3.5 Maxima and Minima for Functions of Two Variables

(a)Maximum value

$f(a,b)$ is a maximum value of $f(x,y)$, if there exists some neighbourhood of the point (a, b) such that for every point (a + h ,b +k) of the neighbourhood.

$$f(a,b) > f(a+h, b+k)$$

(b) Minimum value

$f(a,b)$ is a minimum value of $f(x,y)$,if there exists some neighborhood of the point (a, b) such that for every point (a + h ,b +k) of the neighborhood.

$$f(a,b) < f(a+h, b+k)$$

(c) Extremum value

$f(a,b)$ is said to be an extremum value of $f(x,y)$ if it is either a maximum or minimum.

(d) Necessary conditions for a maximum or a minimum.

$$f_x(a,b) = 0 \ and \ f_y(a,b) = 0$$

Notations : $f_x = \frac{\partial f}{\partial x},\ f_y = \frac{\partial f}{\partial y},\ f_{xx} = \frac{\partial^2 f}{\partial x^2},\ f_{xy} = \frac{\partial^2 f}{\partial x \partial y},\ f_{yy} = \frac{\partial^2 f}{\partial y^2}$

(e) Sufficient conditions:

If $f_x(a,b) = 0 \ and \ f_y(a,b) = 0$ and $f_{xx}(a,b) = A$, $f_{xy}(a,b) = B$, $f_{yy}(a,b) = C$, then

(i) $f(a,b)$ is maximum value if $AC - B^2 > 0 \ and \ A < 0 \ or \ B < 0$

(ii) $f(a,b)$ is minimum value if $AC - B^2 > 0 \ and \ A > 0 \ or \ B > 0$

(iii) $f(a,b)$ is not an extremum (saddle) if $AC - B^2 < 0$

(iv) if $AC - B^2 = 0$ then the test is inconclusive.

(f) Stationary value

A function $f(x,y)$ is said to be stationary at (a, b) or $f(a,b)$ is said to be

Stationary value of $f(x, y)$ if $f_x(a, b) = 0$ and $f_y(a, b) = 0$

Note:

Every extremum value is a stationary value but a stationary value need not be an extremum value

Problems Based on Maxima and Minima for Functions of Two Variables

Example:

Find the extreme values of the function $f(x, y) = x^3 + y^3 - 3x - 12y + 20$

Solution:

Given $f(x, y) = x^3 + y^3 - 3x - 12y + 20$

$f_x = 3x^2 - 3$; $f_y = 3y^2 - 12$

$f_{xx} = 6x = A$, $f_{xy} = 0 = B$, $f_{yy} = 6y = C$

To find the stationary points.

$f_x = 0$	$f_x = 0$
$3x^2 - 3 = 0$ $x^2 - 1 = 0$ $x = \pm 1$	$3y^2 - 12 = 0$ $y^2 - 4 = 0$ $y = \pm 2$

$\therefore$ Stationary points are $(1, 2), (1, -2), (-1, 2), (-1, -2)$

	$(1, 2)$	$(1, -2)$	$(-1, 2)$	$(-1, -2)$
$A = 6x$	$6 > 0$	$6 > 0$	$-6 < 0$	$-6 < 0$
$B = 0$	0	0	0	0
$C = 6y$	12	-12	12	-12
$AC - B^2$	$72 > 0$	$-72 < 0$	$-72 < 0$	$72 > 0$
Conclusion	Min. point	Saddle point	Saddle point	Max. point

$\therefore$ Maximum value of $f(x, y)$ is

$$f(-1,-2) = (-1)^3 + (-2)^3 - 3(-1) - 12(-2) + 20$$
$$= -1 - 8 + 3 + 24 + 20 \quad = \quad 38$$

Minimum value of $f(x, y)$ is

$$f(1, 2) = (1)^3 + (2)^3 - 3(1) - 12(2) + 20 = 2$$

Example:

A flat circular plate is heated so that the temperature at any point (x, y) is $u(x, y) = x^2 + 2y^2 - x$. Find the coldest point on the plate.

Solution:

$$u(x, y) = x^2 + 2y^2 - x$$

$$u_x = 2x - 1 \qquad u_y = 4y$$

$u_x = 0$ ⇨ $2x - 1 = 0$ ⇨ $x = \frac{1}{2}$	$u_y = 0$ $4y = 0$ $y = 0$

$A = u_{xx} = 2$; $C = u_{yy} = 4$ $B = u_{xy} = 0$

$\Delta = AC - B^2 > 0$

U is minimum at $\left(\frac{1}{2}, 0\right)$ and its minimum value is $-\frac{1}{4}$

Example:

Find the maxima and minima of $x^4 + y^4 - 2x^2 + 4xy - 2y^2$

Solution:

$$f(x, y) = x^4 + y^4 - 2x^2 + 4xy - 2y^2$$

$$f_x = 4x^3 - 4x + 4y \quad ; \quad f_y = 4y^3 + 4x - 4y$$

$$f_{xx} = 12x^2 - 4 = A, \; f_{xy} = 4 = B, \; f_{yy} = 12y^2 - 4 = C$$

To find the stationary points.

$f_x = 0$	$f_y = 0$

$4x^3 - 4x + 4y = 0$ $x^3 - x + y = 0 \dots$ (1)	$4y^3 + 4x - 4y = 0$ $y^3 + x - y = 0 \dots$ (2)

$$(1) + (2) \Rightarrow x^3 + y^3 = 0 \quad \Rightarrow x^3 = -y^3 \quad \Rightarrow y = -x$$

$$(1) \Rightarrow x^3 - x - x = 0 \quad \Rightarrow x^3 - 2x = 0 \Rightarrow x(x^2 - 2) = 0$$

$$\Rightarrow x = 0 \ (or) \ (x^2 - 2) = 0$$

$$\Rightarrow x = 0 \ (or) \ x = \pm\sqrt{2}$$

∴ The stationary points are $(0,0), (\sqrt{2}, -\sqrt{2}), (-\sqrt{2}, \sqrt{2})$

	$(0,0)$	$(\sqrt{2}, -\sqrt{2})$	$(-\sqrt{2}, \sqrt{2})$
$A = 12x^2 - 4$	$-4 < 0$	$20 > 0$	$20 > 0$
$B = 4$	4	4	4
$C = 12y^2 - 4$	-4	20	20
$AC - B^2$	0	$384 > 0$	$384 > 0$
Conclusion	Cannot be an extreme point	Minimum point	Minimum point

Minimum at $(\sqrt{2}, -\sqrt{2})$

$$= (\sqrt{2})^4 + (-\sqrt{2})^4 - 2(\sqrt{2})^2 + 4\sqrt{2}(-\sqrt{2}) - 2(-\sqrt{2})^2$$

$$= 4 + 4 - 4 - 8 - 4$$

$$= -8$$

Minimum at $(-\sqrt{2}, \sqrt{2})$

$$= (-\sqrt{2})^4 + (\sqrt{2})^4 - 2(-\sqrt{2})^2 + 4(-\sqrt{2})\sqrt{2} - 2(\sqrt{2})^2$$

$$= 4 + 4 - 8 - 4 - 4 = -8$$

Example:

Examine $f(x,y) = x^3 + y^3 - 12x - 3y + 20$ for its extreme values.

