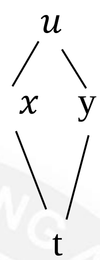
3.2 Total Differentiation

Differentiation of composite functions

Type-I

If $u = f(x,y)$ is a function of x and y where $x = f(t)$ and $y = g(t)$ then

$$\frac{du}{dt} = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt} \quad ...(1)$$



In the differential form, (i) can be written as

$$du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy$$

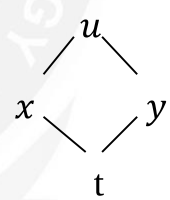
du is called the total differential of u.

Example:

If $= x^2y^3$, where $x = logt$, $y = e^t$ find $\frac{du}{dt}$

Solution:

$$\frac{du}{dt} = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt} \quad ...(1)$$



given $u = x^2y^3$, $x = logt$, $y = e^t$

$$\frac{\partial u}{\partial x} = 2xy^3, \quad \frac{dx}{dt} = \frac{1}{t}, \frac{dy}{dt} = e^t$$

$$\frac{\partial u}{\partial y} = 3x^2y^2$$

$$(1) \Rightarrow \frac{du}{dt} = 2xy^3.\frac{1}{t} + 3x^2y^2.e^t$$

Put $x = logt$, $y = e^t$

$$= 2logt(e^t)^3.\frac{1}{t} + 3(logt)^2(e^t)^2e^t$$

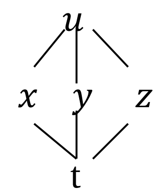
$$= \frac{2logt.e^{3t}}{t} + 3(logt)^2e^{3t}$$

$$\frac{du}{dt} = e^{3t}logt(\frac{2}{t} + 3logt)$$

Example:

If $u = xy + yz + zx$, where $x = \frac{1}{t}$, y = e^t, z = e^{-t} find $\frac{du}{dt}$

Solution:



$$\frac{du}{dt} = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt} + \frac{\partial u}{\partial z}\frac{dz}{dt} \quad ...(1)$$

Given $u = xy + yz + zx$,

$$x = \frac{1}{t}, y = e^{t}, z = e^{-t}$$

$$\frac{\partial u}{\partial x} = y + z, \quad \frac{dx}{dt} = \frac{-1}{t^2}, \frac{dy}{dt} = e^t, \frac{dz}{dt} = -e^{-t}$$

$$\frac{\partial u}{\partial y} = x + z$$

$$\frac{\partial u}{\partial y} = y + x$$

$$(1) \quad \Rightarrow \frac{du}{dt} = (y+z)\left(-\frac{1}{t^2}\right) + (x+z)e^t + (y+x)(-e^{-t})$$

Put $x = \frac{1}{t}$, $y = e^{t}$, $z = e^{-t}$

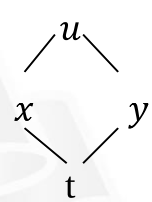
$$= (e^t + e^{-t})\left(-\frac{1}{t^2}\right) + \left(\frac{1}{t} + e^{-t}\right)e^t + \left(e^t + \frac{1}{t}\right)(-e^{-t})$$

$$\frac{du}{dt} = \frac{-(e^t + e^{-t})}{t^2} + \left(\frac{1}{t} + e^{-t}\right)e^t + \left(e^t + \frac{1}{t}\right)(-e^{-t})$$

Example:

If $= xy^2 + x^2y$, where $x = at^2$, $y = 2at$ find $\frac{du}{dt}$

Solution:



$$\frac{du}{dt} = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt} \quad ...(1)$$

Given $u = xy^2 + x^2y$, $x = at^2$, $y = 2at$

$$\frac{\partial u}{\partial x} = y^2 + 2xy, \quad \frac{dx}{dt} = 2at \ , \frac{dy}{dt} = 2a$$

$$\frac{\partial u}{\partial y} = 2xy + x^2$$

$$(1) \quad \Rightarrow \quad \frac{du}{dt} = (y^2 + 2xy)2at + (2xy + x^2)2a$$

$$= (4a^2t^2 + 4a^2t^3)2at + (4a^2t^3 + a^2t^4)2a$$

$$= 8a^3t^3 + 8a^3t^4 + 8a^3t^3 + 2a^3t^4$$

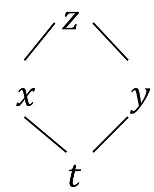
$$= 16a^3t^3 + 10a^3t^4$$

$$= 2a^3t^3(8 + 5t)$$

Example:

If $z = sin^{-1}(x-y)$, where $x = 3t$, $y = 4t^3$ find $\frac{dz}{dt}$

Solution:



