UNIT I - PARTIAL DIFFERENTIAL EQUATIONS

1.4 HOMOGENEOUS LINEAR PDE OF SECOND AND HIGHER ORDER WITH CONSTANT COEFFICIENTS

Homogeneous Linear PDE of second and higher order with constant co-efficient:

Consider the second order homogeneous linear PDE

$$\frac{\partial^2 z}{\partial x^2}+a_1\frac{\partial^2 z}{\partial x\partial y}+a_2\frac{\partial^2 z}{\partial y^2}=f(x,y) \quad ------ (1)$$

Let the differential operator $D=\frac{\partial}{\partial x}$ & $D'=\frac{\partial}{\partial y}$

$$(1)\Rightarrow \left(D^2+a_1DD'+a_2D'^2\right)z=f(x,y) \quad ------ (2)$$

The general solution of equation (2) is

$$\boxed{z=\text{complementary function}+\text{Particular Integral}=\text{C.F}+\text{P.I}}$$

To find complementary Function:

1. Write the Auxiliary equation by putting $D=m, D'=1, z=1,$ & $RHS=0$

   $(2)\Rightarrow m^2+a_1m+a_2=0$

2. Solve the auxiliary equation, we get the roots of *m*. Say the roots are m_1, m_2

3. Comparing the roots of *m* and write the complementary function.

   Case 1: The Roots are real and distinct : say $m_1\neq m_2$

   $$\boxed{C.F=f_1(y+m_1x)+f_2(y+m_2x)}$$

Case 2: The Roots are real and equal : say $m_1 = m_2 = m$

$$\boxed{C.F = f_1(y+mx) + xf_2(y+mx)}$$

Note: If the roots are $m = \alpha \pm i\beta$

then $C.F = f_1[y+(\alpha+i\beta)] + f_2[y+(\alpha-i\beta)]$

To find Particular Integral :

Type : I

If $\boxed{RHS = e^{ax+by}}$ then

$$P.I = \frac{1}{f(D,D')} e^{ax+by}$$

$\boxed{\text{Rule: } D = a \ \& \ D' = b}$

$$P.I = \frac{1}{f(a,b)} e^{ax+by} \quad , \text{Provided Denominator} \neq 0$$

If $\text{Denominator} = 0$, then 1) multiply the numerator by x 2) differentiating denominator partially w.r.to D

$$P.I = \frac{x}{f'(D,D')} e^{ax+by}$$

$$P.I = \frac{1}{f'(a,b)} e^{ax+by}, \ \text{Provided Denominator} \neq 0 \ \because \text{Replace } D = a \ \& \ D' = b$$

Continuing this process until we get $Dr \neq 0$.

Type : II

If $\boxed{RHS = \sin(ax+by) \ \ (or) \ \ RHS = \cos(ax+by)}$ then

$$P.I = \frac{1}{f(D^2, DD', D'^2)} \sin(ax+by)$$

$\boxed{\text{Rule: } D^2 = -(a^2); DD' = (-ab); D'^2 = -(b^2)}$

$$P.I = \frac{1}{f(-a^2, -ab, -b^2)} \sin(ax+by) \quad \text{, Provided Dr} \neq 0$$

Note 1: After substitutions the denominator will be in terms of $D \,\&\, D'$. Multiply and divide by D so that the denominator will have $D^2 \,\&\, DD'$ terms.

Note 2: After substitutions the denominator will be in terms of $D \,\&\, \text{constant}$ terms,

For eg. $P.I = \frac{1}{D-5} \sin(x-2y)$

Take conjugate of denominator with constant term and multiplied with both numerator and denominator.

$$P.I = \frac{1}{D-5} \times \frac{D+5}{D+5} \sin(x-2y) = \frac{1}{D^2-25} \sin(x-2y)$$

Then apply the rule as usual.

Type : III

If $\boxed{RHS = x^m y^n}$ (polynomial type) **then**

$P.I = \frac{1}{f(D,D)} x^m y^n$, we bring this into a standard binomial format, by taking out highest power term of D.

