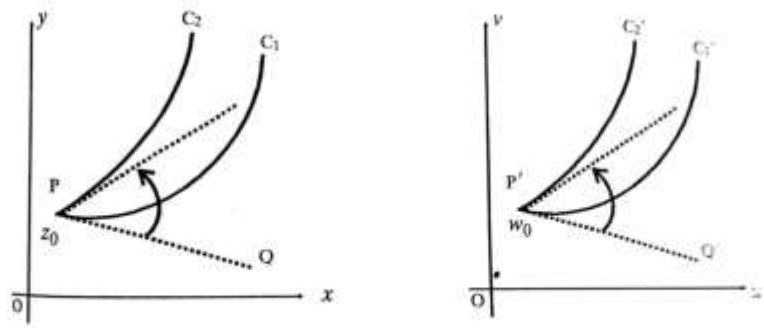
CONFORMAL MAPPING

Definition: Conformal Mapping

A transformation that preserves angels between every pair of curves through a point, both in magnitude and sense, is said to be conformal at that point.



Definition: Isogonal

A transformation under which angles between every pair of curves through a point are preserved in magnitude, but altered in sense is said to be an isogonal at that point.

Note: (i) A mapping $w = f(z)$ is said to be conformal at $z = z_0$, if $f'(z_0) \neq 0$.

Note: (ii) The point, at which the mapping $w = f(z)$ is not conformal,

$(i.e.) f'(z) = 0$ is called a **critical point** of the mapping.

If the transformation $w = f(z)$ is conformal at a point, the inverse transformation $z = f^{-1}(w)$ is also conformal at the corresponding point.

The critical points of $z = f^{-1}(w)$ are given by $\frac{dz}{dw} = 0$. hence the critical point of the transformation $w = f(z)$ are given by $\frac{dw}{dz} = 0$ and $\frac{dz}{dw} = 0$,

Note: (iii) Fixed points of mapping.

Fixed or invariant point of a mapping $w = f(z)$ are points that are mapped onto themselves, are "Kept fixed" under the mapping. Thus they are obtained from $w = f(z) = z$.

The identity mapping $w = z$ has every point as a fixed point. The mapping $w = \bar{z}$ has infinitely many fixed points.

$w = \frac{1}{z}$ has two fixed points, a rotation has one and a translation has none in the complex plane.

Some standard transformations

Translation:

The transformation $w = C + z$, where C is a complex constant, represents a translation.

Let $z = x + iy$

$w = u + iv$ and $C = a + ib$

Given $w = z + C$,

$(i.e.)\ u + iv = x + iy + a + ib$

$\Rightarrow u + iv = (x + a) + i(y + b)$

Equating the real and imaginary parts, we get $u = x + a, v = y + b$

Hence the image of any point $p(x, y)$in the z−plane is mapped onto the point $p'(x + a, y + b)$ in the w-plane. Similarly every point in the z−plane is mapped onto the w plane.

If we assume that the w−plane is super imposed on the z-plane, we observe that the point (x, y) and hence any figure is shifted by a distance $|C| = \sqrt{a^2 + b^2}$ in the direction of C i.e., translated by the vector representing C.

Hence this transformation transforms a circle into an equal circle. Also the corresponding regions in the z and w planes will have the same shape, size and orientation.

Problems based on $w = z + k$

Example: 1 What is the region of the w plane into which the rectangular region in the Z plane bounded by the lines $x = 0, y = 0,\ x = 1\ and\ y = 2$ is mapped under the transformation $\mathbf{w = z + (2 - i)}$

Solution:

Given $w = z + (2 - i)$

$(i.e.)\ u + iv = x + iy + (2 - i) = (x + 2) + i(y - 1)$

Equating the real and imaginary parts

$$u = x + 2,\ v = y - 1$$

Given boundary lines are	transformed boundary lines are
$x = 0$	$u = 0 + 2 = 2$

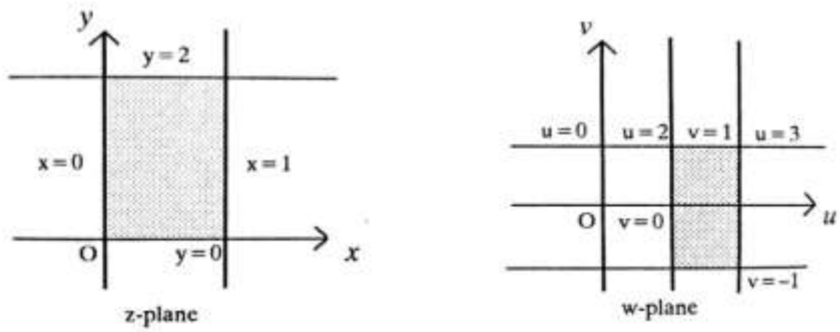
$y = 0$ $\qquad\qquad\qquad\qquad$ $v = 0 - 1 = -1$

$x = 1$ $\qquad\qquad\qquad\qquad$ $u = 1 + 2 = 3$

$y = 2$ $\qquad\qquad\qquad\qquad$ $v = 2 - 1 = 1$

Hence, the lines $x = 0, y = 0, x = 1,$ and $y = 2$ are mapped into the lines $u = 2, v = -1,$

$u = 3,$ *and* $v = 1$ respectively which form a rectangle in the w plane.



Example: 2 Find the image of the circle $|z| = 1$ by the transformation $w = z + 2 + 4i$

Solution:

Given $w = z + 2 + 4i$

$$(i.e.)\ u + iv = x + iy + 2 + 4i$$

$$= (x + 2) + i(y + 4)$$

Equating the real and imaginary parts, we get

$$u = x + 2, v = y + 4,$$

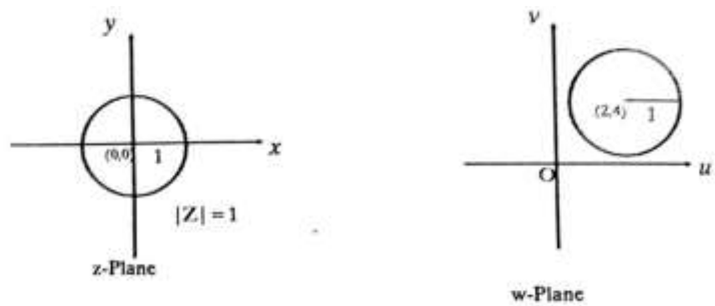
$$x = u - 2, y = v - 4,$$

Given $|z| = 1$

$$(i.e.)\ x^2 + y^2 = 1$$

$$(u-2)^2 + (v-4)^2 = 1$$

Hence, the circle $x^2 + y^2 = 1$is mapped into $(u-2)^2 + (v-4)^2 = 1$ in w plane which is also a circle with centre (2, 4)and radius 1.



2. Magnification and Rotation

The transformation $w = cz$, where c is a complex constant, represents both magnification and rotation.

This means that the magnitude of the vector representing z is magnified by $a = |c|$ and its direction is rotated through angle $\alpha = amp\,(c)$. Hence the transformation consists of a magnification and a rotation.

Problems based on w = cz

Example: 3 Determine the region 'D' of the w-plane into which the triangular region D enclosed by the lines $x = 0, y = 0, x + y = 1$ is transformed under the transformation $w = 2z$.

Solution:

Let $w = u + iv$

$$z = x + iy$$

Given $w = 2z$

$$u + iv = 2(x + iy)$$

$$u + iv = 2x + i2y$$

$$u = 2x \Rightarrow x = \frac{u}{2}, v = 2y \Rightarrow y = \frac{v}{2}$$

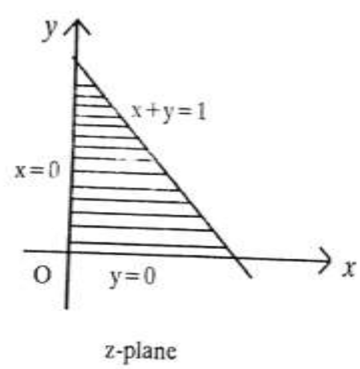
Given region (D) whose boundary lines are		Transformed region D' whose boundary lines are
$x = 0$	$\Rightarrow$	$u = 0$
$y = 0$	$\Rightarrow$	$v = 0$
$x + y = 1$	$\Rightarrow$	$\frac{u}{2} + \frac{v}{2} = 1 [\because x = \frac{u}{2}, y = \frac{v}{2}]$ $(i.e.)\ u + v = 2$

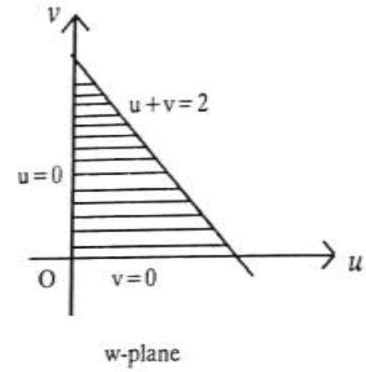
In the z plane the line $x = 0$ is transformed into $u = 0$in the w plane.