$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt} \quad ...(1)$$

Given z $= sin^{-1}(x-y)$, $x = 3t$, $y = 4t^3$

$$\frac{\partial z}{\partial x} = \frac{1}{\sqrt{1-(x-y)^2}} \;, \quad \frac{dx}{dt} = 3 \;, \frac{dy}{dt} = 12t^2$$

$$\frac{\partial z}{\partial y} = \frac{1}{\sqrt{1-(x-y)^2}}(-1)$$

$$(1) \;\Rightarrow\; \frac{dz}{dt} = \frac{3}{\sqrt{1-(x-y)^2}} - \frac{12t^2}{\sqrt{1-(x-y)^2}}$$

$$= \frac{3(1-4t^2)}{\sqrt{1-(x-y)^2}}$$

$$= \frac{3(1-4t^2)}{\sqrt{1-(3t-4t^3)^2}}$$

$$= \frac{3(1-4t^2)}{\sqrt{1-9t^2-16t^6+24t^4}}$$

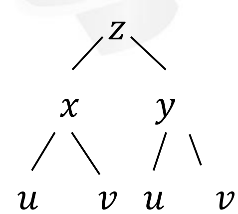
$$= \frac{3(1-4t^2)}{\sqrt{(1-t^2)(1-4t^2)^2}}$$

$$= \frac{3(1-4t^2)}{(1-4t^2)\sqrt{(1-t^2)}}$$

$$= \frac{3}{\sqrt{1-t^2}}$$

Type-II

If $z = f(x,y)$ is a function of x and y where x and y are the functions of u and v such that



$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u}$$

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial v}$$

Example:

If $g(x,y) = \psi(u,v)$ where $u = x^2 - y^2$ and $v = 2xy$ then prove that

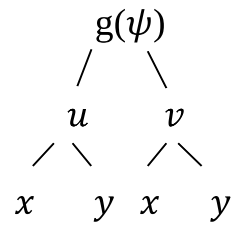
$$\frac{\partial^2 g}{\partial x^2} + \frac{\partial^2 g}{\partial y^2} = 4(x^2+y^2)\left(\frac{\partial^2 \psi}{\partial u^2} + \frac{\partial^2 \psi}{\partial v^2}\right)$$

Solution:

Given $u = x^2 - y^2 , v = 2xy$

$$\frac{\partial u}{\partial x} = 2x, \quad \frac{\partial v}{\partial x} = 2y$$

$$\frac{\partial u}{\partial y} = -2y, \quad \frac{\partial v}{\partial y} = 2x$$



$$\frac{\partial g}{\partial x} = \frac{\partial \psi}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial \psi}{\partial v}\frac{\partial v}{\partial x} \ \ldots (1)$$

$$\frac{\partial g}{\partial y} = \frac{\partial \psi}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial \psi}{\partial v} \ \ldots (2)$$

$$(1) \quad \Rightarrow \quad \frac{\partial g}{\partial x} = \frac{\partial \psi}{\partial u} 2x + \frac{\partial \psi}{\partial v} 2y$$

$$\frac{\partial g}{\partial x} = 2(x\frac{\partial \psi}{\partial u} + y\frac{\partial \psi}{\partial v})$$

$$\frac{\partial^2 g}{\partial x^2} = \frac{\partial}{\partial x}(\frac{\partial g}{\partial x}) = 2(x\frac{\partial}{\partial u} + y\frac{\partial}{\partial v}) \ 2(x\frac{\partial \psi}{\partial u} + y\frac{\partial \psi}{\partial v})$$

$$= 4\left[x^2\frac{\partial^2 \psi}{\partial u^2} + 2xy\frac{\partial^2 \psi}{\partial u \partial v} + y^2\frac{\partial^2 \psi}{\partial v^2}\right] \ldots (3)$$

$$(2) \quad \Rightarrow \frac{\partial g}{\partial y} = \frac{\partial \psi}{\partial u}(-2y) + \frac{\partial \psi}{\partial v} 2x$$

$$\frac{\partial g}{\partial y} = 2(-y\frac{\partial \psi}{\partial u} + x\frac{\partial \psi}{\partial v})$$

$$\frac{\partial^2 g}{\partial y^2} = \frac{\partial}{\partial y}(\frac{\partial g}{\partial y}) = 2(-y\frac{\partial}{\partial u} + x\frac{\partial}{\partial v}) \ 2(-y\frac{\partial \psi}{\partial u} + x\frac{\partial \psi}{\partial v})$$