Solution:

$$f(x,y) = x^3 + y^3 - 12x - 3y + 20$$

$$f_x = 3x^2 - 12 \quad ; \quad f_y = 3y^2 - 3$$

$$f_{xx} = 6x = A \,,\ f_{xy} = 0 = B \,,\ f_{yy} = 6y = C$$

To find the stationary points.

$f_x = 0$	$f_y = 0$
$3x^2 - 12 = 0$ $x^2 - 4 = 0$ $x = \pm 2$	$3y^2 - 3 = 0$ $y^2 - 1 = 0$ $y = \pm 1$

$\therefore$ Stationary points are $(2,1), (2,-1), (-2,1), (-2,-1)$

	$(2,1)$	$(2,-1)$	$(-2,1)$	$(-2,-1)$
$A = 6x$	$12 > 0$	$12 > 0$	$-12 < 0$	$-12 < 0$
$B = 0$	0	0	0	0
$C = 6y$	6	-6	6	-6
$AC - B^2$	$72 > 0$	$-72 < 0$	$-72 < 0$	$72 > 0$
Conclusion	Min. point	Saddle point	Saddle point	Max. point

$\therefore$ Maximum value of $f(x,y)$ is

$$f(-2,-1) = (-2)^3 + (-1)^3 - 12(-2) - 3(-1) + 20$$

$$= -8 - 1 + 24 + 3 + 20 \quad = \quad 38$$

Minimum value of $f(x,y)$ is

$$f(2,1) = (2)^3 + (1)^3 - 12(2) - 3(1) + 20$$

$$= 8 + 1 - 24 - 3 + 20 \quad = 2$$

Example:

Find the maxima and minima values of $x^2 - xy + y^2 - 2x + y$

Solution:

$$f(x, y) = x^2 - xy + y^2 - 2x + y$$

$$f_x = 2x - y - 2 \qquad ; \; f_y = -x + 2y + 1$$

$$f_{xx} = 2 = A\,,\; f_{xy} = -1 = B\,,\; f_{yy} = 2 = C$$

To find the stationary points.

$f_x = 0$	$f_y = 0$
$2x - y - 2 = 0 \ldots (1)$	$-x + 2y + 1 = 0 \ldots (2)$

(1) $\Rightarrow \quad 2x - y = 2$

(1) $\times$ 2 $\Rightarrow -2x + 4y = -2$

$$3y = 0$$

$$\Rightarrow y = 0$$

Substitute in (1), we get $2x - 2 = 0$

$$x - 1 = 0$$

$$x = 1$$

$\therefore$ Stationary point is $(1, 0)$

Now, $(AC - B^2)_{(1,0)} = 3 > 0$

Also, $A > 0, B < 0$

$\therefore$ $(1, 0)$ is a minimum point. $\therefore$ Minimum value of $f(x, y)$ is $= -1$.

Example:

Find the extreme values of $f(x, y) = x^3y^2(1 - x - y)$.

Solution:

Given $f(x, y) = x^3y^2(1 - x - y)$

$$= x^3y^2 - x^4y^2 - x^3y^3$$

$$f_x = 3x^2y^2 - 4x^3y^2 - 3x^2y^3$$

$$f_y = 2x^3y - 2x^4y - 3x^3y^2$$

$$f_{xx} = 6xy^2 - 12x^2y^2 - 6xy^3 = A$$

$$f_{xy} = 6x^2y - 8x^3y - 9x^2y^2 = B$$

$$f_{yy} = 2x^3 - 2x^4 - 6x^3y = C$$

To find the stationary points.

$f_x = 0$	$f_y = 0$
$3x^2y^2 - 4x^3y^2 - 3x^2y^3 = 0$ $x^2y^2(3 - 4x - 3y) = 0$ $=> x = 0, y = 0, 4x + 3y = 3$	$2x^3y - 2x^4y - 3x^3y^2 = 0$ $x^3y\ (2 - 2x - 3y) = 0$ $=> x = 0, y = 0, 2x + 3y = 2$

$4x + 3y = 3 \ldots (1) \qquad 2x + 3y = 2 \ldots (2)$

$(1) - (2) \Rightarrow 2x = 1 \quad ; \quad x = \frac{1}{2}$

$(1) - (2)\text{x } 2 \Rightarrow -3y = -1 \quad ; \quad y = \frac{1}{3}$

$\therefore$ Stationary points are (0,0), $\left(\frac{1}{2}, \frac{1}{3}\right)$, (0,1), $\left(0, \frac{2}{3}\right)$, $\left(\frac{3}{4}, 0\right)$ and (1,0)

Since Put $x = 0$ in (1), we get $3y = 3 \Rightarrow y = 1$, i.e., the point is (0,1)

Put $x = 0$ in (2), we get $3y = 2 \Rightarrow y = \frac{2}{3}$, i.e., the point is $\left(0, \frac{2}{3}\right)$

Put $y = 0$ in (1), we get $4x = 3 \Rightarrow x = \frac{3}{4}$, i.e., the point is $\left(\frac{3}{4}, 0\right)$

Put $y = 0$ in (2), we get $2x = 2 \Rightarrow x = 1$, i.e., the point is (1,0)

Let $6xy^2 - 12x^2y^2 - 6xy^3 = A$

$6x^2y - 8x^3y - 9x^2y^2 = B$

$2x^3 - 2x^4 - 6x^3y = C$

	$(0,0)$	$\left(\frac{1}{2},\frac{1}{3}\right)$	$(0,1)$	$\left(0,\frac{2}{3}\right)$	$\left(\frac{3}{4},0\right)$	$(1,0)$
A	0	$\frac{-1}{9}<0$	0	0	0	0
B	0	$\frac{-1}{12}$	0	0	0	0
C	0	$\frac{-1}{8}$	0	0	$\frac{27}{128}$	0
$AC-B^2$	0	$\frac{1}{144}>0$	0	0	0	0
Conclusion	Inconclusive	Max.point	Inconclusive	Inconclusive	Inconclusive	Inconclusive

Thus, $\left(\frac{1}{2},\frac{1}{3}\right)$ is a maximum point

∴ Maximum value $f\left(\frac{1}{2},\frac{1}{3}\right) = \left(\frac{1}{2}\right)^3\left(\frac{1}{3}\right)^2\left[1-\frac{1}{2}-\frac{1}{3}\right]$

$$= \frac{1}{432}$$

Example:

Find the extreme values of $f(x,y) = x^3y^2(12-x-y)$.

