2.3   ENERGY DENSITY IN ELECTROSTATIC FIELDS

ELECTROSTATIC ENERGY:

The capacitor stores the electrostatic energy equal to work done to build up the charge .If a voltage source is connected across the capacitor, the capacitor charges. Potential is defined as the work done per unit charge.

To determine the energy present in an assembly of charges. We must first determine the amount of work necessary to assemble them.

$$V = \frac{dW}{dQ}$$

The work done

$$dW = V.dQ$$

But

$$V = \frac{Q}{C}$$

Substitute V in dW

$$dW = \frac{Q}{C}dQ$$

The capacitor is charged to the value of Q .The total work done is

$$W = \int_{0}^{Q} \frac{Q}{C} dQ$$

$$W = \frac{1}{C} \int_{0}^{Q} Q \, dQ$$

Integrate the above equation

$$W = \frac{1}{C}\left[\frac{Q^2}{2}\right]_{0}^{Q}$$

$$W = \frac{1}{C}\left[\frac{Q^2}{2}\right]$$

$$W = \frac{1}{C}\frac{Q^2}{2} \; Joules$$

But

$$Q = CV$$

Substitute Q in W

$$W = \frac{1}{C}\frac{Q^2}{2}$$

$$W = \frac{1}{C}\frac{(CV)^2}{2}$$

$$W = \frac{1}{C}\frac{C^2V^2}{2}$$

$$W = \frac{CV^2}{2} \ Joules$$

$$W = \frac{1}{2}CV^2 \ Joules$$

Substitute Q in above equation

$$Q = CV$$

$$W = \frac{CV^2}{2}$$

$$W = \frac{CV \times V}{2}$$

$$W = \frac{Q \times V}{2}$$

$$W = \frac{QV}{2} \ Joules$$

$$W = \frac{1}{2}QV \ Joules$$

ENERGY DENSITY:

Consider a elementary cube of side Δd parallel to the plates of a capacitor as shown in figure 2.3.1
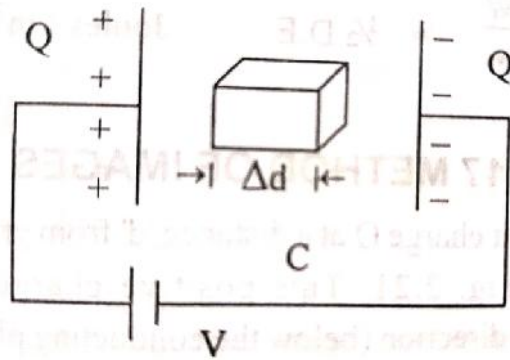
Figure 2.3.1 Energy storage in capacitor

[Source: "Electromagnetic Theory" by Dr.P.Dananjayan, page-2.38]

The capacitance of elemental capacitor is

$$\Delta C = \frac{\varepsilon A}{\Delta d}$$

Side of the cube of is Δd

Area of cube is

$$A = side \times Side$$
$$A = \Delta d \times \Delta d$$
$$A = (\Delta d)^2$$

Substitute A in ΔC

$$\Delta C = \frac{\varepsilon A}{\Delta d}$$
$$A = (\Delta d)^2$$
$$\Delta C = \frac{\varepsilon (\Delta d)^2}{\Delta d}$$
$$\Delta C = \frac{\varepsilon \times (\Delta d) \times (\Delta d)}{(\Delta d)}$$
$$\Delta C = \varepsilon \times (\Delta d)$$
$$\Delta C = \varepsilon \Delta d$$

Energy stored in the elemental capacitor is

$$И = \frac{1}{2} CV^2$$

$$\Delta W = \frac{1}{2}\Delta C(\Delta V)^2$$

But potential difference across the elementary cube is

$$\Delta V = E.\Delta d$$

Where E is the electric field exist in the cube.

The stored energy

$$\Delta W = \frac{1}{2}\Delta C(\Delta V)^2$$

Substitute the values of ΔC and ΔV in energy stored equation.

$$\Delta C = \varepsilon\Delta d$$

$$\Delta V = E.\Delta d$$

$$\Delta W = \frac{1}{2}\varepsilon\Delta d\,(E.\Delta d)^2$$

$$\Delta W = \frac{1}{2}\varepsilon(E)^2(\Delta d)^3$$

$$\Delta W = \frac{1}{2}\varepsilon E^2(\Delta d)^3$$

Side of the cube of is Δd

Elementary volume of cube is

$$V = Area \times Side$$

$$\Delta V = (\Delta d)^2 \times \Delta d$$

$$\Delta V = (\Delta d)^3$$

$$(\Delta d)^3 =$$

Substitute the values of ΔC and ΔV in energy stored equation.

$$\Delta W = \frac{1}{2}\varepsilon E^2\Delta V$$

The energy density is given by

$$\Delta W = \frac{1}{2}\varepsilon E^2\Delta V$$

$$\frac{\Delta W}{\Delta V} = \frac{1}{2}\varepsilon E^2$$

ual magnitude but opposite sign separated by a small distance.