$$= 4\left[y^2\frac{\partial^2 \psi}{\partial u^2} - 2xy\frac{\partial^2 \psi}{\partial u \partial v} + x^2\frac{\partial^2 \psi}{\partial v^2}\right] \ldots (4)$$

$$(3) + (4) \quad \Rightarrow \frac{\partial^2 g}{\partial x^2} + \frac{\partial^2 g}{\partial y^2} = 4\{(x^2 + y^2)\frac{\partial^2 \psi}{\partial u^2} + (x^2 + y^2)\frac{\partial^2 \psi}{\partial u^2}\}$$

$$= 4(x^2 + y^2)(\frac{\partial^2 \psi}{\partial u^2} + \frac{\partial^2 \psi}{\partial v^2})$$

Hence proved.

Example:

If f is a function of x and y then $x = e^{u} sinv$ and $y = e^{u} cosv$, show that

$$\frac{\partial^2 f}{\partial u^2} + \frac{\partial^2 f}{\partial v^2} = (x^2 + y^2)\left(\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}\right) = e^{2u}\left(\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}\right) \quad \textbf{(OR)}$$

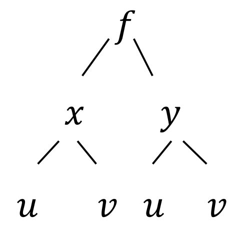
$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = e^{-2u}(\frac{\partial^2 f}{\partial u^2} + \frac{\partial^2 f}{\partial v^2})$$

Solution:

Given $x = e^u sinv \, , y = e^u cosv$

$\frac{\partial x}{\partial u} = e^u sinv, \quad \frac{\partial y}{\partial u} = e^u cosv$

$\frac{\partial x}{\partial v} = e^u cosv, \ \frac{\partial y}{\partial v} = - e^u sinv$



$$\frac{\partial f}{\partial u} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial u} \ \dots (1)$$

$$\frac{\partial f}{\partial v} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial v} \ \dots (2)$$

$$(1) \ \Rightarrow \ \frac{\partial f}{\partial u} = \frac{\partial f}{\partial x} e^u sinv + \frac{\partial f}{\partial y} e^u \, cosv$$

$$\frac{\partial f}{\partial u} = x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y}$$

$$\frac{\partial^2 f}{\partial u^2} = \frac{\partial}{\partial u}\left(\frac{\partial f}{\partial u}\right) = \left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right)\left(x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y}\right)$$

$$= \left[x^2\frac{\partial^2 f}{\partial x^2} + 2xy\frac{\partial^2 f}{\partial x \partial y} + y^2\frac{\partial^2 f}{\partial y^2}\right] \dots (3)$$

$$(2) \ \Rightarrow \ \frac{\partial f}{\partial v} = \frac{\partial f}{\partial x} e^u \mathrm{cosv} - \frac{\partial f}{\partial y} \ e^u \, sinv$$

$$\frac{\partial f}{\partial v} = y\frac{\partial f}{\partial x} - x\frac{\partial f}{\partial y}$$

$$\frac{\partial^2 f}{\partial v^2} = \frac{\partial}{\partial v}\left(\frac{\partial f}{\partial v}\right) = \left(y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}\right)\left(y\frac{\partial f}{\partial x} - x\frac{\partial f}{\partial y}\right)$$

$$= \left[y^2\frac{\partial^2 f}{\partial x^2} - 2xy\frac{\partial^2 f}{\partial x \partial y} + x^2\frac{\partial^2 f}{\partial y^2}\right] \dots (4)$$

$$(3) + (4) \ \Rightarrow \ \frac{\partial^2 f}{\partial u^2} + \frac{\partial^2 f}{\partial v^2} = (x^2 + y^2)\frac{\partial^2 f}{\partial x^2} + (x^2 + y^2)\frac{\partial^2 f}{\partial y^2}$$

$$= (x^2 + y^2)\left(\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}\right)$$

$$[\because x^2 = e^{2u} sin^2 v, y^2 = e^{2u} cos^2 v]$$

$$= e^{2u}\left(\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}\right)$$

$$\therefore \ \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = e^{-2u}\left(\frac{\partial^2 f}{\partial u^2} + \frac{\partial^2 f}{\partial v^2}\right)$$

Example:

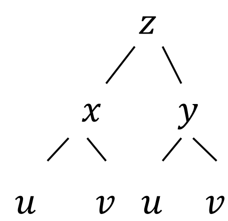
If z is a function of x and y where $x = e^u + e^{-v}$ and $y = e^{-u} - e^v$, show that $\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v} = x\frac{\partial z}{\partial x} - y\frac{\partial z}{\partial y}$