In the z plane the line $y = 0$ is transformed into $v = 0$in the w plane.

In the z plane the line $x + y =$ is transformed into$u + v = 2$

in the w plane.





Example: 4 Find the image of the circle $|z| = \lambda$ under the transformation $w = 5z$.

Solution:

Given $w = 5z$

$$|w| = 5|z|$$

i.e., $|w| = 5\lambda \quad [\because |z| = \lambda]$

Hence, the image of $|z| = \lambda$ in the z plane is transformed into $|w| = 5\lambda$ in the w plane under the transformation $w = 5z$.

Example: 5 Find the image of the circle $|z| = 3$ under the transformation $w = 2z$ [A.U N/D 2012] [A.U N/D 2016 R-13]

Solution:

Given $w = 2z, \ |z| = 3$

$$|w| = (2)|z|$$

$$= (2)(3), \quad \text{Since } |z| = 3$$

$$= 6$$

Hence, the image of $|z| = 3$ in the z plane is transformed into $|w| = 6$ w plane under the transformation $w = 2z$.

Example: 6 Find the image of the region $y > 1$ under the transformation $w = (1 - i)z$. **[Anna, May – 1999]**

Solution:

Given $w = (1 - i)z$.

$$u + v = (1 - i)(x + iy)$$

$$= x + iy - ix + y$$

$$= (x + y) + i(y - x)$$

$$\text{i.e.,} \quad u = x + y, \quad v = y - x$$

$$u + v = 2y \qquad u - v = 2x$$

$$y = \frac{u+v}{2} \qquad x = \frac{u-v}{2}$$

Hence, image region $y > 1$ is $\frac{u+v}{2} > 1$ i.e., $u + v > 2$ in the w plane.

3. Inversion and Reflection

The transformation $w = \frac{1}{z}$ represents inversion w.r.to the unit circle $|z| = 1$, followed by reflection in the real axis.

$$\Rightarrow w = \frac{1}{z}$$

$$\Rightarrow z = \frac{1}{w}$$

$$\Rightarrow x + iy = \frac{1}{u+iv}$$

$$\Rightarrow x + iy = \frac{1}{u^2+v^2}$$

$$\Rightarrow x = \frac{1}{u^2+v^2} \qquad \ldots (1)$$

$$\Rightarrow y = \frac{-v}{u^2+v^2} \qquad \ldots (2)$$

We know that, the general equation of circle in z plane is

$$x^2 + y^2 + 2gx + 2fy + c = 0 \qquad \ldots (3)$$

Substitute, (1) and (2) in (3)we get

$$\frac{u^2}{(u^2+v^2)^2} + \frac{v^2}{(u^2+v^2)^2} + 2g\left(\frac{u}{u^2+v^2}\right) + 2f\left(\frac{-v}{u^2+v^2}\right) + c = 0$$

$$\Rightarrow c(u^2 + v^2) + 2gu - 2fv + 1 = 0 \qquad \ldots (4)$$

which is the equation of the circle in w plane

Hence, under the transformation $w = \frac{1}{z}$ a circle in z plane transforms to another circle in the w plane. When the circle passes through the origin we have $c = 0$ in (3). When $c = 0$, equation (4) gives a straight line.

Problems based on $w = \frac{1}{z}$

Example: 7 Find the image of $|z - 2i| = 2$ under the transformation $w = \frac{1}{z}$

[Anna – May 1999, May 2001] [A.U N/D 2016 R-18]

Solution:

Given $|z - 2i| = 2$ .....(1) is a circle.

Centre = $(0,2)$

radius $= 2$

Given $w = \frac{1}{z} => z = \frac{1}{w}$

(1) $\Rightarrow \left|\frac{1}{w} - 2i\right| = 2$

$\Rightarrow |1 - 2wi| = 2|w|$

$\Rightarrow |1 - 2(u + iv)i| = 2|u + iv|$

$\Rightarrow |1 - 2ui + 2v| = 2|u + iv|$

$\Rightarrow |1 + 2v - 2ui| = 2|u + iv|$

$\Rightarrow \sqrt{(1 + 2v)^2 + (-2u)^2} = 2\sqrt{u^2 + v^2}$

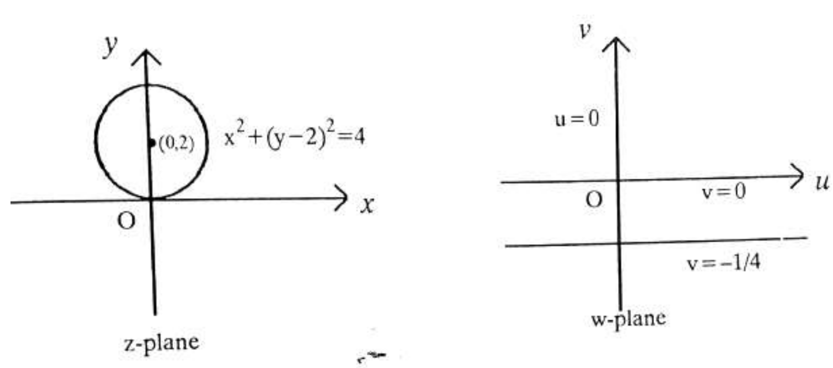
$\Rightarrow (1 + 2v)^2 + 4u^2 = 4(u^2 + v^2)$

$\Rightarrow 1 + 4v^2 + 4v + 4u^2 = 4(u^2 + v^2)$

$\Rightarrow 1 + 4v = 0$

$\Rightarrow v = -\frac{1}{4}$

Which is a straight line in w plane.



Example: 8 Find the image of the circle $|z-1|=1$ in the complex plane under the mapping $w=\frac{1}{z}$ [A.U N/D 2009] [A.U M/J 2016 R-8]

Solution:

Given $|z-1|=1$ .....(1) is a circle.

Centre =(1,0)

radius $=1$

Given $w=\frac{1}{z} \Rightarrow z=\frac{1}{w}$

(1) $\Rightarrow \left|\frac{1}{w}-1\right|=1$

$\Rightarrow |1-w|=|w|$

$\Rightarrow |1-(u+iv)|=|u+iv|$

$\Rightarrow |1-u+iv|=|u+iv|$

$\Rightarrow \sqrt{(1-u)^2+(-v)^2}=\sqrt{u^2+v^2}$

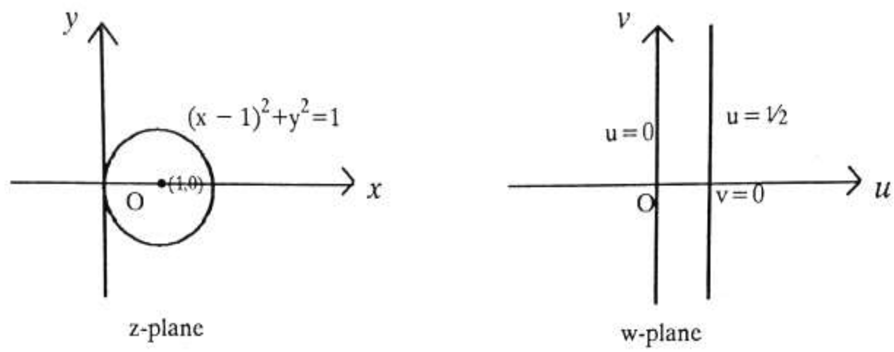
$\Rightarrow (1-u)^2+v^2=u^2+v^2$

$\Rightarrow 1+u^2-2v+v^2=u^2+v^2$

$\Rightarrow 2u=1$

$\Rightarrow u=\frac{1}{2}$

which is a straight line in the w- plane



Example: 9 Find the image of the infinite strips

(i) $\frac{1}{4} < y < \frac{1}{2}$ **(ii)** $0 < y < \frac{1}{2}$ **under the transformation** $w = \frac{1}{z}$

Solution:

Given $w = \frac{1}{z}$ (given)

i.e., $z = \frac{1}{w}$

$$z = \frac{1}{u+iv} = \frac{u-iv}{(u+iv)+(u-iv)} = \frac{u-iv}{u^2+v^2}$$

$$x + iy = \frac{u-iv}{u^2+v^2} = \left[\frac{u}{u^2+v^2}\right] + i\left[\frac{-v}{u^2+v^2}\right]$$

$$x = \frac{u}{u^2+v^2} \dots. (1), y = \frac{-v}{u^2+v^2} \dots. (2)$$

(i) Given strip is $\frac{1}{4} < y < \frac{1}{2}$

when $y = \frac{1}{4}$

$$\frac{1}{4} = \frac{-v}{u^2+v^2} \qquad \text{by (2)}$$

$$\Rightarrow u^2 + v^2 = -4v$$

$$\Rightarrow u^2 + v^2 + 4v = 0$$

$$\Rightarrow u^2 + (v+2)^2 = 4$$

which is a circle whose centre is at $(0, -2)$ in the w plane and radius is 2k.

