UNIT – II – TWO DIMENSIONAL VARIABLES

Let S be the sample space. Let $X = X(s)$ and $Y = Y(s)$ be two functions each assigning a real number to each outcome $s \in S$. Then (X, Y) is a two dimensional random variable.

Let X, Y be a two dimensional discrete random variable for each possible outcome (X_i , Y_j) . We associate a number $P(X_i , Y_j)$ representing $\left[X = x_i , Y = y_j\right]$ and satisfies the following conditions

i) $P\left[x_i, y_j\right] \geq 0$

ii) $\sum\sum p\left(x_i, y_j\right) = 1$

The function $P\left[x_i, y_j\right]$ is called joint probability mass function of x , y

Conditional distribution of X given Y

$$P\left[X = x_i/Y = y_j\right] = \frac{P\left[X = x_i \cap Y = y_j\right]}{P[Y = y_j]}$$

$$= \frac{P\left[X = x_i, Y = y_j\right]}{P[Y = y_j]}$$

$$P\left[Y = y_j/X = x_i\right] = \frac{P\left[Y = y_j \cap X = x_i\right]}{P[X = x_i]}$$

$$= \frac{P\left[X = x_i, Y = y_j\right]}{P[X = x_i]}$$

Test of independent

$$P[X = x_i, Y = y_j] = P[X = x_i], P[Y = y_j]$$

Problems under on Marginal Distribution

1.The joint probability marginal function of X , Y is given by $P(xy) = K(2x + 3y)$, $x = 0, 1, 2$, $y = 1, 2, 3$ find K. Find all the marginal distribution and conditional probability distribution . Also probability distribution X + Y.

Solution:

	1	2	3	$\sum x$
0	3K	6K	9K	18K
1	5K	8K	11K	24K
2	7K	10K	13K	30K
$\sum y$	15K	24K	33K	72K

We know that $\sum\sum P(x, y) = 1$

$\Rightarrow 72K = 1$

$\Rightarrow K = \frac{1}{72}$

Marginal distribution

X	0	1	2
P(X)	$\frac{18}{72}$	$\frac{24}{72}$	$\frac{30}{72}$

Y	1	2	3
P(Y)	$\frac{15}{72}$	$\frac{24}{72}$	$\frac{33}{72}$

Conditional distribution at x given y

$$P[X = 0/Y = 1] = \frac{P[X = 0, Y = 1]}{P[Y = 1]} = \frac{3/72}{15/72} = \frac{1}{5}$$

$$P[X = 2/Y = 2] = \frac{P[X = 2, Y = 2]}{P[Y = 2]} = \frac{6/72}{24/72} = \frac{1}{4}$$

$$P[X = 0/Y = 3] = \frac{P[X = 0, Y = 3]}{P[Y = 3]} = \frac{9/72}{33/72} = \frac{3}{11}$$

$$P[X = 1/Y = 1] = \frac{P[X = 1, Y = 2]}{P[Y = 1]} = \frac{5/72}{15/72} = \frac{1}{3}$$

$$P[X = 1/Y = 2] = \frac{P[X = 1, Y = 2]}{P[Y = 2]} = \frac{8/72}{24/72} = \frac{1}{3}$$

$$P[X = 1/Y = 3] = \frac{P[X = 1, Y = 3]}{P[Y = 3]} = \frac{11/72}{33/72} = \frac{1}{3}$$

$$P[X = 2/Y = 1] = \frac{P[X = 2, Y = 1]}{P[Y = 1]} = \frac{7/72}{15/72} = \frac{7}{15}$$

$$P[X = 2/Y = 2] = \frac{P[X = 2, Y = 2]}{P[Y = 2]} = \frac{10/72}{24/72} = \frac{5}{12}$$

$$P[X = 2/Y = 3] = \frac{P[X = 2, Y = 3]}{P[Y = 3]} = \frac{13/72}{33/72} = \frac{13}{33}$$