Solution:

Given $f(x,y) = x^3y^2(12-x-y)$

$$= 12x^3y^2 - x^4y^2 - x^3y^3$$

$$f_x = 36x^2y^2 - 4x^3y^2 - 3x^2y^3$$

$$f_y = 24x^3y - 2x^4y - 3x^3y^2$$

$$f_{xx} = 72xy^2 - 12x^2y^2 - 6xy^3$$

$$f_{xy} = 72x^2y - 8x^3y - 9x^2y^2 = B$$

$$f_{yy} = 24x^3 - 2x^4 - 6x^3y = C$$

To find the stationary points.

$f_x = 0$	$f_y = 0$
$36x^2y^2 - 4x^3y^2 - 3x^2y^3 = 0$ $x^2y^2(36 - 4x - 3y) = 0$ $=> x = 0, y = 0, 4x + 3y = 36$	$24x^3y - 2x^4y - 3x^3y^2 = 0$ $x^3y\ (24 - 2x - 3y) = 0$ $=> x = 0, y = 0, 2x + 3y = 24$

$$4x + 3y = 36 \dots (1) \qquad 2x + 3y = 24 \dots (2)$$

$(1) - (2) \Rightarrow \quad 2x = 12 \quad ; \quad \therefore\ x = 6$

$\therefore (1) \Rightarrow \quad (4)(6) + 3y = 36$

$$24 + 3y = 36$$

$$3y = 12$$

$$y = 4$$

$\therefore$The Stationary points are $(0,0), (6,4)$

	(0,0)	(6,4)
$72xy^2 - 12x^2y^2 - 6xy^3 = A$	0	$-2304 < 0$
$72x^2y - 8x^3y - 9x^2y^2 = B$	0	$-1728 < 0$
$24x^3 - 2x^4 - 6x^3y = C$	0	$-2592 < 0$
$AC - B^2$	0	$2985984 > 0$
	inconclusive	Max. point

$$A = (72)(6)(16) - (12)(36)(16) - (6)(6)(64)$$

$$= 6912 - 6912 - 2304 = -2304$$

$$B = (72)(36)(4) - 8(216)(4) - 9(36)(16)$$

$$= 10368 - 6912 - 5184 = -1728$$

$$C = 24\,(216) - 2\,(1296) - 6\,(216)\,(4)$$

$$= 5184 - 2592 - 5184 = -2592$$

$$AC - B^2 = (-2304)\,(-2592) - (-1728)2$$

$$= 5971968 - 2985984 = 2985984 > 0$$

Thus (6, 4) is a maximum point

∴ Maximum value $f(x, y) = f\,(6, 4) = (6)3\,(4)2\,(12 - 6 - 4)$

$$= (216)\,(16)\,(2) = 6912.$$

Exercise:

1. Find the extreme points of the following functions:

(i) $2xy - 5x^2 - 2y^2 + 4x + 4y - 4$. **[Ans: $f\left(\frac{2}{3}, \frac{4}{3}\right) = 0$ Maximum]**

(ii) $\frac{1}{x} + xy + \frac{1}{y}$ **[Ans: $f(1, 1) = 3$, Minimum]**

(iii) $x^2 + xy + y^2 + \frac{1}{x} + \frac{1}{y}$ **[A.U 2016]**

[Ans: $f\left(\left(\frac{1}{3}\right)^{1/3}, \left(\frac{1}{3}\right)^{1/3}\right) = 3^{4/3}, Minimum$]

2. Examine the maxima and minima of the following functions.

(i) $x^3 - y^3 - 3xy$ **[Ans: Minimum at (1,1)]**

(ii) $x^3 + 3xy^2 - 15x - 12y$ **[Ans: Maximum at (-2 ,-1)]**

3. Find the extreme values of the function

(i) $x^2 + y^2 + \frac{2}{x} + \frac{2}{y}$ **[Ans: Minimum at (1,1)]**

(ii) $f(x, y) = xy - x^2 - y^2 - 2x - 2y + 4$ **[Ans: Maximum at $(-2, -2) = 8$]**

(ii) $f(x, y) = x^3y^2(a - x - y)$ **[Ans: Maximum at $\left(\frac{a}{2}, \frac{a}{3}\right) = \frac{a^6}{432}$]**

4. Find the extreme points of $f(x, y) = 4xy - x^4 - y^4$ **[Ans: Maximum at $(1, 1) = 2$]**

Lagrange's Method of Undetermined Multipliers

Suppose we require to find the maximum and minimum values of $f(x, y)$ where x, y, z are subject to a constraint equation

$$g(x, y, z) = 0$$

We define a function

$$F(x, y, z, \lambda) = f(x, y, z) + \lambda\ g(x, y, z) \dots (1)$$

Where λ is called Lagrange Multiplier which is independent of x, y, and z.

The necessary conditions for a maximum or minimum are

$$\frac{\partial F}{\partial x} = 0 \dots (2) \qquad \frac{\partial F}{\partial y} = 0 \dots (3) \qquad \frac{\partial F}{\partial z} = 0 \dots (4)$$

Solving the four equations for four unknowns λ, x, y, z, we obtain the point (x, y, z). The point may be a maxima, minima or neither which is decided by the physical consideration.

This method is also applicable when we have more than one constraint equation connecting the variables.