$(i.e)\ P.I = [1 \pm f(D,D')]^{-1} x^m y^n$

This will be expanded by using the formulae

$$(1-x)^{-1} = 1 + x + x^2 + x^3 + \ldots$$

$$(1+x)^{-1} = 1 - x + x^2 - x^3 + \ldots$$

$$(1-x)^{-2} = 1 + 2x + 3x^2 + 4x^3 + \ldots$$

$$(1+x)^{-2} = 1 - 2x + 3x^2 - 4x^3 + \ldots$$

Note:

$$\frac{1}{D} = \int dx \quad , D' = \frac{\partial}{\partial y}$$

Type : IV (Exponential shifted rule)

If $\boxed{RHS = e^{ax+by}\cos(ax+by)}$ (or) $\boxed{RHS = e^{ax+by}\sin(ax+by)}$ (or) $\boxed{RHS = e^{ax+by}x^m y^n}$ then

$$P.I = \frac{1}{f(D,D')}e^{ax+by}\sin(ax+by)$$

$$P.I = e^{ax+by}\frac{1}{f(D+a,D'+b)}\sin(ax+by)$$

Here after apply the rule as we discussed in Type II&III

Type : V

If $\boxed{RHS = y\cos x}$ (or) $\boxed{RHS = y\sin x}$ then

Case 1:

$$P.I = \frac{1}{D-mD'}y\cos x$$

$$P.I = \frac{1}{D-mD'}\int (c-mx)\cos x\,dx \qquad \because Rule : \boxed{y = c-mx}$$

Case 2:

$$P.I = \frac{1}{D+mD'}y\cos x$$

$$P.I = \frac{1}{D+mD'}\int (c+mx)\cos x\,dx \qquad \because Rule : y = c-mx$$

Note: After integration we have to replace $c-mx = y$

1. **Solve** $\left(D^2 - DD' - 20D'^2\right)z = e^{5x+y} + \sin(4x-y)$

Solution:

Given $\left(D^2 - DD' - 20D'^2\right)z = e^{5x+y} + \sin(4x-y)$

To find C.F

The auxiliary equation is

$m^2 - m - 20 = 0$ $\quad \because$ replace $D = m, D' = 1, z = 0$

$(m-5)(m+4) = 0$

$\boxed{m = 5, -4}$

$\therefore \boxed{C.F = f_1(y-4x) + f_2(y+5x)}$ $\because$ The roots are real and distinct $\Rightarrow C.F = f_1(y+mx) + xf_2(y+mx)$

To find P.I

$$P.I = \frac{1}{D^2 - DD' - 20D'^2}\left[e^{5x+y} + \sin(4x-y)\right]$$

$$P.I = P.I_1 + P.I_2$$

$$P.I_1 = \frac{1}{D^2 - DD' - 20D'^2} e^{5x+y}$$ Rule: replace $D = 5$ & $D' = 1$ $\quad$ Type:1

$$= \frac{1}{25-5-20} e^{5x+y} = \frac{1}{0} e^{5x+y}$$ $\because$ Introducing x in Nr.Diff Dr. partially w.r.to D

$$= \frac{x}{2D - D'} e^{5x+y}$$ Rule: replace $D = 5$ & $D' = 1$

$$= \frac{x}{10-1} e^{5x+y}$$

$$\boxed{P.I_1 = \frac{x}{9} e^{5x+y}}$$

$$P.I_2 = \frac{1}{D^2 - DD' - 20D'^2} \sin(4x-y)$$ here $a = 4$ & $b = -1$ $\quad$ (Type:2)

Rule: replace $D^2 = -(a^2) = -16; D'^2 = -(b^2) = -1$ & $DD' = -(ab) = 4$

$$P.I_2 = \frac{1}{-16-4-20(-1)} \sin(4x-y) = \frac{1}{0} \sin(4x-y)$$

$$= \frac{x}{2D - D'} \sin(4x-y)$$ $\because$ Introducing x in Nr.Diff Dr. partially w.r.to D