Conditional distribution at y given x

$$P[Y = 1/X = 0] = \frac{P[Y = 1, X = 0]}{P[X = 0]} = \frac{3/72}{18/72} = \frac{1}{6}$$

$$P[Y = 1/X = 1] = \frac{P[Y = 1, X = 1]}{P[X = 1]} = \frac{5/72}{24/72} = \frac{5}{24}$$

$$P[Y = 1/X = 2] = \frac{P[Y = 1, X = 2]}{P[X = 2]} = \frac{7/72}{30/72} = \frac{7}{30}$$

$$P[Y = 2/X = 0] = \frac{P[Y = 2, X = 0]}{P[X = 0]} = \frac{6/72}{18/72} = \frac{1}{3}$$

$$P[Y = 2/X = 1] = \frac{P[Y = 2, X = 1]}{P[X = 1]} = \frac{8/72}{24/72} = \frac{1}{3}$$

$$P[Y = 2/X = 2] = \frac{P[Y = 2, X = 2]}{P[X = 2]} = \frac{10/72}{30/72} = \frac{1}{3}$$

$$P[Y = 3/X = 0] = \frac{P[Y = 3, X = 0]}{P[X = 0]} = \frac{9/72}{18/72} = \frac{1}{2}$$

$$P[Y = 3/X = 1] = \frac{P[Y = 3, X = 1]}{P[X = 1]} = \frac{11/72}{24/72} = \frac{11}{24}$$

$$P[Y = 3/X = 2] = \frac{P[Y = 3, X = 2]}{P[X = 2]} = \frac{13/72}{30/72} = \frac{13}{30}$$

Distribution function of $x + y$

1	P_{01}	$3/72$
2	$P_{02} + P_{11}$	$11/72$
3	$P_{03} + P_{12} + P_{21}$	$24/72$
4	$P_{13} + P_{22}$	$21/72$
5	P_{23}	$13/72$

2. The joint distribution of X and Y is given by $f(x,y) = \frac{x+y}{21}, x = 1,2,3 ; y = 1,2$.

Find the marginal distributions.

Solution:

Given $f(x,y) = \frac{x+y}{21}, x = 1,2,3 ; y = 1,2$

$\Rightarrow f(1,1) = \frac{2}{21}, f(1,2) = \frac{3}{21}, f(2,1) = \frac{3}{21}, f(2,2) = \frac{4}{21}, f(3,1) = \frac{4}{21}, f(3,2) = \frac{5}{21}$

The marginal distributions are given in the table.

<table>
<tr><td colspan="2" rowspan="2"></td><td colspan="3">X</td><td rowspan="2">$P_Y(y) = P(Y = y)$</td></tr>
<tr><td>1</td><td>2</td><td>3</td></tr>
<tr><td rowspan="2">Y</td><td>1</td><td>$\frac{2}{21}$ P(1,1)</td><td>$\frac{3}{21}$ P(1,2)</td><td>$\frac{4}{21}$ P(1,3)</td><td>$\frac{9}{21}$</td></tr>
<tr><td>2</td><td>$\frac{3}{21}$ P(2,1)</td><td>$\frac{4}{21}$ P(2,2)</td><td>$\frac{5}{21}$ P(2,3)</td><td>$\frac{12}{21}$</td></tr>
<tr><td colspan="2">$P_X(x) = P(X = x)$</td><td>$\frac{5}{21}$</td><td>$\frac{7}{21}$</td><td>$\frac{9}{21}$</td><td>1</td></tr>
</table>

The marginal distribution of X

$P_X(1) = P(X = 1) = \frac{5}{21}$, $P_X(2) = P(X = 2) = \frac{7}{21}$, $P_X(3) = P(X = 3) = \frac{9}{21}$

The marginal distribution of Y

$P_Y(1) = P(Y = 1) = \frac{9}{21}$, $P_Y(2) = P(Y = 2) = \frac{12}{21}$

Problems under on Conditional Distribution

1. The two dimensional random variable (X, Y) has the joint probability mass function $f(x, y) = \frac{x+2y}{27}, x = 0, 1, 2 ; y = 0, 1, 2$. Find the conditional distribution of Y for X= x. Also find the conditional distribution of Y / X = 1

Solution:

We know that the conditional probability distribution of Y for $X = x$ is

$f\left(\frac{y}{x}\right) = \frac{f(x,y)}{f(x)}$ where $f(x, y)$ is the joint probability function of X and y.