when $y = \frac{1}{2}$

$$\frac{1}{2} = \frac{-v}{u^2+v^2} \qquad \text{by (2)}$$

$$\Rightarrow \; u^2 + v^2 = -2v$$

$$\Rightarrow u^2 + v^2 + 2v = 0$$

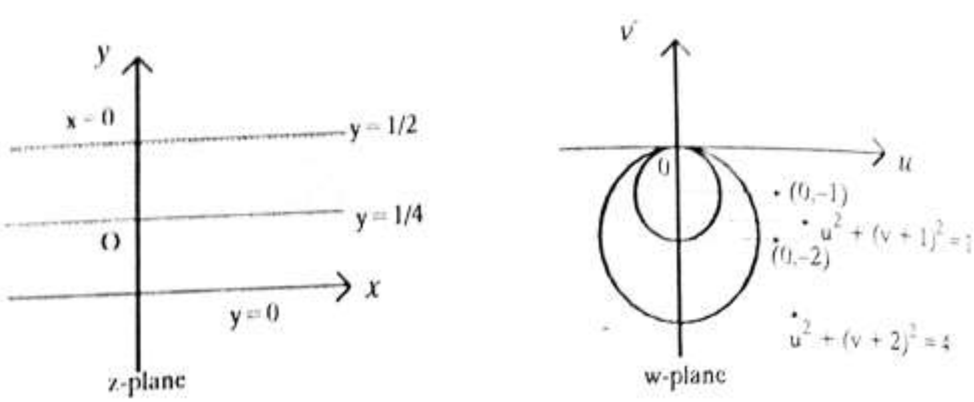
$$\Rightarrow u^2 + (v+1)^2 = 0$$

$$\Rightarrow u^2 + (v+1)^2 = 1 \qquad \ldots\ldots(3)$$

which is a circle whose centre is at $(0, -1)$ in the w plane and unit radius

Hence the infinite strip $\frac{1}{4} < y < \frac{1}{2}$ is transformed into the region in between circles

$u^2 + (v+1)^2 = 1$ and $\quad u^2 + (v+2)^2 = 4$ in the w plane.



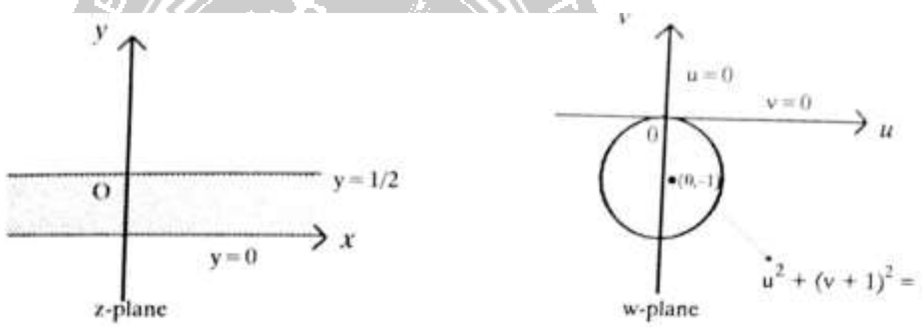
ii) Given strip is $0 < y < \frac{1}{2}$

when $y = 0$

$\Rightarrow v = 0$ by (2)

when $y = \frac{1}{2}$ we get $u^2 + (v+1)^2 = 1$ by (3)

Hence, the infinite strip $0 < y < \frac{1}{2}$ is mapped into the region outside the circle $u^2 + (v+1)^2 = 1$ in the lower half of the w plane.



Example: 10 Find the image of $x = 2$ under the transformation $w = \frac{1}{z}$. [Anna – May 1998]

Solution:

Given $w = \frac{1}{z}$

i.e., $z = \frac{1}{w}$

$$z = \frac{1}{u+iv} = \frac{u-iv}{(u+iv)+(u-iv)} = \frac{u-iv}{u^2+v^2}$$

$$x + iy = \left[\frac{u}{u^2+v^2}\right] + i\left[\frac{-v}{u^2+v^2}\right]$$

i.e., $x = \frac{u}{u^2+v^2}$ .... (1), $y = \frac{-v}{u^2+v^2}$ ... (2)

Given $x = 2$ in the z plane.

$$\therefore 2 = \frac{u}{u^2+v^2} \qquad \text{by (1)}$$

$$2(u^2 + v^2) = u$$

$$u^2 + v^2 - \frac{1}{2}u = 0$$

which is a circle whose centre is $\left(\frac{1}{4}, 0\right)$ and radius $\frac{1}{4}$

$\therefore x = 2$ in the z plane is transformed into a circle in the w plane.

Example: 11 What will be the image of a circle containing the origin(i.e., circle passing through the origin) in the XY plane under the transformation $w = \frac{1}{z}$? [Anna – May 2002]

Solution:

Given $w = \frac{1}{z}$

i.e., $z = \frac{1}{w}$

$$z = \frac{1}{u+iv} = \frac{u-iv}{(u+iv)+(u-iv)} = \frac{u-iv}{u^2+v^2}$$

$$x + iy = \left[\frac{u}{u^2+v^2}\right] + i\left[\frac{-v}{u^2+v^2}\right]$$

i.e., $x = \frac{u}{u^2+v^2}$ ... (1),

$y = \frac{-v}{u^2+v^2}$ ... (2)

Given region is circle $x^2 + y^2 = a^2$ in z plane.

Substitute, (1) and (2), we get

$$\left[\frac{u^2}{(u^2+v^2)^2} + \frac{v^2}{(u^2+v^2)^2}\right] = a^2$$

$$\left[\frac{u^2+v^2}{(u^2+v^2)^2}\right] = a^2$$

$$\frac{1}{(u^2+v^2)} = a^2$$

$$u^2 + v^2 = \frac{1}{a^2}$$

Therefore the image of circle passing through the origin in the XY −plane is a circle passing through the origin in the w − plane.

Example: 12 Determine the image of $1 < x < 2$ under the mapping $w = \frac{1}{z}$

Solution:

Given $w = \frac{1}{z}$

i.e., $z = \frac{1}{w}$

$$z = \frac{1}{u+iv} = \frac{u-iv}{(u+iv)+(u-iv)} = \frac{u-iv}{u^2+v^2}$$

$$x + iy = \left[\frac{u}{u^2+v^2}\right] + i\left[\frac{-v}{u^2+v^2}\right]$$

i.e., $x = \frac{u}{u^2+v^2}$ ....(1), $y = \frac{-v}{u^2+v^2}$ ....(2)

Given $1 < x < 2$

When $x = 1$

$\Rightarrow 1 = \frac{u}{u^2+v^2}$ by ....(1)

$\Rightarrow u^2 + v^2 = u$

$\Rightarrow u^2 + v^2 - u = 0$

which is a circle whose centre is $\left(\frac{1}{2}, 0\right)$ and is $\frac{1}{2}$

When $x = 2$

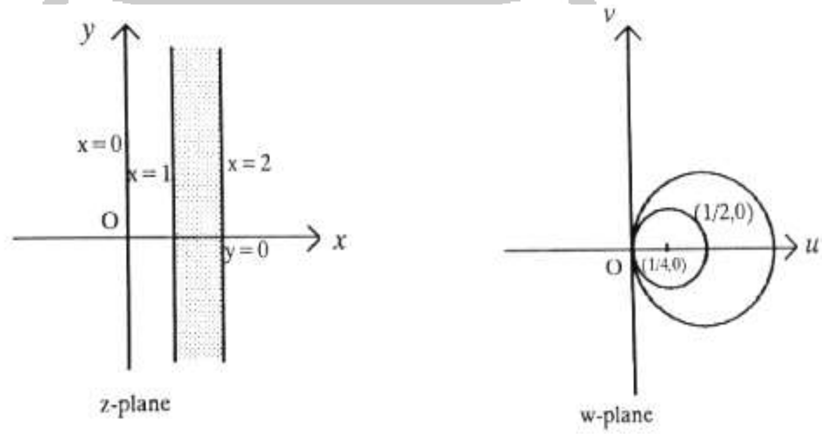
$\Rightarrow 2 = \frac{u}{u^2+v^2}$ by ....(1)

$\Rightarrow u^2 + v^2 = \frac{u}{2}$

$\Rightarrow u^2 + v^2 - \frac{u}{2} = 0$

which is a circle whose centre is $\left(\frac{1}{4}, 0\right)$ and is $\frac{1}{4}$

Hence, the infinite strip $1 < x < 2$ is transformed into the region in between the circles in the $w -$ plane.