$$= \frac{x}{2D - D'} \times \frac{D}{D} \sin(4x-y)$$

	$=\dfrac{xD}{2D^2-DD'}\sin(4x-y)$ $=\dfrac{xD}{2(-16)-4}\sin(4x-y)$ $=\dfrac{xD\left(\sin(4x-y)\right)}{-32-4}$ $P.I_2=\dfrac{4x\cos(4x+3)}{-36}=\dfrac{-1}{9}x\cos(4x+3) \qquad \because \dfrac{d}{dx}(\sin nx)=n\cos nx$ $\boxed{P.I_2 \dfrac{-1}{9}x\cos(4x+3)}$ $\boxed{P.I=\dfrac{x}{9}e^{5x+y}-\dfrac{1}{9}x\cos(4x+3)=\dfrac{x}{9}\left[e^{5x+y}-\cos(4x+3)\right]}$ The general solution is $z=C.F+P.I$ $\boxed{z=f_1(y-4x)+f_2(y+5x)+\dfrac{x}{9}\left[e^{5x+y}-\cos(4x+3)\right]}$
2.	Solve $\dfrac{\partial^3 z}{\partial x^3}-2\dfrac{\partial^3 z}{\partial x^2\partial x}=e^{x+2y}+4\sin(x+y)$ **Solution: same as previous problem** **Hint:** Given $\dfrac{\partial^3 z}{\partial x^3}-2\dfrac{\partial^3 z}{\partial x^2\partial x}=e^{x+2y}+4\sin(x+y)$ $\left(D^3-2D^2D'\right)z=e^{x+2y}+4\sin(x+y) \qquad \because D=\dfrac{\partial}{\partial x}\ \ \&\ D'=\dfrac{\partial}{\partial y}$ $\boxed{m=0,0,2}$ $C.F=f_1(y)+xf_2(y)+f_2(y+2x)$

	$P.I = \frac{-1}{3}e^{x+2y} - 4\cos(x+2y)$
3.	**Solve** $\left(D^2 + DD' - 6D'^2\right)z = x^2y + e^{3x+y}$ **Solution:** Given $\left(D^2 + DD' - 6D'^2\right)z = x^2y + e^{3x+y}$ **To find C.F** The auxiliary equation is $m^2 + m - 6 = 0 \qquad \because$ replace $D = m, D' = 1, z = 0$ $(m-2)(m+3) = 0$ $\boxed{m = 2, -3}$ $\therefore \boxed{C.F = f_1(y-3x) + f_2(y+2x)} \quad \because$ The roots are real and distinct $\Rightarrow C.F = f_1(y+m_1x) + f_2(y+m_2x)$ **To find P.I** $P.I = \frac{1}{D^2 + DD' - 6D'^2}\left[x^2y + e^{3x+y}\right]$ $P.I = P.I_1 + P.I_2$ $P.I_1 = \frac{1}{D^2 + DD' - 6D'^2}x^2y \qquad$ (Type:3) $= \frac{1}{D^2\left[1 + \left(\frac{DD' - 6D'^2}{D^2}\right)\right]}x^2y$ $= \frac{1}{D^2}\left[1 + \left(\frac{D'}{D} - \frac{6D'^2}{D^2}\right)\right]^{-1}x^2y$ $= \frac{1}{D^2}\left[1 - \left(\frac{D'}{D} - \frac{6D'^2}{D^2}\right) + \ldots\right]x^2y \qquad \because (1+x)^{-1} = 1 - x + x^2 - x^3 + \ldots$

$$= \frac{1}{D^2}\left[1 - \frac{D'}{D} + \frac{6D'^2}{D^2}\right]x^2 y \qquad \because D' = \frac{\partial}{\partial y}$$

$$= \frac{1}{D^2}\left[x^2 y - \frac{D'(x^2 y)}{D} + \frac{6D'^2(x^2 y)}{D^2}\right]$$

$$= \frac{1}{D^2}\left[x^2 y - \frac{x^2}{D}\right]$$

$$= \frac{1}{D^2}\left[x^2 y - \int x^2 dx\right] \qquad \because \frac{1}{D} = \int dx$$

$$= \frac{1}{D^2}\left[x^2 y - \frac{x^3}{3}\right]$$

$$= \frac{1}{D}\int\left(x^2 y + \frac{x^3}{3}\right)dx$$

$$= \int\left(\frac{x^3 y}{3} - \frac{x^4}{12}\right)dx$$

$$\boxed{P.I_1 = \frac{x^4 y}{12} - \frac{x^5}{60}}$$

$$P.I_2 = \frac{1}{D^2 + DD' - 6D'^2}e^{3x+y}$$ Rule: replace $D = a = 3$ & $D' = b = 1$ Type:1