To find $f(x, y)$ Marginal Distributions

Given $f(x, y) = \frac{x+2y}{27}, x = 0,1,2\,; y = 0,1,2.$

		X: 0	X: 1	X: 2	$P_Y(y) = P(Y = y)$
Y	0	0 P(0,0)	$\frac{1}{27}$ P(1,0)	$\frac{2}{27}$ P(2,0)	$\frac{3}{27}$ P(Y=0)
	1	$\frac{2}{27}$ P(0,1)	$\frac{3}{27}$ P(1,1)	$\frac{4}{27}$ P(2,1)	$\frac{9}{27}$ P(Y=1)
	2	$\frac{4}{27}$ P(0,2)	$\frac{5}{27}$ P(1,2)	$\frac{6}{27}$ P(2,2)	$\frac{15}{27}$ P(Y=2)
$P_X(x) = P(X = x)$		$\frac{6}{27}$ P(X=0)	$\frac{9}{27}$ P(X=1)	$\frac{12}{27}$ P(X=2)	1

The Conditional Probability of Y for X= x is given by $f\left(\frac{y}{x}\right) = \frac{f(x,y)}{f(x)}$

By using the above table we get the conditional probability of Y for $X = x$ as follows

When $x = 0$,

$$P[Y=0/X=0]=\frac{P[X=0,Y=0]}{P[X=0]}=\frac{0}{6/27}=0$$

$$P[Y=1/X=0]=\frac{P[X=0,Y=1]}{P[X=0]}=\frac{2/27}{6/27}=\frac{1}{3}$$

$$P[Y=2/X=0]=\frac{P[X=0,Y=2]}{P[X=0]}=\frac{4/27}{6/27}=\frac{2}{3}$$

When $x = 1$,

$$P[Y=0/X=1]=\frac{P[X=1,Y=0]}{P[X=1]}=\frac{1/27}{9/27}=\frac{1}{9}$$

$$P[Y=1/X=1]=\frac{P[X=1,Y=1]}{P[X=1]}=\frac{3/27}{9/27}=\frac{1}{3}$$

$$P[Y=2/X=1]=\frac{P[X=1,Y=2]}{P[X=1]}=\frac{5/27}{9/27}=\frac{5}{9}$$

When $x = 2$,

$$P[Y=0/X=2]=\frac{P[X=2,Y=0]}{P[X=2]}=\frac{2/27}{12/27}=\frac{1}{6}$$

$$P[Y=1/X=2]=\frac{P[X=2,Y=1]}{P[X=2]}=\frac{4/27}{12/27}=\frac{1}{3}$$

$$P[Y=2/X=2]=\frac{P[X=2,Y=2]}{P[X=2]}=\frac{6/27}{12/27}=\frac{1}{2}$$

Table of f(y/x)			
X \ Y	0	1	2
0	0	$\frac{1}{3}$	$\frac{2}{3}$
1	$\frac{1}{9}$	$\frac{1}{3}$	$\frac{5}{9}$
2	$\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{2}$

The conditional distribution of Y given X = 1 is given in the table

Y	Table of $f(y/x = 1)$
0	$P[Y = 0/X = 1] = \frac{P[X = 1, Y = 0]}{P[X = 1]} = \frac{1/27}{9/27} = \frac{1}{9}$
1	$P[Y = 1/X = 1] = \frac{P[X = 1, Y = 1]}{P[X = 1]} = \frac{3/27}{9/27} = \frac{1}{3}$
2	$P[Y = 2/X = 1] = \frac{P[X = 1, Y = 2]}{P[X = 1]} = \frac{5/27}{9/27} = \frac{5}{9}$

2. The joint probability mass function of X and Y is

X \ Y	0	1	2
0	**0.10**	**0.04**	**0.02**
1	**0.08**	**0.20**	**0.06**
2	**0.06**	**0.14**	**0.30**

Find the M.D.F of X and Y. Also $P(X \leq 1, Y \leq 1)$ and check if X and Y are independent.

Solution:

The marginal distributions are given in the table below

X \ Y	0	1	2	P(X = x)
0	0.10	0.04	0.02	0.16
1	0.08	0.20	0.06	0.34
2	0.06	0.14	0.30	0.50
P(Y = y)	0.24	0.38	0.38	1

Now, $P(X \leq 1, Y \leq 1) = p(0,0) + p(1,0) + p(0,1) + p(1,1)$

$= 0.1 + 0.08 + 0.04 + 0.2 \quad = 0.42$

To test X and Y are independent

P(X = 0) P (Y = 0) = 0.16 X 0.24 $\neq$ 0.1

$\therefore P(X = 0)P(Y = 0) \neq P(X = 0, Y = 0)$

$\therefore$ X and Y are independent.