Problems Based on Lagrange's Method of Undetermined Multipliers

Example:

Find the minimum value of $x^2 + y^2 + z^2$ subject to the condition $\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = 1$

Solution:

Let the auxiliary function 'F' be

$$F(x,y,z,\lambda) = (x^2 + y^2 + z^2) + \lambda\left(\frac{1}{x} + \frac{1}{y} + \frac{1}{z} - 1\right)$$

Where λ is Lagrange Multiplier

$\frac{\partial F}{\partial x} = 2x + \lambda\left(\frac{-1}{x^2}\right)$ $= 2x - \frac{\lambda}{x^2}$	$\frac{\partial F}{\partial y} = 2y + \lambda\left(\frac{-1}{y^2}\right)$ $= 2y - \frac{\lambda}{y^2}$	$\frac{\partial F}{\partial z} = 2z + \lambda\left(\frac{-1}{z^2}\right)$ $= 2z - \frac{\lambda}{z^2}$

For a minimum at (x, y, z) we have

$\frac{\partial F}{\partial x} = 0$	$\frac{\partial F}{\partial y} = 0$	$\frac{\partial F}{\partial z} = 0$
$2x - \frac{\lambda}{x^2} = 0$ $2x = \frac{\lambda}{x^2}$ $x^3 = \frac{\lambda}{2}$ $x = \left(\frac{\lambda}{2}\right)^{\frac{1}{3}}$ ….(1)	$2y - \frac{\lambda}{y^2} = 0$ $2y = \frac{\lambda}{y^2}$ $y^3 = \frac{\lambda}{2}$ $y = \left(\frac{\lambda}{2}\right)^{\frac{1}{3}}$ ….(2)	$2z - \frac{\lambda}{z^2} = 0$ $2z = \frac{\lambda}{z^2}$ $z^3 = \frac{\lambda}{2}$ $z = \left(\frac{\lambda}{2}\right)^{\frac{1}{3}}$ ….(3)

From (1), (2) and (3), we get

$$x = y = z$$

Given: $\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = 1$

$$\therefore 3\ \frac{1}{x} = 1$$

$$\therefore 3 = x$$

$$\therefore \Rightarrow y = 3 \text{ and } z = 3$$

∴ (3, 3, 3) is the point where minimum values occur.

The minimum value is $3^2 + 3^2 + 3^2 = 9 + 9 + 9 = 27$.

Example:

A rectangular box open at the top, is to have a volume of 32cc. find the dimensions of the box that requires the least material for its construction.

Solution:

Let x, y, z be the length, breadth and height of the box.

Surface area = $xy + 2yz + 2zx$

Volume = $xyz = 32$

Let the auxiliary function F be

$F(x, y, z, \lambda) = (xy + 2yz + 2zx) + \lambda(xyz - 32)$

Where λ is Lagrange Multiplier

$\frac{\partial F}{\partial x} = y + 2z + \lambda yz$	$\frac{\partial F}{\partial y} = x + 2z + \lambda xz$	$\frac{\partial F}{\partial z} = 2x + 2y + \lambda xy$

When F is extremum

$\frac{\partial F}{\partial x} = 0$	$\frac{\partial F}{\partial y} = 0$	$\frac{\partial F}{\partial z} = 0$
$y + 2z + \lambda yz = 0$ $\Rightarrow$ y + 2 z = $-\lambda yz$ $\Rightarrow \frac{1}{z} + \frac{2}{y} = -\lambda$ …(1)	$x + 2z + \lambda zx = 0$ $\Rightarrow$ x + 2 z = $-\lambda yz$ $\Rightarrow \frac{1}{z} + \frac{2}{x} = -\lambda$ …(2)	$2x + 2y + \lambda xy = 0$ $\Rightarrow$ 2x + 2 y = $-\lambda xy$ $\Rightarrow \frac{2}{y} + \frac{2}{x} = -\lambda$ …(3)

From (1) and (2), we get $\frac{1}{z} + \frac{2}{y} = \frac{1}{z} + \frac{2}{x}$ $\frac{2}{y} = \frac{2}{x}$ $x = y$ ... (4)	From (2) and (3), we get $\frac{1}{z} + \frac{2}{x} = \frac{2}{y} + \frac{2}{x}$ $\frac{1}{z} = \frac{2}{y}$ $y = 2z$ ... (5)

From (4) and (5), we get

$$x = y = 2z$$

Volume = $xyz = 32$ $\Rightarrow$ (2z) (2z) z = 32 $\Rightarrow$ 4 $z^3 = 32$

$$z^3 = \frac{32}{4} = 8 \Rightarrow z = 2 \qquad i.e.\ x = 4,\ y = 4, z = 2$$

$\therefore$ Cost minimum when $x = 4, y = 4, z = 2$

Example:

A rectangular box open at the top is to have a given capacity K. Find the dimensions of the box requiring least material for its construction.

Solution:

Let x, y, z be the dimensions of the box.

Surface area = $x\,y + 2yz + 2zx$

Volume = $xyz = K$

Let the auxiliary function F be

$$F(x, y, z, \lambda) = (xy + 2yz + 2zx) + \lambda(xyz - k)$$

Where λ is Lagrange Multiplier

$\frac{\partial F}{\partial x} = y + 2z + \lambda yz$	$\frac{\partial F}{\partial y} = x + 2z + \lambda zx$	$\frac{\partial F}{\partial z} = 2x + 2y + \lambda xy$

When F is extremum.

$\frac{\partial F}{\partial x} = 0$	$\frac{\partial F}{\partial y} = 0$	$\frac{\partial F}{\partial z} = 0$
$y + 2z + \lambda yz = 0$ $\Rightarrow y + 2z = -\lambda yz$ $\Rightarrow \frac{1}{z} + \frac{2}{y} = - \ldots(1)$	$x + 2z + \lambda zx = 0$ $\Rightarrow x + 2z = -\lambda zx$ $\Rightarrow \frac{1}{z} + \frac{2}{x} = -\lambda \ldots(2)$	$2x + 2y + \lambda xy = 0$ $\Rightarrow 2x + 2y = -\lambda xy$ $\Rightarrow \frac{2}{y} + \frac{2}{x} = -\lambda \ldots(3)$

From (1) and (2), we get $\frac{1}{z} + \frac{2}{y} = \frac{1}{z} + \frac{2}{x}$ $\frac{2}{y} = \frac{2}{x}$ $x = y \quad \ldots\ldots(5)$	From (2) and (3), we get $\frac{1}{z} + \frac{2}{x} = \frac{2}{y} + \frac{2}{x}$ $\frac{1}{z} = \frac{2}{y}$ $y = 2z \quad \ldots.(6)$

From (4) and (5), we get

$$x = y = 2z$$

∴ Volume = xyz = k ⇒ (2z) (2z) z = k ⇒ $4\,z^3 = k$

$$4\,z^3 = k \quad \Rightarrow \; z^3 = \frac{k}{4}$$

$$z = \left(\frac{k}{4}\right)^{\frac{1}{3}}; \quad x = 2\left(\frac{k}{4}\right)^{\frac{1}{3}} \quad ; \quad y = 2\left(\frac{k}{4}\right)^{\frac{1}{3}}$$

∴ Value of minimum = $xy + 2yz + 2zx$

$$= 4\left(\frac{k}{4}\right)^{\frac{1}{3}} + 4\left(\frac{k}{4}\right)^{\frac{1}{3}} + 4\left(\frac{k}{4}\right)^{\frac{1}{3}}$$

$$= 12\left(\frac{k}{4}\right)^{\frac{2}{3}}$$

$$= 3\,(2\,k)^{2/3}$$

Example:

Find the point on the plane a x + b y + c z =p at which $f = x^2 + y^2 + z^2$ has a stationary value and find the stationary value of f , using Lagrange's method of multipliers.