An electric dipole or simply dipole is nothing but two equal and opposite charges are separated by a very small distance. The product of charge and spacing is called **Electric Dipole Moment.**

Let Q_1 and Q_2 be the two charges separated by a small distance d. The product of charge Q and spacing d is called dipole moment.

Let P **be any point at distance of** r_1, r_2 **and** r from $+Q, -Q$ and midpoint of dipole respectively as shown in figure 2.2.1.

Figure 2.2.1 Evaluation of the electric potential V due to a line charge

[Source: "Elements of Electromagnetics" by Matthew N.O.Sadiku, page-114]

$$n = Qd$$

Potential at P due to $+Q$ is

$$V_1 = \frac{+Q}{4\pi\varepsilon r_1}$$

$$V_1 = \frac{Q}{4\pi\varepsilon r_1}$$

Potential at P due to $-Q$ is

$$V_2 = \frac{-Q}{4\pi\varepsilon r_2}$$

The resultant potential at P

$$V = V_1 + V_2$$

Substitute V_1 and V_2 in above expression

$$V = \frac{Q}{4\pi\varepsilon r_1} + \frac{-Q}{4\pi\varepsilon r_2}$$

Take $\dfrac{Q}{4\pi\varepsilon}$ common as outside

$$V = \frac{Q}{4\pi\varepsilon r_1} + \frac{-Q}{4\pi\varepsilon r_2}$$

$$V = \frac{Q}{4\pi\varepsilon r_1} + \frac{-Q}{4\pi\varepsilon r_2}$$

$$V = \frac{Q}{4\pi\varepsilon}\left[\frac{1}{r_1} + \frac{-1}{r_2}\right]$$

$$V = \frac{Q}{4\pi\varepsilon}\left[\frac{1}{r_1} - \frac{1}{r_2}\right]$$

If the point P is too far from the dipole, the distance r_1 and r_2 are written as.

$$r_1 = r - \frac{d}{2}\cos\theta$$

$$r_2 = r + \frac{d}{2}\cos\theta$$

Potential at P due to dipole

$$V = \frac{Q}{4\pi\varepsilon}\left[\frac{1}{r_1} - \frac{1}{r_2}\right]$$

Substitute r_1 and r_2 in above expression

$$V = \frac{Q}{4\pi\varepsilon}\left[\frac{1}{r_1} - \frac{1}{r_2}\right]$$

$$V = \frac{Q}{4\pi\varepsilon}\left[\left(\frac{1}{r - \frac{d}{2}\cos\theta}\right) - \left(\frac{1}{r + \frac{d}{2}\cos\theta}\right)\right]$$

Take LCM

$$V = \frac{Q}{4\pi\varepsilon}\left[\frac{\left(r + \frac{d}{2}\cos\theta\right) - \left(r - \frac{d}{2}\cos\theta\right)}{\left(r - \frac{d}{2}\cos\theta\right) \times \left(r + \frac{d}{2}\cos\theta\right)}\right]$$

$$V = \frac{Q}{4\pi\varepsilon}\left[\frac{r + \frac{d}{2}\cos\theta - r + \frac{d}{2}\cos\theta}{\left(r - \frac{d}{2}\cos\theta\right) \times \left(r + \frac{d}{2}\cos\theta\right)}\right]$$

$$V = \frac{Q}{4\pi\varepsilon}\left[\frac{r + \frac{d}{2}\cos\theta - r + \frac{d}{2}\cos\theta}{\left(r - \frac{d}{2}\cos\theta\right) \times \left(r + \frac{d}{2}\cos\theta\right)}\right]$$

$$V = \frac{Q}{4\pi\varepsilon}\left[\frac{+\frac{d}{2}\cos\theta + \frac{d}{2}\cos\theta}{\left(r - \frac{d}{2}\cos\theta\right) \times \left(r + \frac{d}{2}\cos\theta\right)}\right]$$

$$V = \frac{Q}{4\pi\varepsilon}\left[\frac{\frac{2d}{2}\cos\theta}{\left(r - \frac{d}{2}\cos\theta\right) \times \left(r + \frac{d}{2}\cos\theta\right)}\right]$$

$$V = \frac{Q}{4\pi\varepsilon}\left[\frac{d\cos\theta}{\left(r - \frac{d}{2}\cos\theta\right) \times \left(r + \frac{d}{2}\cos\theta\right)}\right]$$