Example: 13 Show the transformation $w = \frac{1}{z}$ transforms all circles and straight lines in the $z -$ plane into circles or straight lines in the $w -$ plane.

[A.U N/D 2007, J/J 2008, N/D 200] [A.U N/D 2016 R-13]

Solution:

Given $w = \frac{1}{z}$

i.e., $z = \frac{1}{w}$

Now, $w = u + iv$

$$z = \frac{1}{w} = \frac{1}{u+iv} = \frac{u-iv}{u+iv+u-iv} = \frac{u-iv}{u^2+v^2}$$

$$\text{i.e.,} \quad x + iy = \frac{u}{u^2+v^2} + i\frac{v}{u^2+v^2}$$

$$x = \frac{u}{u^2+v^2} \quad \ldots.(1), \qquad y = \frac{-v}{u^2+v^2} \quad \ldots(2)$$

The general equation of circle is

$$a(x^2+y^2) + 2gx + 2fy + c = 0 \quad \ldots(3)$$

$$a\left[\frac{u^2}{(u^2+v^2)^2} + \frac{v^2}{(u^2+v^2)^2}\right] + 2g\left[\frac{u}{u^2+v^2}\right] + 2f\left[\frac{-v}{u^2+v^2}\right] + c = 0$$

$$a\frac{(u^2+v^2)}{(u^2+v^2)^2} + 2g\frac{u}{u^2+v^2} - 2f\frac{v}{u^2+v^2} + c = 0$$

The transformed equation is

$$c(u^2+v^2) + 2gu - 2fv + a = 0 \quad \ldots(4)$$

(i) $a \neq 0, c \neq 0$ $\Rightarrow$ circles not passing through the origin in $z -$ plane map into circles not passing through the origin in the $w -$ plane.

(ii) $a \neq 0, c = 0 \Rightarrow$ circles through the origin in $z-$plane map into straight lines not through the origin in the $w-$plane.

(iii) $a = 0, c \neq 0 \Rightarrow$ the straight lines not through the origin in $z-$plane map onto circles through the origin in the $w-$plane.

(iv) $a = 0, c = 0 \Rightarrow$ straight lines through the origin in $z-$plane map onto straight lines through the origin in the $w-$plane.

Example: 14 Find the image of the hyperbola $x^2 - y^2 = 1$ under the transformation $w = \frac{1}{z}$. [A.U M/J 2010, M/J 2012]

Solution:

Given $w = \frac{1}{z}$

$$x + iy = \frac{1}{Re^{i\phi}}$$

$$x + iy = \frac{1}{R}e^{-i\phi} = \frac{1}{R}[\cos\phi - i\sin\phi]$$

$$x = \frac{1}{R}\cos\phi,\ y = -\frac{1}{R}\sin\phi$$

Given $x^2 - y^2 = 1$

$$\Rightarrow \left[\frac{1}{R}\cos\phi\right]^2 - \left[\frac{-1}{R}\sin\phi\right]^2 = 1$$

$$\frac{cos^2\phi - sin^2\phi}{R^2} = 1$$

$$\cos 2\phi = R^2 \qquad i.e., R^2 = \cos 2\phi$$

which is lemniscate

4. Transformation $w = z^2$

Problems based on $w = z^2$

Example: 15 Discuss the transformation $w = z^2$. [Anna – May 2001]

Solution:

Given $w = z^2$

$$u + iv = (x + iy)^2 = x^2 + (iy)^2 + i2xy = x^2 - y^2 + i2xy$$

$$i.e., u = x^2 - y^2 \quad ....(1), \quad v = 2xy \quad ....(2)$$

Elimination:

$$(2) \Rightarrow x = \frac{v}{2y}$$

$$(1) \Rightarrow u = \left(\frac{v}{2y}\right)^2 - y^2$$

$$\Rightarrow u = \frac{v^2}{4y^2} - y^2$$

$$\Rightarrow 4uy^2 = v^2 - 4y^4$$

$$\Rightarrow 4uy^2 + 4y^4 = v^2$$

$$\Rightarrow y^2[4u + 4y^2] = v^2$$

$$\Rightarrow 4y^2[u + y^2] = v^2$$

$$\Rightarrow v^2 = 4y^2(y^2 + u)$$

when $y = c\ (\neq 0)$, we get

$$v^2 = 4c^2(u+c^2)$$

which is a parabola whose vertex at $(-c^2, 0)$ and focus at (0,0)

Hence, the lines parallel to X-axis in the z plane is mapped into family of confocal parabolas in the w plane.

when $y = 0$, we get $v^2 = 0$ i.e., $v = 0$, $u = x^2$ i.e., $u > 0$

Hence, the line $y = 0$, in the z plane are mapped into $v = 0$, in the w plane.

Elimination:

$$(2) \Rightarrow y = \frac{v}{2x}$$

$$(1) \Rightarrow u = x^2 - \left(\frac{v}{2x}\right)^2$$

$$\Rightarrow u = x^2 - \frac{v^2}{4x^2}$$

$$\Rightarrow \frac{v^2}{4x^2} = x^2 - u$$

$$\Rightarrow v^2 = (4x^2)(x^2 - u)$$

when $x = c\ (\neq 0)$, we get $v^2 = 4c^2(c^2 - u) = -4c^2(u - c^2)$

which is a parabola whose vertex at $(c^2, 0)$ and focus at (0,0) and axis lies along the u −axis and which is open to the left. Hence, the lines parallel to y axis in the z plane are mapped into confocal parabolas in the w plane when $x = 0$, we get $v^2 = 0$. i.e., $v = 0, u = -y^2$ i.e., $u < 0$

i.e., the map of the entire y axis in the negative part or the left half of the u −axis.

Example: 16 Find the image of the hyperbola $x^2 - y^2 = 10$ under the transformation $w = z^2$ if

$\boldsymbol{w = u + iv}$ **[Anna – May 1997]**

Solution:

Given $w = z^2$

$$u + iv = (x + iy)^2$$

$$= x^2 - y^2 + i2xy$$

$$i.e., u = x^2 - y^2 \quad \ldots\ldots(1)$$

$$v = 2xy \quad \ldots\ldots(2)$$

Given $x^2 - y^2 = 10$

i.e., $u = 10$

Hence, the image of the hyperbola $x^2 - y^2 = 10$ in the z plane is mapped into $u = 10$ in the w plane which is a straight line.

Example: 17 Determine the region of the $\boldsymbol{w}$ plane into which the circle $|\boldsymbol{z - 1}| = \boldsymbol{1}$ is mapped by the transformation $\boldsymbol{w = z^2}$.

Solution:

In polar form $z = re^{i\theta}, w = Re^{i\phi}$

Given $|z - 1| = 1$

i.e., $\left|re^{i\theta} - 1\right| = 1$

$$\Rightarrow |r\cos\theta + i\, r\sin\theta| = 1$$

$$\Rightarrow |(r\cos\theta - 1) + i\, r\sin\theta| = 1$$

$$\Rightarrow (r\cos\theta - 1)^2 + (r\sin\theta)^2 = 1^2$$

$$\Rightarrow r^2 \cos^2\theta + 1 - 2r\cos\theta + r^2 sin^2\theta = 1$$

$$\Rightarrow r^2[\cos^2\theta + sin^2\theta = 2r\cos\theta$$

$$\Rightarrow r^2 = 2r\cos\theta$$

$$\Rightarrow r = 2\cos\theta \quad \ldots(1)$$

Given $w = z^2$

$$Re^{i\phi} = (re^{i\theta})^2$$

$$Re^{i\phi} = r^2 e^{i2\theta}$$

$$\Rightarrow R = r^2, \qquad \phi = 2\theta$$

$$(1) \qquad \Rightarrow r^2 = (2\cos\theta)^2$$

$$\Rightarrow r^2 = 4cos^2\theta$$

$$= 4\left[\frac{1+\cos 2\theta}{2}\right]$$

$$r^2 = 2[1+\cos 2\theta]$$

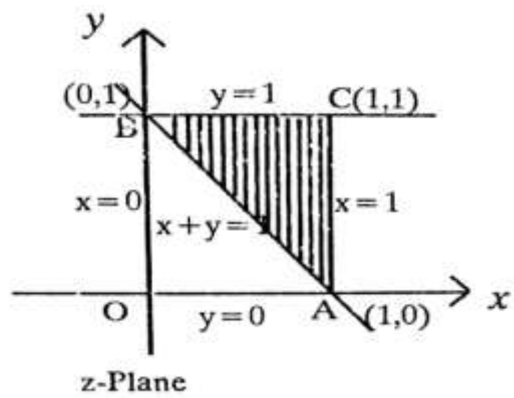
$$R = 2[1+\cos\phi] \qquad by\ (2),$$

which is a Cardioid

Example: 18 Find the image under the mapping $w = z^2$ of the triangular region bounded by $y = 1$, $x = 1$, and $x + y = 1$ and plot the same. [Anna, Oct., - 1997]