Solution:

Given $\boldsymbol{x} = e^{u} + e^{-v}$; $y = e^{-u} - e^{v}$

$$\frac{\partial x}{\partial u} = e^{u}, \qquad \frac{\partial y}{\partial u} = -e^{-u}$$

$$\frac{\partial x}{\partial v} = -e^{-v}, \qquad \frac{\partial y}{\partial v} = -e^{v}$$



$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u} \ \ldots (1)$$

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial v} \ \ldots (2)$$

$$(1) \Rightarrow \frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}e^{u} - \frac{\partial z}{\partial y}e^{-u}$$

$$(2) \Rightarrow \frac{\partial z}{\partial v} = -\frac{\partial z}{\partial x}e^{-v} - \frac{\partial z}{\partial y}e^{v}$$

$$(1) - (2) \Rightarrow \frac{\partial z}{\partial u} - \frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}(e^{u} + e^{-v}) - \frac{\partial z}{\partial y}(e^{-u} - e^{v})$$

$$= x\frac{\partial z}{\partial x} - y\frac{\partial z}{\partial y}$$

Hence proved.

Example:

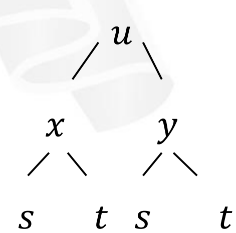
If $\boldsymbol{u = e^{x} siny \ where \ x = st^2}$ and $\boldsymbol{y = s^2 t}$ find $\boldsymbol{\frac{\partial u}{\partial s} \ and \ \frac{\partial u}{\partial t}}$

Solution:

Given $u = e^{x} siny$, $x = st^2$, $y = s^2 t$

$$\frac{\partial u}{\partial x} = e^{x} siny , \quad \frac{\partial x}{\partial t} = 2st$$

$$\frac{\partial u}{\partial y} = e^{x} cosy , \quad \frac{\partial y}{\partial t} = s^2$$



$$\frac{\partial u}{\partial s} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial s} \ldots (1)$$

$$\frac{\partial u}{\partial t} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial t} \ \ldots (2)$$

$$(1) \Rightarrow \frac{\partial u}{\partial s} = e^{x} siny \text{ x } t^2 + e^{x} cosy \text{ x } 2st$$

$$(2) \Rightarrow \frac{\partial u}{\partial t} = e^{x} siny \text{ x } 2st + e^{x} cosy \text{ x } s^2$$

Example:

If $u = \left(\frac{y-x}{xy}, \frac{z-x}{xz}\right)$ show that $x^2\frac{\partial u}{\partial x} + y^2\frac{\partial u}{\partial y} + z^2\frac{\partial u}{\partial z} = 0$

Solution:

Let r = $\frac{y}{xy} - \frac{x}{xy} = \frac{1}{x} - \frac{1}{y}$, s = $\frac{z}{xz} - \frac{x}{xz} = \frac{1}{x} - \frac{1}{z}$

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial u}{\partial s}\frac{\partial s}{\partial x} \ \ldots(1)$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial u}{\partial s}\frac{\partial s}{\partial y} \ \ldots(2)$$

$$\frac{\partial u}{\partial z} = \frac{\partial u}{\partial r}\frac{\partial r}{\partial z} + \frac{\partial u}{\partial s}\frac{\partial s}{\partial z} \ \ldots(3)$$

Here r = $\frac{1}{x} - \frac{1}{y}$, s = $\frac{1}{x} - \frac{1}{z}$

$$\frac{\partial r}{\partial x} = -\frac{1}{x^2}, \quad \frac{\partial s}{\partial x} = -\frac{1}{x^2}$$

$$\frac{\partial r}{\partial y} = \frac{1}{y^2}, \quad \frac{\partial s}{\partial y} = 0$$

$$\frac{\partial r}{\partial z} = 0, \quad \frac{\partial s}{\partial z} = \frac{1}{z^2}$$

$$(1) \Rightarrow \frac{\partial u}{\partial x} = -\frac{1}{x^2}\frac{\partial u}{\partial r} - \frac{1}{x^2}\frac{\partial u}{\partial s}$$

$$x^2\frac{\partial u}{\partial x} = -\frac{\partial u}{\partial r} - \frac{\partial u}{\partial s}$$

$$(2) \Rightarrow \frac{\partial u}{\partial y} = \frac{1}{y^2}\frac{\partial u}{\partial r} + 0\frac{\partial u}{\partial s}$$