Solution:

Let the auxiliary function F be

F(x,y,z,λ) = $x^2 + y^2 + z^2 + \lambda(ax + by + cz - p)$

Where λ is Lagrange Multiplier

$\frac{\partial F}{\partial x} = 2x + \lambda a$	$\frac{\partial F}{\partial y} = 2y + \lambda\, b$	$\frac{\partial F}{\partial z} = 2z + \lambda\, c$

When F is extremum.

$\frac{\partial F}{\partial x} = 0$	$\frac{\partial F}{\partial y} = 0$	$\frac{\partial F}{\partial z} = 0$
$2x + \lambda\, a = 0$ $\Rightarrow 2\,x = -\lambda\, a$ $\Rightarrow \frac{x}{a} = \frac{-\lambda}{2}$ ... (1)	$2y + \lambda\, b = 0$ $\Rightarrow 2\;y = -\lambda\; b$ $\Rightarrow \frac{y}{b} = \frac{-\lambda}{2}$ ... (2)	$2z + \lambda\, a = 0$ $\Rightarrow 2z = -\lambda\, a$ $\Rightarrow \frac{z}{c} = \frac{-\lambda}{2}$ .... (3)

From (1), (2) & (3), we get

$$\frac{x}{a} = \frac{y}{b} = \frac{z}{c}$$

$$\frac{a\,x}{a^2} = \frac{b\,y}{b^2} = \frac{c\,z}{c^2}$$

$$\Rightarrow \quad \frac{a\,x}{a^2} = \frac{b\,y}{b^2} = \frac{c\,z}{c^2} \Rightarrow \frac{a\,x + by + cz}{a^2 + b^2 + c^2} = \frac{p}{a^2 + b^2 + c^2}$$

$$x = \frac{ap}{a^2 + b^2 + c^2} \; ; \quad y = \frac{bp}{a^2 + b^2 + c^2} \; ; \quad z = \frac{cp}{a^2 + b^2 + c^2}$$

Stationary value of $f = x^2 + y^2 + z^2$

$$= \left(\frac{a\,p}{a^2 + b^2 + c^2} \right)^2 + \left(\frac{b\,p}{a^2 + b^2 + c^2} \right)^2 + \left(\frac{c\,p}{a^2 + b^2 + c^2} \right)^2$$

$$= \frac{a^2\,p^2 + b^2\,p^2 + c^2\,p^2}{(a^2 + b^2 + c^2)^2}$$

$$= \frac{(a^2+b^2+c^2)p^2}{(a^2+b^2+c^2)^2} = \frac{p^2}{a^2 + b^2 + c^2}$$

Example:

Find the greatest and the least distances of the point (3, 4, 12) from the unit sphere whose centre is at the origin.

Solution:

The equation of the unit sphere is $x^2 + y^2 + z^2 = 1$

Distance between (3, 4, 12) to any point of the sphere is

$$d = \sqrt{(x-3)^2 + (y-4)^2 + (z-12)^2}$$

$$Let \;\; f = (x-3)^2 + (y-4)^2 + (z-12)^2$$

Let the auxiliary function F be

$$F(x, y, z, \lambda) = (x-3)^2 + (y-4)^2 + (z-12)^2 + \lambda\,(x^2 + y^2 + z^2 - 1) \quad \dots (1)$$

Where λ is Lagrange multiplier

$\frac{\partial F}{\partial x} = 2\,(x-3) + 2x\lambda$	$\frac{\partial F}{\partial y} = 2\,(y-4) + 2y\lambda$	$\frac{\partial F}{\partial z} = 2\,(z-12) + 2z\lambda$

To find the stationary values

$F_x = 0$	$F_y = 0$	$F_z = 0$
$\Rightarrow 2\,(x-3) + 2x\lambda = 0$ $\Rightarrow x - 3 + x\lambda = 0$ $\Rightarrow (1+\lambda)\,x = 3$ $\Rightarrow x = \frac{3}{1+\lambda}$ $\Rightarrow \frac{x}{3} = \frac{1}{1+\lambda}$ ... (1)	$\Rightarrow 2\,(y-4) + 2y\lambda = 0$ $\Rightarrow y - 4 + y\lambda = 0$ $\Rightarrow (1+\lambda)\,y = 4$ $\Rightarrow y = \frac{4}{1+\lambda}$ $\Rightarrow \frac{y}{4} = \frac{1}{1+\lambda}$ ... (1)	$\Rightarrow 2\,(z-12) + 2z\lambda = 0$ $\Rightarrow z - 12 + z\lambda = 0$ $\Rightarrow (1+\lambda)\,z = 12$ $\Rightarrow z = \frac{12}{1+\lambda}$ $\Rightarrow \frac{z}{12} = \frac{1}{1+\lambda}$ ... (1)

From (1), (2) & (3), we get

$$\frac{x}{3} = \frac{y}{4} = \frac{z}{12} \quad i.e., \; x = \frac{3z}{12} \quad , y = \frac{4z}{12}$$

$$\therefore \; x^2 + y^2 + z^2 = 1$$

$$\Rightarrow \left(\frac{3z}{12}\right)^2 + \left(\frac{4z}{12}\right)^2 + z^2 = 1$$

$$\Rightarrow \frac{9}{144} z^2 + \frac{16}{144} z^2 + z^2 = 1$$

$$\Rightarrow \quad 9z^2 + 16\,z^2 + 144\,z^2 = 144$$

$$\Rightarrow \quad 169\,z^2 = 144$$

$$\Rightarrow \quad z^2 = \frac{144}{169} \qquad \therefore \; z = \pm \frac{12}{13}$$

$$\therefore \; x = \frac{3}{12}\,z = \frac{3}{12}\left(\frac{12}{13}\right) = \pm \frac{3}{13}$$

$$\therefore \; y = \frac{4}{12}\,z = \frac{4}{12}\left(\frac{12}{13}\right) = \pm \frac{4}{13}$$

Hence, the two points are $\left(\frac{3}{13}, \frac{4}{13}, \frac{12}{13}\right)$ and $\left(\frac{-3}{13}, \frac{-4}{13}, \frac{-12}{13}\right)$

$$\therefore \text{ Minimum distance } = \sqrt{\left(3 - \frac{3}{13}\right)^2 + \left(4 - \frac{4}{13}\right)^2 + \left(12 - \frac{12}{13}\right)^2}$$

$$= \quad 12$$

Maximum distance = $\sqrt{\left(3+\frac{3}{13}\right)^2+\left(4+\frac{4}{13}\right)^2+\left(12+\frac{12}{13}\right)^2} = 14$

Example:

Find the dimensions of the rectangular box without top of maximum capacity with surface area 432 square metre.