Solution:

In Z-plane given lines are $y = 1$, $x = 1$, $x + y = 1$



Given $w = z^2$

$$u + iv = (x + iy)^2$$

$$u + iv = x^2 - y^2 + 2xyi$$

Equating the real and imaginary parts, we get

$$u = x^2 - y^2 \quad ....(1)$$

$$v = 2xy \quad ...(2)$$

When $x = 1$	When $y = 1$
$(1) \Rightarrow u = 1 - y^2 \quad ....(3)$ $(2) \Rightarrow v = 2y \quad ...(4)$	$(1) \Rightarrow u = x^2 - 1 \quad ....(5)$ $(2) \Rightarrow v = 2x \quad ...(6)$
$(4) \Rightarrow v^2 = 4y^2$ $v^2 = 4(1 - u)\ by\ (3)$	$(6) \Rightarrow v^2 = 4x^2$ $= 4(u + 1) by\ (5)$

i.e., $v^2 = -4(u-1)$	

when $x + y = 1$

$$(1) \Rightarrow u = (x+y)(x-y)$$

$$u = x - y \qquad [\because x + y = 1]$$

$$u = \sqrt{(x+y)^2 - 4xy}$$

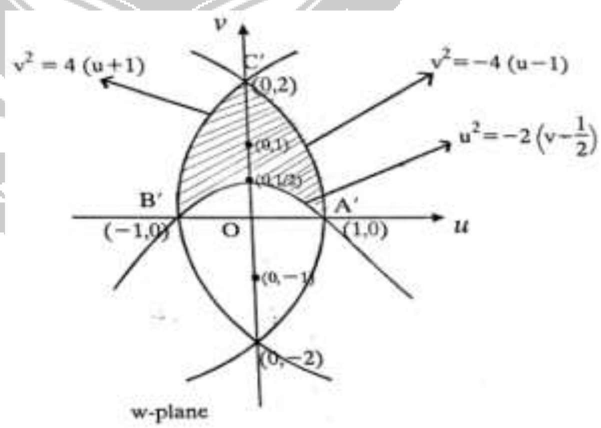
$$u = \sqrt{1 - 2v}$$

$$u^2 = 1 - 2v = -2\left(v - \frac{1}{2}\right)$$

$\therefore$ The image of $x = 1$ is $v^2 = -4(u-1)$

The image of $y = 1$ is $v^2 = 4(u+1)$

The image of $x + y = 1$ is $u^2 = -2\left(v - \frac{1}{2}\right)$



$v^2 = -4(u-1)$		
u	0	1
v	± 2	0

$v^2 = 4(u+1)$		
u	0	-1
v	± 2	0

$u^2 = -2\left(v - \frac{1}{2}\right)$			
u	0	1	-1
v	1/2	0	0

Problems based on critical points of the transformation

Example: 19 Find the critical points of the transformation $w^2 = (z - \alpha)(z - \beta)$. [A.U Oct., 1997] [A.U N/D 2014] [A.U M/J 2016 R-13]

Solution:

Given $w^2 = (z-\alpha)(z-\beta)$ ...(1)

Critical points occur at $\frac{dw}{dz} = 0$ and $\frac{dz}{dw} = 0$

Differentiation of (1) w. r. to z, we get

$$\Rightarrow 2w\frac{dw}{dz} = (z-\alpha) + (z-\beta)$$

$$= 2z - (\alpha + \beta)$$

$$\Rightarrow \frac{dw}{dz} = \frac{2z-(\alpha+\beta)}{2w} \quad \dots(2)$$

Case $(i)\,\frac{dw}{dz}=0$

$$\Rightarrow \frac{2z-(\alpha+\beta)}{2w}=0$$

$$\Rightarrow 2z-(\alpha+\beta)=0$$

$$\Rightarrow 2z=\alpha+\beta$$

$$\Rightarrow \ z=\frac{\alpha+\beta}{2}$$

Case $(ii)\,\frac{dz}{dw}=0$

$$\Rightarrow \frac{2w}{2z-(\alpha+\beta)}=0$$

$$\Rightarrow \frac{w}{z-\frac{\alpha+\beta}{2}}=0$$

$$\Rightarrow w=0 \ \Rightarrow (z-\alpha)\,(z-\beta)=0$$

$$\Rightarrow z=\alpha,\beta$$

$\therefore$ The critical points are $\frac{\alpha+\beta}{2},\alpha$ and β.

Example: 20 Find the critical points of the transformation $w=z^2+\frac{1}{z^2}$. [A.U A/M 2017 R-13]

Solution:

Given $w=z^2+\frac{1}{z^2}$ ... (1)

Critical points occur at $\frac{dw}{dz}=0$ and $\frac{dz}{dw}=0$

Differentiation of (1) w. r. to z, we get

$$\Rightarrow \frac{dw}{dz} = 2z - \frac{2}{z^3} = \frac{2z^4-2}{z^3}$$

Case $(i)\,\frac{dw}{dz} = 0$

$$\Rightarrow \frac{2z^4-2}{z^3} = 0 \Rightarrow 2z^4 - 2 = 0$$

$$\Rightarrow z^4 - 1 = 0$$

$$\Rightarrow z = \pm 1, \pm i$$

Case $(ii)\,\frac{dz}{dw} = 0$

$$\Rightarrow \frac{z^3}{2z^4-2} = 0 \Rightarrow z^3 = 0 \Rightarrow z = 0$$

∴ The critical points are ± 1 , $\pm i$, 0

Example: 21 Find the critical points of the transformation $w = z + \frac{1}{z}$

Solution:

Given $w = z + \frac{1}{z}$ ...(1)

Critical points occur at $\frac{dw}{dz} = 0$ and $\frac{dz}{dw} = 0$

Differentiation of (1) w. r. to z, we get

$$\Rightarrow \frac{dw}{dz} = 1 - \frac{1}{z^2} = \frac{z^2-1}{z^2}$$

Case $(i)\,\frac{dw}{dz} = 0$

$$\Rightarrow \frac{z^2-1}{z^3} = 0 \Rightarrow z^2 - 1 = 0 \Rightarrow z = \pm 1$$

Case $(ii)\,\frac{dz}{dw} = 0$

$$\Rightarrow \frac{z^3}{z^2-1} = 0 \Rightarrow z^2 = 0 \Rightarrow z = 0$$

$\therefore$ The critical points are $0, \pm 1$.

Example: 22 Find the critical points of the transformation $w = 1 + \frac{2}{z}$. [A.U N/D 2013 R-08]

Solution:

Given $w = 1 + \frac{2}{z}$ ... (1)

Critical points occur at $\frac{dw}{dz} = 0$ and $\frac{dz}{dw} = 0$

Differentiation of (1) w. r. to z, we get

$$\Rightarrow \frac{dw}{dz} = \frac{-2}{z^2}$$

Case (i) $\frac{dw}{dz} = 0$

$$\Rightarrow \frac{-2}{z^2} = 0$$

Case (ii) $\frac{dz}{dw} = 0$

$$\Rightarrow \frac{z^2}{2} = 0 \Rightarrow z = 0$$

$\therefore$ The critical points is $z = 0$

Example: 23 Prove that the transformation $w = \frac{z}{1-z}$ maps the upper half of the z plane into the upper half of the w plane. What is the image of the circle $|z| = 1$ under this transformation. [Anna, May – 2001]

Solution:

Given $|z| = 1$ is a circle

Centre = $(0,0)$

Radius = 1

Given $w = \frac{z}{1-z}$

$$\Rightarrow z = \frac{w}{w+1}$$

$$\Rightarrow |z| = \left|\frac{w}{w+1}\right| = \frac{|w|}{|w+1|}$$

Given $|z| = 1$

$$\Rightarrow \frac{|w|}{|w+1|} = 1$$

$$\Rightarrow |w| = |w+1|$$

$$\Rightarrow |u+iv| = |u+iv+1|$$

$$\Rightarrow \sqrt{u^2+v^2} = \sqrt{(u+1)^2+v^2}$$

$$\Rightarrow u^2+v^2 = (u+1)^2+v^2$$

$$\Rightarrow u^2+v^2 = u^2+2u+1+v^2$$

$$\Rightarrow 0 = 2u+1$$

$$\Rightarrow u = \frac{-1}{2}$$

Further the region $|z| < 1$ transforms into $u > \frac{-1}{2}$

BILINEAR TRANSFORMATION

Introduction

The transformation $w = \frac{az+b}{cz+d}, ad - bc \neq 0$ where a, b, c, d are complex numbers, is called a bilinear transformation.