$$y^2\frac{\partial u}{\partial y} = \frac{\partial u}{\partial r}$$

$$(3) \Rightarrow \quad \frac{\partial u}{\partial z} = 0\frac{\partial u}{\partial r} + \frac{1}{z^2}\frac{\partial u}{\partial s}$$

$$z^2\frac{\partial u}{\partial z} = \frac{\partial u}{\partial s}$$

$$(1) + (2) + (3) \Rightarrow x^2\frac{\partial u}{\partial x} + y^2\frac{\partial u}{\partial y} + z^2\frac{\partial u}{\partial z} = -\frac{\partial u}{\partial r} - \frac{\partial u}{\partial s} + \frac{\partial u}{\partial r} + \frac{\partial u}{\partial s}$$

$$= 0$$

Hence proved.

Example:

If $u = f(x-y, y-z, z-x)$ show that $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0$

Solution:

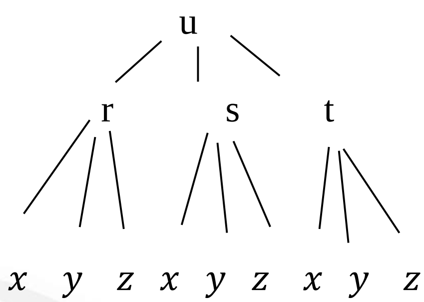
Let $r = x - y, s = y - z, t = z - x$

$\therefore \; u = f(r, s, t)$ where $r = x - y, s = y - z, t = z - x$

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial u}{\partial s}\frac{\partial s}{\partial x} + \frac{\partial u}{\partial t}\frac{\partial t}{\partial x} \;\ldots(1)$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial u}{\partial s}\frac{\partial s}{\partial y} + \frac{\partial u}{\partial t}\frac{\partial t}{\partial y} \;\ldots(2)$$

$$\frac{\partial u}{\partial z} = \frac{\partial u}{\partial r}\frac{\partial r}{\partial z} + \frac{\partial u}{\partial s}\frac{\partial s}{\partial z} + \frac{\partial u}{\partial t}\frac{\partial t}{\partial z} \;\ldots(3)$$



Given $r = x - y, \quad s = y - z, \quad t = z - x$ x y z x y

$$\frac{\partial r}{\partial x} = 1, \quad \frac{\partial s}{\partial x} = 0, \quad \frac{\partial t}{\partial x} = -1$$

$$\frac{\partial r}{\partial y} = -1, \quad \frac{\partial s}{\partial y} = 1, \quad \frac{\partial t}{\partial y} = 0$$

$$\frac{\partial r}{\partial z} = 0, \quad \frac{\partial s}{\partial z} = -1, \quad \frac{\partial t}{\partial z} = 1$$

$(1) \Rightarrow \quad \frac{\partial u}{\partial x} = \frac{\partial u}{\partial r}1 + \frac{\partial u}{\partial s}0 + \frac{\partial u}{\partial t}(-1)$

$= \frac{\partial u}{\partial r} - \frac{\partial u}{\partial t}$

$(2) \Rightarrow \quad \frac{\partial u}{\partial y} = \frac{\partial u}{\partial r}(-1) + \frac{\partial u}{\partial s}1 + \frac{\partial u}{\partial t}0$

$= -\frac{\partial u}{\partial r} + \frac{\partial u}{\partial s}$

$(3) \Rightarrow \frac{\partial u}{\partial z} = \frac{\partial u}{\partial r}0 + \frac{\partial u}{\partial s}(-1) + \frac{\partial u}{\partial t}1$

$= -\frac{\partial u}{\partial s} + \frac{\partial u}{\partial t}$

$(1) + (2) + (3) \Rightarrow \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = \frac{\partial u}{\partial r} - \frac{\partial u}{\partial t} - \frac{\partial u}{\partial r} + \frac{\partial u}{\partial s} - \frac{\partial u}{\partial s} + \frac{\partial u}{\partial t}$

= 0

Hence proved

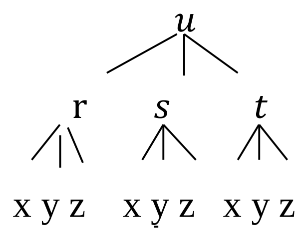
Example:

If u = $f\left(\frac{x}{y}, \frac{y}{z}, \frac{z}{x}\right)$ show that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = 0$

Solution:

Let $r = \frac{x}{y}, s = \frac{y}{z}, t = \frac{z}{x}$

$\therefore\ u = f(r,s,t)$ where $r = \frac{x}{y}, s = \frac{y}{z}, t = \frac{z}{x}$



