3.4 MOMENT OF INERTIA

The total ***area*** of a shape is found by integrating the differential elements of area over the entire shape.

$$A = \int_A dA.$$

The limit on this integral is indicated with an **A** to indicate that the integration is carried out over the entire area. The resulting value will have units of [length]^2 and does not depend on the position of the shape on the coordinate plane.

The ***area moment of inertia***, is defined by these two equations.

$$I_x = \int_A y^2\, dA \qquad I_y = \int_A x^2\, dA$$

In recognition of the similarity, the area moments of inertia are also known as the ***second moments of area***. We will use the terms moment of inertia and second moment interchangeably. These two quantities are sometimes designated as *rectangular* moments of inertia to distinguish them from the polar moment of inertia. Like the first moment, the second moment of area provides a measure of the distribution of area around an axis, but in this case the distance to each element is squared. This gives increased importance to portions of the area which are far from the axis. Squaring the distance means that identical elements on opposite sides of the axis *both* contribute to the sum rather than cancel each other out as they do in the first moment. As a result, the moment of inertia is always a positive quantity.

Two identical shapes can have completely different moments of inertia, depending on how the shape is distributed around the axis. A shape with most of its area close to the axis has a smaller moment of inertia than the same shape would if its area was distributed farther from the axis. This is a non-linear effect, because when the distance term is doubled, the contribution of that element to the sum increases fourfold.

Mass Moment of Inertia

$$T = I\alpha.$$

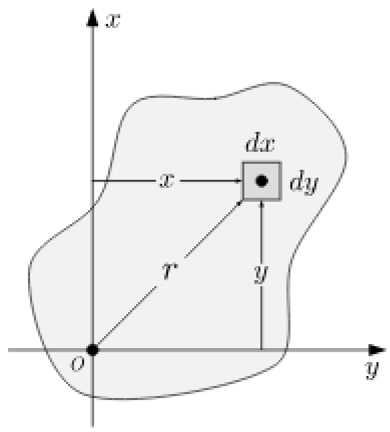
This formula is the rotational analog of Newton's second law F = ma. Here, the I represents the ***mass moment of inertia,*** which is the three-dimensional measure of a rigid body's resistance to rotation around an axis. Mass moment of inertia plays the same role for angular motion as *mass* does for linear motion.

Mass moment of inertia is defined by an integral equation that the differential area dA is replaced with a differential element of mass dm. The integration is conducted over a three dimensional physical object instead of a two dimensional massless area.

The units of mass moment of inertia are, [mass][length]^2, in contrast to area moment of inertia's units of [length]^4

Polar Moment of Inertia

The ***polar moment of inertia*** is defined as



$$J_O = \int_A r^2 \, dA$$

and has units of [length]^4

The polar moment of inertia is a another measure of the distribution of an area but, in this case, about a point at the origin rather than about an axis. One important application of this value is to quantify the resistance of a shaft to torsion or twisting due to the shape of its cross-section.

The polar moment of inertia describes the distribution of the area of a body with respect to a point in the plane of the body. Alternately, the point can be considered to be where a perpendicular axis crosses the plane of the body. The subscript on the symbol j indicates the point or axis.
There is a particularly simple relationship between the polar moment of inertia and the rectangular moments of inertia. Referring to the figure, apply the Pythagorean theorem $r^2 = x^2 + y^2$ to the definition of polar moment of inertia to get

$$\begin{aligned} J_O &= \int_A r^2 \, dA \\ &= \int_A (x^2 + y^2) \, dA \\ &= \int_A x^2 dA + \int_A y^2 dA \\ J_O &= I_x + I_y. \end{aligned}$$

Radius of Gyration

The radius of gyration is an alternate way of expressing the distribution of area away from an axis which combines the effects of the moments of inertia and cross sectional area.

The radius of gyration can be thought of as the radial distance to a thin strip which has the same area and the same moment of inertia around a specific axis as the original shape. Compared to the moment of inertia, the radius of gyration is easier to visualize since it's a distance, rather than a distance to the fourth power.

The radius of gyration, k and the corresponding moment of inertia I are related, and both must refer to the same axis. If one is known, the other is easily found.

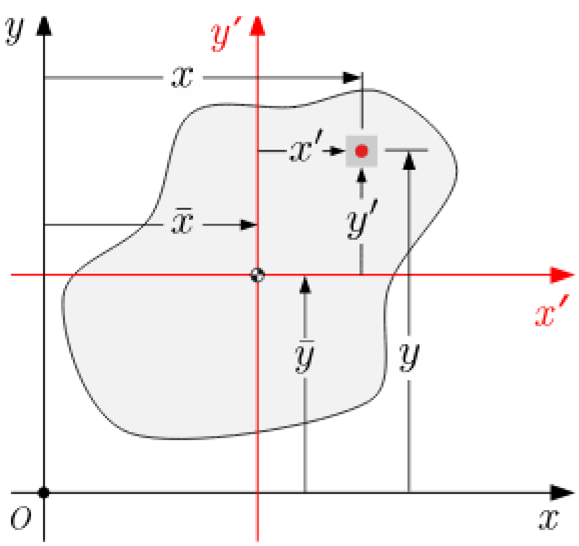
The radius of gyration with respect to the x and y axes and the origin are given by these formulas

$$k_x = \sqrt{\frac{I_x}{A}} \qquad k_y = \sqrt{\frac{I_y}{A}} \qquad k_o = \sqrt{\frac{J_o}{A}}.$$

Product of Inertia

The ***product of inertia*** is another integral property of area, and is defined as

$$I_{xy} = \int_A xy\, dA.$$



The parallel axis theorem for products of inertia is

$$I_{xy} = \bar{I}_{x'y'} + A\bar{x}\bar{y}.$$

Unlike the rectangular moments of inertia, which are always positive, the product of inertia may be either positive, negative, or zero, depending on the object's shape and the orientation of the coordinate axes. The product of inertia will be zero for symmetrical objects when a coordinate axis is also an axis of symmetry.

If the product of inertia is not zero it is always possible to rotate the coordinate system until it is, in which case the new coordinate axes are called the ***principle axes***. When the coordinate axes are oriented in the principle directions, the centroidal moments of inertia are maximum about one axis and minimum about the other, but neither is necessarily zero. The principle directions determine the best way to orient a beam to for maximum stiffness, and how much asymmetrical beams, like channels and angles, will twist when a load is applied.