Formula $A^2 - B^2 = (A + B)(A - B)$

$$V = \frac{Q}{4\pi\varepsilon}\left[\frac{d\cos\theta}{\left((r)^2 - \left(\frac{d}{2}\cos\theta\right)^2\right)}\right]$$

$$\frac{d}{2} \ll r^2$$

$$V = \frac{Q}{4\pi\varepsilon}\left[\frac{d\cos\theta}{((r)^2)}\right]$$

$$V = \frac{Qd\cos\theta}{4\pi\varepsilon r^2}$$

Substitute Qd expression in above equation

$$n = Qd$$

$$V = \frac{m\cos\theta}{4\pi\varepsilon r^2}$$

This shows that the potential is directly proportional to the dipole moment and inversely proportional to the square of the distance.

EQUIPOTENTIAL PLOTS OR EQUIPOTENTIAL SURFACE:

In an electric field ,there are many points at which the electric potential is same.This is because,the potential is a scalar quantity which depends on the distance between the point at which potential is to be obtained and the location of the charge. There can be

number of points which can be located at the same distance from the charge. All such points are at the same electric potential. If the surface is imagined, joining all such points which are at the same potential, then such a surface is called **equipotential surface.**

Any surface on which the potential is the same thought is known as an **equipotential surface.**

The intersection of an eqipotential surface and a plane results in a path or line known as **equipotential line.** No work is done in moving a charge from one point to another along an eqipotential line or surface $(V_A - V_B = 0)$ and hence

$$\int_L E dl = 0$$

On the line surface. The lines of force or flux lines (or the direction of E) are always normal to equipotential surface .Equipotential surface for point charge and a dipole are shown in figure 2.2.2 .The direction of E is everywhere normal to the equipotential lines. Consider a point charge located at the origin of a sphere. The potential at a point which is at a radial distance r from the point charge is given by

Figure 2.2.2 Evaluation of the electric potential V due to a line charge

[Source: "Elements of Electromagnetics" by Matthew N.O.Sadiku, page-114]

$$V = \frac{Q}{4\pi\varepsilon_0 r}$$

At all points which are at a distance r from Q the potential at a point which is at a radial distance r from the point charge is given by.
Similarly at $r = r_1, r = r_2$ there exists other equipotential surfaces, in an electric field of point charge ,in the form of concentric sphere as shown in figure 2.2.3
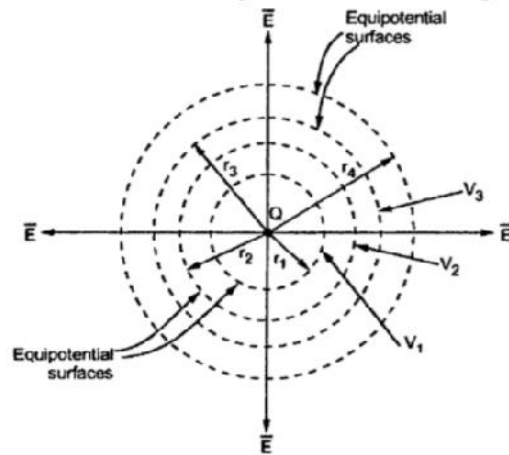
Figure 2.2.3 Equipotential Surfaces

[Source: " Electromagnetic Theory" by U.A.Bakshi, page-4-27]

V is inversely proportional to distance .Thus V_1 at equipotential surface at $r = r_1$ is highest and it goes on decreasing, as the distance r increasing. Thus $V_1 > V_2 > V_3$.As we move away from the charge,The E decreases the hence potential of equipotential surfaces goes on decreasing. While potential of equipotential surface goes on decreasing. While potential of equipotential surface goes on increasing as we move against the direction of electric field.

For a uniform electric field E the equipotential surfaces are perpendicular to E and are equispaces for fixed increment of voltages. Thus if we move a charge along a circular path of radius r_1 as shown in a_φ direction, the work done is zero. This is because E and dL are perpendicular .Thus E and equipotential surface are at right angles each other.

For non uniform filed, the field lines tends to diverge in the direction of decreasing E.Hence equipotential surfaces are still perpendiculat to E but are not equispaced,for fixed increment of voltages.