$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial u}{\partial s}\frac{\partial s}{\partial x} + \frac{\partial u}{\partial t}\frac{\partial t}{\partial x} \ \ldots (1)$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial u}{\partial s}\frac{\partial s}{\partial y} + \frac{\partial u}{\partial t}\frac{\partial t}{\partial y} \ \ldots (2)$$

$$\frac{\partial u}{\partial z} = \frac{\partial u}{\partial r}\frac{\partial r}{\partial z} + \frac{\partial u}{\partial s}\frac{\partial s}{\partial z} + \frac{\partial u}{\partial t}\frac{\partial t}{\partial z} \ \ldots (3)$$

Given $r = \frac{x}{y}, \quad s = \frac{y}{z}, \quad t = \frac{z}{x}$

$$\frac{\partial r}{\partial x} = \frac{1}{y}, \quad \frac{\partial s}{\partial x} = 0, \quad \frac{\partial t}{\partial x} = -\frac{z}{x^2}$$

$$\frac{\partial r}{\partial y} = -\frac{x}{y^2}, \quad \frac{\partial s}{\partial y} = \frac{1}{z}, \quad \frac{\partial t}{\partial y} = 0$$

$$\frac{\partial r}{\partial z} = 0, \quad \frac{\partial s}{\partial z} = -\frac{y}{z^2}, \frac{\partial t}{\partial z} = \frac{1}{x}$$

$$(1) \Rightarrow \quad \frac{\partial u}{\partial x} = \frac{\partial u}{\partial r}\frac{1}{y} + \frac{\partial u}{\partial s}0 + \frac{\partial u}{\partial t}\left(-\frac{z}{x^2}\right)$$

$$= \frac{1}{y}\frac{\partial u}{\partial r} - \frac{z}{x^2}\frac{\partial u}{\partial t}$$

$$x\frac{\partial u}{\partial x} = \frac{x}{y}\frac{\partial u}{\partial r} - \frac{z}{x}\frac{\partial u}{\partial t} \ \ldots (4)$$

$$(2) \Rightarrow \quad \frac{\partial u}{\partial y} = \frac{\partial u}{\partial r}\left(-\frac{x}{y^2}\right) + \frac{\partial u}{\partial s}\frac{1}{z} + \frac{\partial u}{\partial t}0$$

$$= -\frac{x}{y^2}\frac{\partial u}{\partial r} + \frac{1}{z}\frac{\partial u}{\partial s}$$

$$y\frac{\partial u}{\partial y} = -\frac{x}{y}\frac{\partial u}{\partial r} + \frac{y}{z}\frac{\partial u}{\partial s} \ \ldots (5)$$

$$(3) \Rightarrow \quad \frac{\partial u}{\partial z} = \frac{\partial u}{\partial r}0 + \frac{\partial u}{\partial s}\left(-\frac{y}{z^2}\right) + \frac{\partial u}{\partial t}\frac{1}{x}$$

$$= -\frac{y}{z^2}\frac{\partial u}{\partial s} + \frac{1}{x}\frac{\partial u}{\partial t}$$

$$z\frac{\partial u}{\partial z} = -\frac{y}{z}\frac{\partial u}{\partial s} + \frac{z}{x}\frac{\partial u}{\partial t} \ \ldots (6)$$

(4) + (5) + (6)

$$\Rightarrow x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = \frac{x}{y}\frac{\partial u}{\partial r} - \frac{z}{x}\frac{\partial u}{\partial t} - \frac{x}{y}\frac{\partial u}{\partial r} + \frac{y}{z}\frac{\partial u}{\partial s} - \frac{y}{z}\frac{\partial u}{\partial s} + \frac{z}{x}\frac{\partial u}{\partial t}$$

$$= 0$$

Hence Proved

Example:

Prove that $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = \frac{\partial^2 z}{\partial u^2} + \frac{\partial^2 z}{\partial v^2}$ **where** $x = u\cos\alpha - v\sin\alpha$, $y = u\sin\alpha + v\cos\alpha$

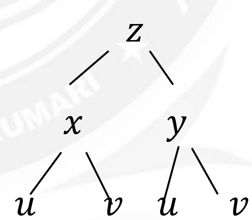
(OR)

By changing independent variables u **and** v **to** x **and** y **by means of the relation** $x = u\cos\alpha - v\sin\alpha$, $y = u\sin\alpha + v\cos\alpha$ **, show that** $\frac{\partial^2 z}{\partial u^2} + \frac{\partial^2 z}{\partial v^2}$ **transforms into** $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2}$

Solution:

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u} \quad \dots (1)$$

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial v} \quad \dots (2)$$



$x = u\cos\alpha - v\sin\alpha,\ y = u\sin\alpha + v\cos\alpha$

$\frac{\partial x}{\partial u} = \cos\alpha,\qquad \frac{\partial y}{\partial u} = \sin\alpha$

$\frac{\partial x}{\partial v} = -\sin\alpha,\qquad \frac{\partial y}{\partial v} = \cos\alpha$

$$(1) \Rightarrow \frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\cos\alpha + \frac{\partial z}{\partial y}\sin\alpha$$

$$\frac{\partial^2 z}{\partial u^2} = \frac{\partial}{\partial u}\left(\frac{\partial z}{\partial u}\right) = \left(\frac{\partial}{\partial x}\cos\alpha + \frac{\partial}{\partial y}\sin\alpha\right)\left(\frac{\partial z}{\partial x}\cos\alpha + \frac{\partial z}{\partial y}\sin\alpha\right)$$

$$= \cos^2\alpha\ \frac{\partial^2 z}{\partial x^2} + \ \frac{\partial^2 z}{\partial x^2}\sin^2\alpha + 2\ \frac{\partial^2 z}{\partial x\partial y}\cos\alpha\sin\alpha \quad \dots (3)$$

$$(2) \Rightarrow \frac{\partial z}{\partial v} = -\frac{\partial z}{\partial x}\sin\alpha + \frac{\partial z}{\partial y}\cos\alpha$$

$$\frac{\partial^2 z}{\partial v^2} = \frac{\partial}{\partial v}\left(\frac{\partial z}{\partial v}\right) = \left(-\frac{\partial}{\partial x} sin\alpha + \frac{\partial}{\partial y} cos\alpha\right)\left(-\frac{\partial z}{\partial x} sin\alpha + \frac{\partial z}{\partial y} cos\alpha\right)$$

$$= sin^2\alpha \ \frac{\partial^2 z}{\partial x^2} + \ \frac{\partial^2 z}{\partial x^2} cos^2\alpha - 2 \ \frac{\partial^2 z}{\partial x \partial y} cos\alpha sin\alpha \ \dots (4)$$

$$(3) + (4) \quad \Rightarrow \quad \frac{\partial^2 z}{\partial u^2} + \frac{\partial^2 z}{\partial v^2}$$

$$= \ cos^2\alpha \ \frac{\partial^2 z}{\partial x^2} + \ \frac{\partial^2 z}{\partial x^2} sin^2\alpha + 2 \ \frac{\partial^2 z}{\partial x \partial y} cos\alpha sin\alpha + sin^2\alpha \ \frac{\partial^2 z}{\partial x^2} +$$

$$\frac{\partial^2 z}{\partial x^2} cos^2\alpha - 2 \ \frac{\partial^2 z}{\partial x \partial y} cos\alpha sin\alpha$$

$$= \ \left(\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2}\right)\left(\ cos^2\alpha + \ sin^2\alpha \ \right)$$

$$= \ \frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2}$$

Hence proved.

Differentiation of Implicit Function

If $u = f(x, y) = c$ be a given implicit function of x and y then

$$du = \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy \ \dots (1)$$

$$\frac{du}{dx} = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}\frac{dy}{dx}$$

Given $u = c \quad \Rightarrow \quad \frac{du}{dx} = 0$

$$0 = \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy$$

$$\frac{\partial u}{\partial y} dy = -\frac{\partial u}{\partial x} dx$$

$$\frac{dy}{dx} = -\frac{\frac{\partial u}{\partial x}}{\frac{\partial u}{\partial y}}$$

Example:

Find $\frac{dy}{dx}$ if $x^3 + y^3 = 3axy$

Solution:

Let $f = x^3 + y^3 - 3axy$

$$\frac{\partial f}{\partial x} = 3x^2 - 3ay$$

$$\frac{\partial f}{\partial y} = 3y^2 - 3ax$$

$$\frac{dy}{dx} = -\frac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial y}} \quad = -\frac{3(x^2-ay)}{3(y^2-ax)}$$

$$\frac{dy}{dx} = -\frac{(x^2-ay)}{(y^2-ax)}$$

Example:

Find $\frac{dy}{dx}$ when $ysinx = xcosy$

Solution:

Let $f = ysinx - xcosy$

$$\frac{\partial f}{\partial x} = ycosx - cosy$$

$$\frac{\partial f}{\partial y} = sinx + xsiny$$

$$\frac{dy}{dx} = -\frac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial y}}$$

$$\frac{dy}{dx} = -\frac{(ycosx-cosy)}{(sinx+xsiny)}$$

Example:

If $u = xlog(xy)$ where $x^3+y^3+3xy=1$ then find $\frac{du}{dx}$

Solution:

$$\frac{du}{dx} = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}\frac{dy}{dx} \; ...(1)$$

$$u = x(logx + logy) = xlogx + xlogy$$

$$\frac{\partial f}{\partial x} = x\left(\frac{1}{x}\right) + logx + logy$$

$$\frac{\partial f}{\partial y} = x\left(\frac{1}{y}\right)$$

Given $x^3+y^3+3xy = 1$

Differentiating with respect to 'x'

$$3x^2 + 3y^2\frac{dy}{dx} + 3\left(x\frac{dy}{dx} + y \times 1\right) = 0$$

$$3x^2 + 3y^2\frac{dy}{dx} + 3x\frac{dy}{dx} + 3y = 0$$

$$\frac{dy}{dx}(3y^2+3x) = -3(x^2+y)$$

$$\frac{dy}{dx} = \frac{-3(x^2+y)}{3(y^2+x)}$$

$$\frac{dy}{dx} = \frac{-(x^2+y)}{(y^2+x)}$$

$$(1) \Rightarrow \frac{du}{dx} = 1 + logx + logy - \frac{x}{y}\left(\frac{x^2+y}{y^2+x}\right)$$

Exercise:

1. If $= sin(xy^2)$, where $x = logt , y = e^t$ find $\frac{du}{dt}$

Ans : $y^2\left(\frac{1}{t^2} + 2x\right)\cos(xy^2)$

2. If $= x^3y^2 + x^2y^3$, where $x = at^2 , y = 2at$ find $\frac{du}{dt}$

Ans: $8a^5t^6(4t+7)$

3. If $z = x^2 + y^2$, where $x = t^3 , y = 1 + t^2$ find $\frac{dz}{dt}$

Ans: $6t^5 + 4t^3 + 4t$

4. If $u = y^4x^3z^2$, where $x = t^2 , y = t^3 , z = t4$ find $\frac{du}{dt}$

5. If z is a function of x and y and u and v are other two variables such that

   u = $\ell x + my$, v = $\ell y - mx$ prove that $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = (\ell^2 + m^2)\left(\frac{\partial^2 z}{\partial u^2} + \frac{\partial^2 z}{\partial v^2}\right)$

6. Prove that $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = \frac{\partial^2 z}{\partial u^2} + \frac{\partial^2 z}{\partial v^2}$ where $x = ucos\alpha - vsin\alpha,$

   $y = usin\alpha + vcos\alpha$

7. If u is a function of x and y where $x = e^r cos\theta, y = e^r sin\theta$

   $x\frac{\partial u}{\partial \theta} + y\frac{\partial u}{\partial r} = e^{2r}\frac{\partial u}{\partial y}$

8. If φ is a function of u and v also x and y then $u = e^x siny$ and $v = e^x cosy$ prove that

$$\frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2} = (u^2 + v^2)\left(\frac{\partial^2 \varphi}{\partial u^2} + \frac{\partial^2 \varphi}{\partial v^2}\right)$$

9. If $g(s,t) = f(s^2 - t^2, t^2 - s^2)$ show that $t\frac{\partial g}{\partial s} + s\frac{\partial g}{\partial t} = 0$

10. If $w = f(u, v)$ where u $= x + y$, $v = x - y$ then show that $\frac{\partial w}{\partial x} + \frac{\partial w}{\partial y} = 2\frac{\partial w}{\partial u}$

11. If $u = f(x^2 + 2yz, y^2 + 2xz)$ then prove that

$$(y^2 - zx)\frac{\partial u}{\partial x} + (x^2 - yz)\frac{\partial u}{\partial y} + (z^2 - xy)\frac{\partial u}{\partial z} = 0$$

12. Find $\frac{dy}{dx}$ for the following:

a) $(cosx)^y = (\text{siny})^x$ **Ans:** $\frac{-ytanx+logsiny}{logcosx-xcoty}$

b) $x^y = y^x$ **Ans:** $\frac{-yx^{y-1}+y^x logy}{xy^{x-1}-x^y logx}$

c) $3x^2 + xy - y^2 + 4x - 2y + 1 = 0$ **Ans:** $\frac{-(6x+y+4)}{x-2y-2}$

d) $ax^2 + by^2 + 2hxy + 2gx + 2fy + c = 0$ **Ans:** $\frac{-(2ax+2hy+2g)}{2by+2hx+2f}$