Solution:

Let x, y, z be the length, breadth and height of the box.

Surface area = $xy + 2yz + 2zx = 432$

Volume $= xyz$

Let the auxiliary function F be

F($, y, z$,λ)= $xyz + \lambda\,(xy + 2yz + 2zx - 432\,)$

$\frac{\partial F}{\partial x} = yz + \lambda\,(y + 2z)$	$\frac{\partial F}{\partial y} = xz + \lambda\,(x + 2z)$	$\frac{\partial F}{\partial z} = xy + \lambda\,(2y + 2x)$

To find the stationary value.

$F_x = 0$	$F_y = 0$	$F_z = 0$
$yz + \lambda\,(y + 2z) = 0$ $\Rightarrow\ yz = -\lambda\,(y + 2z)$ $\Rightarrow\ \frac{y+2z}{yz} = \frac{-1}{\lambda}$ $\Rightarrow\ \frac{1}{z} + \frac{2}{y} = -\frac{1}{\lambda}\ \ldots(1)$	$xz + \lambda\,(x + 2z) = 0$ $\Rightarrow\ xz = -\lambda\,(x + 2z)$ $\Rightarrow\ \frac{x+2z}{xz} = \frac{-1}{\lambda}$ $\Rightarrow\ \frac{1}{z} + \frac{2}{x} = -\frac{1}{\lambda}\ \ldots(2)$	$xy + \lambda\,(2y + 2x) = 0$ $\Rightarrow\ xy = -\lambda\,(2y + 2x)$ $\Rightarrow\ \frac{2y+2x}{xy} = \frac{-1}{\lambda}$ $\Rightarrow\ \frac{2}{x} + \frac{2}{y} = -\frac{1}{\lambda}\ \ldots(3)$

$From\ (1)\ \&\ (2), we\ \ get$	$From\ (2)\ \&\ (3), we\ \ get$
$\frac{1}{z}+\frac{2}{y}=\frac{1}{z}+\frac{2}{x}$	$\frac{1}{z}+\frac{2}{x}=\frac{2}{y}+\frac{2}{x}$
$\Rightarrow \frac{2}{y}=\frac{2}{x}$	$\Rightarrow \frac{1}{z}=\frac{2}{y}$
$\Rightarrow x=y \quad \ldots(4)$	$\Rightarrow y=2z \quad \ldots(5)$

From (4) & (5), we get $x = y = 2z$

Surface area = $xy + 2yz + 2zx = 432$

$$(2z)(2z) + 2\,(2z)z + 2z\,(2z) = 432$$

$$4z^2 + 4z^2 + 4z^2 = 432$$

$$12z^2 = 432$$

$$z^2 = 36 \ \therefore z = 6$$

$$\therefore x = 12\,, y = 12, \quad z = 6 \qquad by\,(6)$$

Thus, the dimension of the box is 12, 12, 6.

Maximum volume = 12 x 12 x 6 = 864 cubic metres.

Example:

Find the foot of the perpendicular from the origin on the plane

$\boldsymbol{2x + 3y - z - 5 = 0}$

Solution:

Let A be (0, 0, 0)

Let the required point B be (x, y, z)

$$AB = d = \sqrt{x^2 + y^2 + z^2}$$

$$(i.e.,) \quad f = d^2 = x^2 + y^2 + z^2 \quad \ldots(A)$$

$$\emptyset = 2x + 3y - z - 5 = 0 \qquad \ldots(B)$$

Let the auxiliary function F be

$$\mathrm{F}(x, y, z, \lambda) = x^2 + y^2 + z^2 - d^2 + \lambda\,(2x + 3y - z - 5)$$

$\frac{\partial F}{\partial x} = 2x + 2\lambda$	$\frac{\partial F}{\partial y} = 2y + 3\lambda$	$\frac{\partial F}{\partial z} = 2z - \lambda$

To find the stationary value.

$F_x = 0$	$F_y = 0$	$F_z = 0$
$2x + 2\lambda = 0$ $\Rightarrow 2x = -2\lambda$ $\Rightarrow x = -\lambda \quad \dots(1)$	$2y + 3\lambda = 0$ $\Rightarrow 2y = -3\lambda$ $\Rightarrow \frac{2}{3}y = -\lambda \quad \dots(2)$	$2z - \lambda = 0$ $\Rightarrow 2z = \lambda$ $\Rightarrow -2z = -\lambda \quad \dots(3)$

From (1), (2) & (3), we get

$$x = \frac{2}{3}y = -2z \quad \dots(4)$$

$$(B) \Rightarrow \quad 2(-2z) + 3(-3z) - z - 5 = 0$$

$$\Rightarrow \quad -4z - 9z - z - 5 = 0$$

$$\Rightarrow \quad -14z = 5$$

$$\Rightarrow \quad z = \frac{-5}{14}$$

$$(4) \Rightarrow x = -2\left(\frac{-5}{14}\right) = \frac{5}{7}$$

$$(4) \Rightarrow y = \left(\frac{3}{2}\right) x = \left(\frac{3}{2}\right) \frac{5}{7} = \frac{15}{14}$$

Hence the extreme value occurs at $x = \frac{5}{7}$, $y = \frac{15}{14}$, $z = \frac{-5}{14}$

∴ The required point is $\left(\frac{5}{7}, \frac{15}{14}, \frac{-5}{14}\right)$ on the plane.

Example:

The temperature $u(x, y, z)$ at any point in space is $u = 400xyz^2$ .Find the highest temperature on surface of the sphere $x^2 + y^2 + z^2 = 1$.

Solution:

$$u = f = 400xyz^2 \quad \dots(A)$$

$$\emptyset = x^2 + y^2 + z^2 - 1 = 0 \quad \dots(B)$$

Let the auxiliary function F be

$$F(x, y, z, \lambda) = 400xyz^2 + \lambda(x^2 + y^2 + z^2 - 1)$$

$\frac{\partial F}{\partial x} = 400yz^2 + \lambda(2x)$	$\frac{\partial F}{\partial y} = 400xz^2 + \lambda(2y)$	$\frac{\partial F}{\partial z} = 800xyz + \lambda(2z)$

To find the stationary value.