This transformation was first introduced by A.F. Mobius, So it is also called Mobius transformation.

A bilinear transformation is also called a linear fractional transformation because $\frac{az+b}{cz+d}$ is a fraction formed by the linear fuctions $az - b$ and $cz + d$.

Theorem: 1 Under a bilinear transformation no two points in z plane go to the same point in w plane.

Proof:

Suppose z_1 and z_2 go to the same point in the w plane under the transformation $w = \frac{az+b}{cz+d}$.

Then $\frac{az_1+b}{cz_1+d} = \frac{az_2+b}{cz_2+d}$

$$\Rightarrow (az_1 + b)(cz_2 + d) = (az_2 + b)(cz_1 + d)$$

$$i.e., \quad (az_1 + b)(cz_2 + d) - (az_2 + b)(cz_1 + d) = 0$$

$$\Rightarrow acz_1 z_2 + adz_1 + bcz_2 + bd - acz_1 z_2 - adz_2 - bcz_1 - bd = 0$$

$$\Rightarrow (ad - bc)(z_1 - z_2) = 0$$

$$\text{or} \quad z_1 = z_2 \qquad [\because ad - bc \neq 0]$$

This implies that no two distinct points in the z plane go to the same point in w plane. So, each point in the z plane go to a unique point in the w plane.

Theorem: 2 The bilinear transformation which transforms z_1, z_2, z_3, into w_1, w_2, w_3 is $\frac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_2-w_1)} = \frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}$

Proof:

If the required transformation $w = \frac{az+b}{cz+d}$.

$$\Rightarrow w - w_1 = \frac{az+b}{cz+d} - \frac{az_1+b}{cz_1+d} = \frac{(ad-bc)(z-z_1)}{(cz+d)(cz_1+d)}$$

$$\Rightarrow (cz+d)(cz_1+d)(w-w_1) = (ad-bc)(z-z_1)$$

$$\Rightarrow (cz_2+d)(cz_3+d)(w_2-w_3) = (ad-bc)(z_2-z_3)$$

$$\Rightarrow (cz+d)(cz_3+d)(w-w_3) = (ad-bc)(z-z_3)$$

$$\Rightarrow (cz_2+d)(cz_1+d)(w_2-w_1) = (ad-bc)(z_2-z_1)$$

$$\Rightarrow \frac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_2-w_1)} = \frac{\left[\frac{(ad-bc)(z-z_1)}{(cz+d)(cz_1+d)}\right]\left[\frac{(ad-bc)(z_2-z_3)}{(cz_2+d)(cz_3+d)}\right]}{\left[\frac{(ad-bc)(z-z_3)}{(cz+d)(cz_3+d)}\right]\left[\frac{(ad-bc)(z_2-z_1)}{(cz_2+d)(cz_1+d)}\right]}$$

$$= \frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}$$

Now, $\frac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_2-w_1)} = \frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}$ ... (1)

Let : $A = \frac{w_2-w_3}{w_2-w_1}, B = \frac{z_2-z_3}{z_2-z_1}$

$$(1) \Rightarrow \frac{w-w_1}{w-w_3}A = \frac{z-z_1}{z-z_3}B$$

$$\frac{wA-w_1A}{w-w_3} = \frac{zB-z_1B}{z-z_3}$$

$$\Rightarrow wAz - wAz_3 - w_1Az + w_1Az_3 = wBz - wz_1B - w_3zB + w_3z_1B$$

$$\Rightarrow w[(A-B)z + (Bz_1 - Az_3)] = (Aw_1 - Bw_3)z + (Bw_3z_1 - Aw_1z_3)$$

$$\Rightarrow w = \frac{(Aw_1 - Bw_3)z + (Bw_3z_1 - Aw_1z_3)}{(A-B)z + (Bz_1 - Az_3)}$$

$\frac{az+b}{cz+d}$, Hence $a = Aw_1 - Bw_3, b = Bw_3z_1 - Aw_1z_3, c = A - B, d = Bz_1 - Az_3$

Cross ratio

Definition:

Given four point z_1, z_2, z_3, z_4 in this order, the ratio $\frac{(z-z_1)(z_3-z_4)}{(z_2-z_3)(z_4-z_1)}$ is called the cross ratio of the points.

Note: (1) $w = \frac{az+b}{cz+d}$ can be expressed as $cwz + dw - (az + b) = 0$

It is linear both in w and z that is why, it is called bilinear.

Note: (2) This transformation is conformal only when $\frac{dw}{dz} \neq 0$

$$i.e., \frac{ad - bc}{(cz + d)^2} \neq 0$$

$$i.e., ad - bc \neq 0$$

If $ad - bc \neq 0$, every point in the z plane is a critical point.

Note: (3) Now, the inverse of the transformation $w = \frac{az+b}{cz+d}$ is $z = \frac{-dw+b}{cw-a}$ which is also a bilinear transformation except $w = \frac{a}{c}$.

Note: (4) Each point in the plane except $z = \frac{-d}{c}$ corresponds to a unique point in the w plane.

The point $z = \frac{-d}{c}$ corresponds to the point at infinity in the w plane.

Note: (5) The cross ratio of four points

$\frac{(w_1 - w_2)(w_3 - w_4)}{(w_2 - w_3)(w_4 - w_1)} = \frac{(z_1 - z_2)(z_3 - z_4)}{(z_2 - z_3)(z_4 - z_1)}$ is invariant under bilinear transformation.

Note: (6) If one of the points is the point at infinity the quotient of those difference which involve this points is replaced by 1.

Suppose $z_1 = \infty$, then we replace $\frac{z-z_1}{z_2-z_1}$ by 1 (or)Omit the factors involving ∞

Example: 1 Find the fixed points of $w = \frac{2zi+5}{z-4i}$.

Solution:

The fixed points are given by replacing $w\ by\ z$

$$z = \frac{2zi+5}{z-4i}$$

$$z^2 - 4iz = 2zi + 5\ ;\ z^2 - 6iz - 5 = 0$$

$$z = \frac{6i \pm \sqrt{-36+20}}{2} \qquad \therefore z = 5i, i$$

Example: 2 Find the invariant points of $w = \frac{1+z}{1-z}$

Solution:

The invariant points are given by replacing $w\ by\ z$

$$z = \frac{1+z}{1-z}$$

$$\Rightarrow z - z^2 = 1 + z$$

$$\Rightarrow z^2 = -1$$

$$\Rightarrow z = \pm i$$

Example: Obtain the invariant points of the transformation $w = 2 - \frac{2}{z}$. [Anna, May 1996]

Solution:

The invariant points are given by

$$z = 2 - \frac{2}{z}\ ; \qquad z = \frac{2z-2}{z}$$

$$z^2 = 2z - 2; \qquad z^2 - 2z + 2 = 0$$

$$z = \frac{2 \pm \sqrt{4-8}}{2} = \frac{2 \pm \sqrt{-4}}{2} = \frac{2 \pm 2i}{2} = 1 \pm i$$

Example: 4 Find the fixed point of the transformation $w = \frac{6z-9}{z}$. [A.U N/D 2005]

Solution:

The fixed points are given by replacing $w = z$

i.e., $w = \frac{6z-9}{z} \Rightarrow z = \frac{6z-9}{z}$

$$\Rightarrow z^2 = 6z - 9$$

$$\Rightarrow z^2 - 6z + 9 = 0$$

$$\Rightarrow (z-3)^2 = 0$$

$$\Rightarrow z = 3{,}3$$

The fixed points are $3, 3$.

