

$$\int_c \; f(z)dz = 2\pi i\ (sum\ of\ residues)$$

$$= 2\pi i \left[\frac{1}{6}\right]$$

$$\therefore \int_c \; z^2 e^{1/z}\, dz = \frac{\pi i}{3}$$