$F_x = 0$	$F_y = 0$	$F_z = 0$
$400yz^2 + \lambda(2x) = 0$ $400yz^2 = -\lambda(2x)$ $\frac{200yz^2}{x} = -\lambda \quad ...(1)$	$400xz^2 + \lambda(2y) = 0$ $400xz^2 = -\lambda(2y)$ $\frac{200xz^2}{y} = -\lambda \quad ...(2)$	$800xyz + \lambda(2z) = 0$ $800xyz = -\lambda(2z)$ $400xy = -\lambda \quad ...(3)$

From (1) & (2) , we get $y^2 = x^2 \; .....(4)$

From (2) & (3) , we get $z^2 = 2y^2 \; .....(5)$

From (4) & (5), we get

$$x^2 = y^2 = \frac{1}{2}z^2 \quad ...(6)$$

$$(B) \Rightarrow \frac{1}{2}z^2 + \frac{1}{2}z^2 + z^2 - 1 = 0$$

$$\Rightarrow \quad 2z^2 = 1$$

$$\Rightarrow z^2 = \frac{1}{2} \Rightarrow \quad z = \pm\frac{1}{2}$$

$$(6) \qquad \Rightarrow x^2 = \frac{1}{2}\left(\frac{1}{2}\right) = \frac{1}{4} \qquad \Rightarrow x = \pm\frac{1}{2}$$

$$(6) \qquad \Rightarrow y^2 = \frac{1}{2}\left(\frac{1}{2}\right) = \frac{1}{4} \qquad \Rightarrow y = \pm\frac{1}{2}$$

$\therefore \; u = 400xyz^2$, we select x, y, z to be positive

$$\Rightarrow u = 400\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)$$

$$\Rightarrow u = 50$$

$\therefore$ Maximum temperature is 50

Example:

Find the maximum volume of the largest rectangular parallelepiped that can be inscribed in an ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ **[A.U May 1998]**

Solution:

Let a vertex of such parallelepiped be (x, y, z)

Then all the vertices will be ($\pm x, \pm y, \pm z$)

Then, the sides of the solid be $2x, 2y, 2z$ (lengths)

Hence, the volume $V = (2x)(2y)(2z) = 8xyz$

Let $f = 8xyz$

$$\emptyset = \frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}-1=0$$

Let the auxiliary function F be

$$F(x, y, z, \lambda) = 8xyz + \lambda\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}-1\right)$$

$\frac{\partial F}{\partial x} = 8yz + \lambda\frac{2x}{a^2}$	$\frac{\partial F}{\partial y} = 8xz + \lambda\frac{2y}{b^2}$	$\frac{\partial F}{\partial z} = 8xy + \lambda\frac{2z}{c^2}$

To find the stationary value.

$F_x = 0$	$F_y = 0$	$F_z = 0$
$8yz + \lambda\frac{2x}{a^2} = 0$ $\Rightarrow 8yz = -\lambda\frac{2x}{a^2}$ X[ly] $\frac{x}{2} \Rightarrow \frac{4xyz}{-\lambda} = \frac{x^2}{a^2}$ ... (1)	$8xz + \lambda\frac{2y}{b^2} = 0$ $\Rightarrow 8xz = -\lambda\frac{2y}{b^2}$ X[ly] $\frac{y}{2} \Rightarrow \frac{4xyz}{-\lambda} = \frac{y^2}{b^2}$ ... (2)	$8xy + \lambda\frac{2z}{c^2} = 0$ $\Rightarrow 8xy = -\lambda\frac{2z}{c^2}$ X[ly] $\frac{z}{2} \Rightarrow \frac{4xyz}{-\lambda} = \frac{z^2}{c^2}$ ... (3)

From (1), (2) & (3), we get

$$\frac{x^2}{a^2} = \frac{y^2}{b^2} = \frac{z^2}{c^2} \ldots (4)$$

Given $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$

$$\Rightarrow \frac{3x^2}{a^2} = 1 \text{ by (4)}$$

$$\Rightarrow x^2 = \frac{a^2}{3} \Rightarrow x = \frac{a}{\sqrt{3}}$$

Similarly, $= \frac{b}{\sqrt{3}}$; $z = \frac{c}{\sqrt{3}}$

The extremum point is $\left(\frac{a}{\sqrt{3}}, \frac{b}{\sqrt{3}}, \frac{c}{\sqrt{3}}\right)$

Maximum volume V= $8\left(\frac{a\,bc}{3\sqrt{3}}\right)$

Example:

Find the shortest and the longest distances from the point $(1, 2, -1)$ to the sphere $x^2 + y^2 + z^2 = 24$, using Lagrange's method of constrained maxima and minima.

Solution:

Let (x, y, z) be any point on the sphere.

Distance of the point (x, y, z) from $(1, 2, -1)$ is

$$d = \sqrt{(x-1)^2 + (y-2)^2 + (z+1)^2}$$

$$d^2 = (x-1)^2 + (y-2)^2 + (z+1)^2$$

Subject to constraint $x^2 + y^2 + z^2 - 24 = 0$

Here, $f = (x-1)^2 + (y-2)^2 + (z+1)^2$ and $\emptyset = x^2 + y^2 + z^2 - 24$

Let the auxiliary function F be

$$F(x, y, z, \lambda) = (x-1)^2 + (y-2)^2 + (z+1)^2 + \lambda\,(x^2 + y^2 + z^2 - 24)$$

$\frac{\partial F}{\partial x} = 2(x-1) + 2\lambda x$	$\frac{\partial F}{\partial y} = 2(y-2) + 2\lambda y$	$\frac{\partial F}{\partial z} = 2(z+1) + 2\lambda z$

To find the stationary value.

$F_x = 0$	$F_y = 0$	$F_z = 0$
$2(x-1)+2\lambda x = 0$ $\Rightarrow (x-1)+\lambda x = 0$ $\Rightarrow (1+\lambda)x = 1$ $\Rightarrow x = \frac{1}{(1+\lambda)} \ldots (1)$	$2(y-2)+2\lambda y = 0$ $\Rightarrow (y-2)+\lambda y = 0$ $\Rightarrow (1+\lambda)y = 2$ $\Rightarrow \frac{y}{2} = \frac{1}{(1+\lambda)} \ldots (2)$	$2(z+1)+2\lambda z = 0$ $\Rightarrow (z+1)+\lambda z = 0$ $\Rightarrow (1+\lambda)z = -1$ $\Rightarrow \frac{z}{-1} = \frac{1}{(1+\lambda)} \ldots (3)$

From (1), (2) & (3), we get

$$x = \frac{y}{2} = \frac{z}{-1} \ldots (4) \qquad x = -z \ldots (5) \quad y = -2z \ldots (6)$$

Given: $x^2 + y^2 + z^2 = 24$

$(-z)^2 + (-2z)^2 + (z)^2 = 24$ by (5) & (6)

$z^2 + 4z^2 + z^2 = 24$

$6z^2 = 24 \quad \therefore \quad z^2 = 4$

$\therefore z = \pm 2$

$(5) \Rightarrow x = -z$

If $z = 2$, then $x = -2$

If $z = -2$, then $x = 2$

$(6) \Rightarrow y = -2z$

If $z = 2$, then $y = -4$; If $z = -2$, then $y = 4$

$\therefore$ The stationary points are $(2,4,-2)$ and $(-2,-4,2)$

When the point is $(2,4,-2)$, we get $d = \sqrt{(1)^2 + (2)^2 + (-1)^2} = \sqrt{6}$

When the point is $(-2,-4,2)$, we get $d = \sqrt{(-3)^2 + (-6)^2 + (3)^2} = 3\sqrt{6}$

$\therefore$ Shortest and longest distances are $\sqrt{6}$ and $3\sqrt{6}$ respectively.