Example: 5 Find the invariant points of the transformation $w = \frac{2z+6}{z+7}$. [A.U M/J 2009]

Solution:

The invariant (fixed) points are given by

$$w = \frac{2z+6}{z+7}$$

$$\Rightarrow z^2 + 7z = 2z + 6$$

$$\Rightarrow z^2 + 5z - 6 = 0$$

$$\Rightarrow (z+6)(z-1) = 0$$

$$\Rightarrow z = -6, z = 1$$

Example: 6 Find the invariant points of $f(z) = z^2$. [A.U M/J 2014 R-13]

Solution:

The invariant points are given by $z = w = f(z)$

$$\Rightarrow z = z^2$$

$$\Rightarrow z^2 - z = 0$$

$$\Rightarrow z(z-1) = 0$$

$$\Rightarrow z = 0, \quad z = 1$$

Example 7 Find the invariant points of a function $f(z) = \frac{z^3+7z}{7-6zi}$. [A.U D15/J16 R-13]

Solution:

Given $w = f(z) = \frac{z^3+7z}{7-6zi}$

The invariant points are given by

$$\Rightarrow z = \frac{z^3+7z}{7-6zi}$$

$$\Rightarrow 7 - 6zi = z^2 + 7$$

$$\Rightarrow -6zi = z^2 \Rightarrow z^2 + 6zi = 0 \Rightarrow z(z+6i) = 0$$

$$\Rightarrow z = 0,\ z = -6i$$

PROBLEMS BASED ON BILINEAR TRANSFORMATION

Example: 8 Find the bilinear transformation that maps the points $z = 0, -1, i$ into the points $w = i, 0, \infty$ respectively. [A.U. A/M 2015 R-13, A.U N/D 2013, N/D 2014]

Solution:

Given $z_1 = 0,\ \ z_2 = -1,\ \ z_3 = i,$

$w_1 = i,\ \ w_2 = 0,\ \ w_3 = \infty,$

Let the required transformation be

$$\frac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_2-w_1)} = \frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}$$

[omit the factors involving w_3, since $w_3 = \infty$]

$$\Rightarrow \frac{w-w_1}{w_2-w_1} = \frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}$$

$$\Rightarrow \frac{w-i}{0-i} = \frac{(z-0)(-1-i)}{(z-i)(-1-0)}$$

$$\Rightarrow \frac{w-i}{-i} = \frac{z}{(z-i)}(1+i)$$

$$\Rightarrow w-i = \frac{z}{(z-i)}(-i+1)$$

$$\Rightarrow w = \frac{z}{(z-i)}(-i+1)+i = \frac{-iz+z+iz+1}{(z-i)} = \frac{z+1}{z-i}$$

Aliter: Given $z_1 = 0,\ \ z_2 = -1,\ \ z_3 = i,$

$w_1 = i,\ \ w_2 = 0,\ \ w_3 = \infty,$

Let the required transformation be

$$w = \frac{az+b}{cz+d} \dots (1),\ ad-bc \neq 0$$

$$i = \frac{b}{d}$$

$w_1 = \frac{az_1+b}{cz_1+d}$	$w_2 = \frac{az_2+b}{cz_2+d}$	$w_3 = \frac{az_3+b}{cz_3+d}$
$i = \frac{b}{d}$	$0 = \frac{-a+b}{-c+d}$	$\frac{1}{0} = \frac{ai+b}{ci+d}$
$b = di$	$\Rightarrow -a+b=0$	$\Rightarrow ci+d=0$
	$\Rightarrow a=b$	$\Rightarrow d=-ci$

$$\therefore a = b = di = c$$

$$\therefore (1) \Rightarrow w = \frac{az+a}{az+\frac{a}{i}} = \frac{z+1}{z+\frac{1}{i}} = \frac{z+1}{z-i}$$

Example: 9 Find the bilinear transformation that maps the points $\infty, i, 0$ onto $0, i, \infty$ respectively[Anna, May 1997] [A.U N/D 2012] [A.U A/M 2017 R-08]

Solution:

Given $z_1 = \infty,\ \ z_2 = i,\ \ z_3 = 0, w_1 = 0,\ \ w_2 = i,\ \ w_3 = \infty,$

Let the required transformation be

$$\frac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_2-w_1)} = \frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}$$

[omit the factors involving z_1, and w_3, since $z_1 = \infty, w_3 = \infty$]

$$\Rightarrow \frac{w-w_1}{w_2-w_1} = \frac{(z_2-z_3)}{z-z_3}$$

$$\Rightarrow \frac{w-0}{i-0} = \frac{i-0}{z-0}$$

$$\Rightarrow w = \frac{-1}{z}$$

Example: 10 Find the bilinear transformation which maps the points $1, i, -1$ onto the points $0, 1, \infty$, show that the transformation maps the interior of the unit circle of the $z-$ plane onto the upper half of the $w-$ plane. [A.U. May 2001] [A.U M/J 2014] [A.U D15/J16 R-13]

Solution:

Given $z_1 = 1,\ \ z_2 = i,\ \ z_3 = -1$

$w_1 = 0,\ \ w_2 = 1,\ \ w_3 = \infty,$

Let the transformation be

$$\frac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_2-w_1)} = \frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}$$

[Omit the factors involving w_3, since , $w_3 = \infty$]

$$\Rightarrow \frac{w-w_1}{w_2-w_1} = \frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}$$

$$\Rightarrow \frac{w-0}{1-0} = \frac{(z-1)(i+1)}{(z+1)(i-1)} \qquad \because \left[\left(\frac{i+1}{i-1}\right)\left(\frac{i+1}{i+1}\right)\right] = \left[\frac{i^2+i+i+1}{i^2-i^2}\right] = \left[\frac{2i}{-2}\right] = -i$$

$$\Rightarrow w = \frac{(z-1)(i+1)}{(z+1)(i-1)}$$

$$= \frac{z-1}{z+1}[-i]$$

$$\Rightarrow w = \frac{(-i)z+i}{(1)z+1}\left[\because w = \frac{az+b}{cz+d}, ad-bc \neq 0 \text{ Form}\right]$$

To find z:

$$\Rightarrow wz + w = -iz + i$$

$$\Rightarrow wz + iz = -w + i$$

$$\Rightarrow z[w+i] = -w + i$$

$$\Rightarrow z = \frac{(w-i)}{w+i}$$

To prove: $|z| < 1$ maps $v > 0$

$$\Rightarrow |z| < 1$$

$$\Rightarrow \left|\frac{-(w-i)}{w+i}\right| < 1$$

$$\Rightarrow \left|\frac{w-i}{w+i}\right| < 1$$

$$\Rightarrow |w-i| < |w+i|$$

$$\Rightarrow |u+iv-i| < |u+iv+i|$$

$$\Rightarrow |u+i(v-1)| < |u+i(v+i)|$$

$$\Rightarrow u^2+(v-1)^2 < u^2+(v+1)^2$$

$$\Rightarrow (v-1)^2 < (v+1)^2$$

$$\Rightarrow \ v^2-2v+1 < v^2+2v+1$$

$$\Rightarrow \ -4v < 0$$

$$\Rightarrow v > 0$$

Example: 11 Determine the bilinear transformation that maps the points $-1, 0, 1$, in the z plane onto the points $0, i, 3i$ in the w plane. [Anna, May 1999]

Solution:

Given $z_1 = -1, \ \ z_2 = 0, \ \ z_3 = 1,$

$w_1 = 0, \ \ w_2 = i, \ \ w_3 = 3i,$

Let the required transformation be

$$\frac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_2-w_1)} = \frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}$$

$$\Rightarrow \ \frac{(w-0)(i-3i)}{(w-3i)(i-0)} = \frac{[z-(-1)][0-1]}{(z-1)[0-(-1)]}$$

$$\Rightarrow \frac{w(-2i)}{(w-3i)(i)} = \frac{(z+1)(-1)}{(z-1)(1)}$$

$$\Rightarrow \frac{-2w}{w-3i} = \frac{z+1}{z-1}$$

$$\Rightarrow \frac{2w}{w-3i} = \frac{z+1}{z-1}$$

$$\Rightarrow 2wz - 2w = wz + w - 3zi - 3i$$

$$\Rightarrow 2wz - 2w - wz - w = -3i(z+1)$$

$$\Rightarrow w[2z - 2 - z - 1] = -3i(z+1)$$

$$\Rightarrow w[z-3] = -3i(z+1)$$

$$\Rightarrow w = -3i\frac{(z+1)}{(z-3)}$$

Note: Either image or object or both are infinity should not apply the following Aliter method.