Example:

Find the minimum values of x^2yz^3 subject to the condition $2x + y + 3z = a$.

Solution:

Let $f = x^2yz^3$

$\emptyset = 2x + y + 3z - a = 0$

Let the auxiliary function F be

$$F(x, y, z, \lambda) = x^2yz^3 + \lambda(2x + y + 3z - a)$$

$\frac{\partial F}{\partial x} = 2xyz^3 + \lambda 2$	$\frac{\partial F}{\partial y} = x^2z^3 + \lambda$	$\frac{\partial F}{\partial z} = 3x^2yz^2 + 3\lambda$

To find the stationary value.

$F_x = 0$	$F_y = 0$	$F_z = 0$
$2xyz^3 + \lambda 2 = 0$ $xyz^3 = -\lambda \ \ldots (1)$	$x^2z^3 + \lambda = 0$ $x^2z^3 = -\lambda \ldots (2)$	$3x^2yz^2 + 3\lambda = 0$ $x^2yz^2 = -\lambda \ \ldots (3)$

From (1) & (2), we get	From (2) & (3), we get
$xyz^3 = x^2z^3$ $x = y \ \ldots (4)$	$x^2z^3 = x^2yz^2$ $z = y \ \ldots (5)$

From (4) & (5), we get

$$x = y = z \ldots (6)$$

Given: $2x + y + 3z = a$

$\Rightarrow 2z + z + 3z = a$

$\Rightarrow 6z = a$

$\Rightarrow z = \frac{a}{6}$ $\quad (6) \Rightarrow x = y = z = \frac{a}{6}$

$\therefore$ The stationary point is $\left(\frac{a}{6}, \frac{a}{6}, \frac{a}{6}\right)$

Hence, Minimum value of $f = x^2yz^3$

$$= \left(\frac{a}{6}\right)^2 \left(\frac{a}{6}\right) \left(\frac{a}{6}\right)^3 = \left(\frac{a}{6}\right)^6$$

Example:

Find the maximum value of $x^my^nz^p$, when $x + y + z = a$. [A.U. Jan.2009]

Solution:

Let $f = x^m y^n z^p$

$\emptyset = x + y + z - a = 0$

Let the auxiliary function F be

$F(x, y, z, \lambda) = x^m y^n z^p + \lambda(x + y + z - a)$

$\frac{\partial F}{\partial x} = mx^{m-1}y^n z^p + \lambda$	$\frac{\partial F}{\partial y} = nx^m y^{n-1} z^p + \lambda$	$\frac{\partial F}{\partial z} = px^m y^n z^{p-1} + \lambda$

To find the stationary value.

$F_x = 0$	$F_y = 0$	$F_z = 0$
$mx^{m-1}y^n z^p + \lambda = 0$ $\Rightarrow mx^{m-1}y^n z^p = -\lambda$ $\frac{mx^m y^n z^p}{x} = -\lambda \dots (1)$	$nx^m y^{n-1} z^p + \lambda = 0$ $\Rightarrow nx^m y^{n-1} z^p = -\lambda$ $\frac{nx^m y^n z^p}{y} = -\lambda \dots (2)$	$px^m y^n z^{p-1} + \lambda = 0$ $\Rightarrow px^m y^n z^{p-1} = -\lambda$ $\frac{px^m y^n z^p}{z} = -\lambda \dots (3)$

From (1), (2) & (3), we get

$$\frac{mx^m y^n z^p}{x} = \frac{nx^m y^n z^p}{y} = \frac{px^m y^n z^p}{z}$$

$$\div x^m y^n z^p \Rightarrow \frac{m}{x} = \frac{n}{y} = \frac{p}{z} \dots (4)$$

$$x = \frac{m}{p} z \dots (5) \quad y = \frac{n}{p} z \dots (6)$$

Given: $x + y + z = a$

$$\Rightarrow \frac{m}{p} z + \frac{n}{p} z + z = a$$

$$\Rightarrow \left[\frac{m}{p} + \frac{n}{p} + 1\right] z = a$$

$$\Rightarrow \left[\frac{m+n+p}{p}\right] z = a$$

$$\Rightarrow z = \frac{ap}{m+n+p}$$

$$(5) \Rightarrow x = \frac{m}{p} \frac{ap}{m+n+p} = \frac{am}{m+n+p}$$

$$(5) \Rightarrow y = \frac{n}{p} \frac{ap}{m+n+p} = \frac{an}{m+n+p}$$

$\therefore$ The stationary point is $\left(\frac{am}{m+n+p}, \frac{an}{m+n+p}, \frac{ap}{m+n+p}\right)$

$$\therefore \text{ The Maximum value of } f = \left(\frac{am}{m+n+p}\right)^m \left(\frac{an}{m+n+p}\right)^n \left(\frac{ap}{m+n+p}\right)^p$$

$$\text{Max. Value of } f = \frac{a^{m+n+p}\, m^m n^n p^p}{(m+n+p)^{m+n+p}}$$

Exercise:

1. Prove that the rectangular solid of maximum volume which can be inscribed in a sphere is a cube**.**

2. Find the maximum and minimum value of $x^2 + y^2 + z^2$ subject to the condition $x + y + z = 3a$ **.** **[Ans: the minimum value is $3a^2$]**

3. Find the maximum and minimum distances from the origin to the curve $5x^2 + 6xy + 5y - 8 = 0$

[Ans: Maximum distance = 2; minimum distance = 1]

4. Determine the greatest and the smallest values of xy on the ellipse $\frac{x^2}{8} + \frac{y^2}{2} = 1$

[Ans: Greatest distance = 2 and small distance = -2]

5. Find the length of the shortest line from the point $\left[0,0,\frac{25}{9}\right]$ to the surface $z = xy$

[Ans: Distance = $\frac{\sqrt{41}}{3}$]