Aliter:

Given $z_1 = -1, \;\; z_2 = 0, \;\; z_3 = 1,$

$w_1 = 0, \;\; w_2 = i, \;\; w_3 = 3i,$

Let the required transformation be

$$\frac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_2-w_1)} = \frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}$$

Let $A = \frac{w_2-w_3}{w_2-w_1} = \frac{i-3i}{i-0} = \frac{-2i}{i} = -2$

$$B = \frac{z_2-z_3}{z_2-z_1} = \frac{0-1}{0+1} = -1$$

$$\Rightarrow a = Aw_1 - Bw_3 = 0 + 3i = 3i$$

$$\Rightarrow b = Bw_3z_1 - Aw_1z_3 = (-1)(3i)(-1) - 0 = 3i$$

$$\Rightarrow c = A - B = (-2) - (-1) = -1$$

$$\Rightarrow d = Bz_1 - Az_3 = (-1)(-1) - (-2)(1) = 3$$

We know that, $w = \frac{az+b}{cz+d}, ad - bc \neq 0$

$$\therefore w = \frac{(3i)+z(3i)}{(-1)z+3}$$

Example: 12 Find the bilinear transformation which maps the points $-2, 0, 2$ into the points $w = 0, 1, -i$ respectively.[Anna, May 2002]

Solution:

Given $z_1 = -1, \ z_2 = 0, \ z_3 = 2,$

$w_1 = 0, \ w_2 = i, \ w_3 = -i,$

Let the required transformation be

$$\frac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_2-w_1)} = \frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}$$

$$\text{Let } A = \frac{w_2-w_3}{w_2-w_1} = \frac{i+i}{i-0} = \frac{2i}{i} = 2$$

$$B = \frac{z_2-z_3}{z_2-z_1} = \frac{0-2}{0+2} = -1$$

$$\Rightarrow a = Aw_1 - Bw_3 = (2)(0) - (-1)(-1) = -i$$

$$\Rightarrow b = Bw_3z_1 - Aw_1z_3 = (-1)(-i)(-2) - (2)(0)(2) = -2i$$

$$\Rightarrow c = A - B = 2 - (-1) = 3$$

$$\Rightarrow d = Bz_1 - Az_3 = (-1)(-1) - (2)(2) = -2$$

We know that, $w = \frac{az+b}{cz+d}, ad - bc \neq 0$

$$\therefore w = \frac{(-i)z+(-2i)}{3z+(-2)}$$

Example: 13 Find the bilinear transformation which maps $z = 1, i, -1$ respectively onto $w = i, 0, -i$. Hence find the fixed points. [A.U, May 2001] [A.U April 2016 R-15 U.D]

Solution:

Given $z_1 = 1, \; z_2 = i, \; z_3 = -1,$

$w_1 = i, \; w_2 = 0, \; w_3 = -i,$

Let the required transformation be

$$\frac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_2-w_1)} = \frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}$$

$$Let \quad A = \frac{w_2-w_3}{w_2-w_1} = \frac{0+i}{0-i} = -1$$

$$B = \frac{z_2-z_3}{z_2-z_1} = \frac{i+1}{i-1} = -i$$

$$\Rightarrow a = Aw_1 - Bw_3 = (-1)(i) - (-i)(-i) = -i + 1$$

$$\Rightarrow b = Bw_3z_1 - Aw_1z_3 = (-i)(-i)(1) - (-1)(i)(-1) = -1 - i$$

$$\Rightarrow c = A - B = (-1) - (-i) = -1 + i$$

$$\Rightarrow d = Bz_1 - Az_3 = (-i)(1) - (-1)(-1) = -i - 1$$

We know that, $w = \frac{az+b}{cz+d}, ad - bc \neq 0$

$$\therefore w = \frac{(-i+1)z+(-1-i)}{(-1+i)z+(-i-1)} = \frac{iz+1}{(-i)z+1}$$

Example: 14 Find the bilinear transformation which maps $z = 0$ onto $w = -i$ and has -1 and 1 as the invariant points. Also show that under this

transformation the upper half of the z plane maps onto the interior of the unit circle in the w plane. [A.U A/M 2017 R-13]

Solution:

Given $z_1 = 0,\ \ z_2 = -1,\ \ z_3 = 1,$

$w_1 = -i,\ \ w_2 = -1,\ \ w_3 = 1,$

Let the required transformation be

$$\frac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_2-w_1)} = \frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}$$

$$Let \quad A = \frac{w_2-w_3}{w_2-w_1} = \frac{-1-1}{-1+i} = \frac{-2}{-1+i} = 1+i$$

$$B = \frac{z_2-z_3}{z_2-z_1} = \frac{-1-1}{-1-0} = 2$$

$$\Rightarrow a = Aw_1 - Bw_3 = (1+i)(-i) - 2(1) = -i + 1 - 2 = -i - 1$$

$$\Rightarrow b = Bw_3z_1 - Aw_1z_3 = (2)(1)(0) - (1+i)(-i)(1) = i - 1$$

$$\Rightarrow c = A - B = (1+i) - 2 = i - 1$$

$$\Rightarrow d = Bz_1 - Az_3 = (2)(0) - (1+i)(1) = -(1+i)$$

We know that, $w = \frac{az+b}{cz+d}, ad - bc \neq 0$

$$\therefore w = \frac{(-i+1)z+(i-1)}{(i-1)z+(-1-i)} = \frac{z+(-i)}{(-i)z+1}$$

We know that, $z = \frac{-dw+b}{cw-a} = \frac{-w-i}{-iw-1} = \frac{w+i}{1+wi}$

$$z = \frac{u+iv+i}{1(u+iv)i}$$

$$= \frac{u+iv+i}{1+iu-v} = \frac{u+iv+i}{(1-v)+iu}$$

$$= \left[\frac{u+iv+i}{(1-v)+iu}\right]\left[\frac{1-v-iu}{(1-v)-iu}\right]$$

$$= \frac{u-uv-iu^2+iv-iv^2+uv+i-iv+u}{(1-v)^2+u^2}$$

$$x+iy = \frac{2u+i[-u^2-v^2+1]}{(1-v)^2+u^2}$$

$$\Rightarrow \quad y \quad = \frac{1-u^2-v^2}{(1-v)^2+u^2}$$

Upper half of the z −plane

$$\Rightarrow y \geq 0$$

$$\Rightarrow \frac{1-u^2-v^2}{(1-v)^2+u^2} \geq 0$$

$$\Rightarrow 1-u^2-v^2 \geq 0$$

$$\Rightarrow 1 \geq u^2+v^2$$

$$\Rightarrow u^2+v^2 \leq 1$$

Therefore the upper half of the z −plane maps onto the interior of the unit circles in the w-plane.

Example: 15 Find the Bilinear transformation that maps the points $1+i, -i, 2-i$ of the z −plane into the points $0, 1, i$ of the w-plane.

[A.U M/J 2007, N/D 2007]

Solution:

Given $z_1 = 1+i \qquad w_1 = 0$

$z_2 = -i \qquad w_2 = 1$

$$z_3 = 2 - i \qquad\qquad w_3 = i$$

Let the required transformation be

$$\frac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_2-w_1)} = \frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}$$

$$Let \quad A = \frac{w_2-w_3}{w_2-w_1} = \frac{1-i}{1-0} = 1 - i = \frac{1-i}{1+2i}(1+2i) = \frac{3+i}{1+2i}$$

$$B = \frac{z_2-z_3}{z_2-z_1} = \frac{-i-2+i}{-i-1-i} = \frac{-2}{-1-2i} = \frac{2}{1+2i}$$

$$\Rightarrow a = Aw_1 - Bw_3 = \left(\frac{3+i}{1+2i}\right)(0) - \left(\frac{2}{1+2i}\right)(i) = \frac{-2i}{1+2i}$$

$$\Rightarrow b = Bw_3z_1 - Aw_1z_3 = \left(\frac{2}{1+2i}\right)(i)(1+i) - 0 = \frac{-2+2i}{1+2i}$$

$$\Rightarrow c = A - B = \frac{3+i}{1+2i} - \frac{2}{1+2i} = \frac{1+i}{1+2i}$$

$$\Rightarrow d = Bz_1 - Az_3 = \left(\frac{2}{1+2i}\right)(1+i) - \left(\frac{3+i}{1+2i}\right)(2-i) = \frac{-5+3i}{1+2i}$$

We know that, $w = \frac{az+b}{cz+d}, ad - bc \neq 0$

$$\Rightarrow \ w = \frac{\left(\frac{-2i}{1+2i}\right)z + \left(\frac{2i-2}{1+2i}\right)}{\left(\frac{1+i}{1+2i}\right)z + \left(\frac{3i-5}{1+2i}\right)}$$

$$\Rightarrow w = \frac{(-2i)z+(2i-2)}{(1+i)z+(3i-5)}$$

Verification:

(i) If $z = 1 + i$,then

$$w = \frac{(-2i)(1+i)+(2i-2)}{(1+i)(1+i)+(3i-5)}$$

$$= \frac{-2i+2+2i-2}{(1+i)(1+i)+(3i-5)} \quad = 0$$

(ii) If $z = -i$, then

$$w = \frac{(-2i)(-i)+(2i-2)}{(1+i)(-i)+(3i-5)}$$

$$= \frac{-2+2i-2}{-i+1+3i-5} = \frac{2i-4}{2i-4} = 1$$

(iii)If $z = -i$, then

$$w = \frac{(-2i)(2-i)+(2i-2)}{(1+i)(2-i)+(3i-5)} = \frac{-4i-2+2i-2}{2-i+2i+1+3i-5}$$

$$= \frac{-2i-4}{4i-2} = \frac{-i-2}{2i-1} \times \frac{2i+1}{2i+1}$$

$$= \frac{2-i-4i-2}{-4-1} = \frac{-5i}{-5} = i$$