$$= \frac{1}{9 + 3 - 6(1)}e^{3x+y} = \frac{1}{6}e^{3x+y}$$

$$\boxed{P.I_2 = \frac{1}{6}e^{3x+y}}$$

$$\boxed{P.I = \frac{x^4 y}{12} - \frac{x^5}{60} + \frac{1}{6}e^{3x+y}}$$

The general solution is

$z = C.F + P.I$

	$\boxed{z = f_1(y-3x) + f_2(y+2x) + \frac{x^4 y}{12} - \frac{x^5}{60} + \frac{1}{6}e^{3x+y}}$
4.	**Solve** $\left(D^2 + 2DD' + D'^2\right)z = x^2 y + e^{x-y}$ **Solution: same as previous problem** **Hint:** $m = -1, -1$ $C.F = f_1(y-x) + xf_2(y-x)$ $P.I = \frac{x^4 y}{12} - \frac{x^5}{30} + \frac{x^2}{2}e^{x-y}$
5.	**Solve** $\left(D^2 - 6DD' + 5D'^2\right)z = xy + e^x \sinh y$ **Solution:** Given $\left(D^2 - 6DD' + 5D'^2\right)z = xy + e^x \sinh y$ $= xy + e^x\left(\frac{e^y - e^{-y}}{2}\right)$ $\quad \because \boxed{\sinh\theta = \frac{e^\theta - e^{-\theta}}{2}}$, here $\theta = y$ $= xy + \left(\frac{e^x e^y - e^x e^{-y}}{2}\right)$ $\left(D^2 - 6DD' + 5D'^2\right)z = xy + \frac{e^{x+y}}{2} - \frac{e^{x-y}}{2}$ $\quad \because e^a e^b = e^{a+b}$ **To find C.F** The auxiliary equation is $m^2 - 6m + 5 = 0$ $\quad \because$ replace $D = m, D' = 1, z = 0$ $(m-1)(m-5) = 0$ $\boxed{m = 1, 2}$ $\therefore \boxed{C.F = f_1(y+x) + f_2(y+5x)}$ $\quad \because$ The roots are real and distinct $\Rightarrow C.F = f_1(y+m_1 x) + xf_2(y+m_2 x)$

To find P.I

$$P.I = \frac{1}{D^2 - 6DD' + 5D'^2}\left[xy + \frac{e^{x+y}}{2} - \frac{e^{x-y}}{2}\right]$$

$$P.I = P.I_1 + P.I_2 - P.I_3 \quad -----(1)$$

To find P.I$_1$

$$P.I_1 = \frac{1}{D^2 - 6DD' + 5D'^2}\, xy \qquad \text{(Type:3)}$$

$$= \frac{1}{D^2\left[1 + \left(\frac{-6DD' + 5D'^2}{D^2}\right)\right]}\, xy$$

$$= \frac{1}{D^2}\left[1 + \left(\frac{-6D'}{D} + \frac{5D'^2}{D^2}\right)\right]^{-1} xy$$

$$= \frac{1}{D^2}\left[1 - \left(\frac{-6D'}{D} + \frac{5D'^2}{D^2}\right) + ...\right] xy \quad \because (1+x)^{-1} = 1 - x + x^2 - x^3 + ...$$

$$= \frac{1}{D^2}\left[1 + \frac{6D'}{D} - \frac{5D'^2}{D^2} + ...\right] xy$$

$$= \frac{1}{D^2}\left[xy + \frac{6x}{D} - \frac{5(0)}{D^2} + ...\right] xy \qquad \because D'(y) = \frac{\partial}{\partial y}(y) = 1 \ \& \ D'^2(y) = \frac{\partial^2}{\partial y^2}(y) = 0$$

$$= \frac{1}{D^2}\left[xy + 6\int x\,dx\right] = \frac{1}{D^2}\left[xy + \frac{6x^2}{2}\right] \qquad \because \frac{1}{D} = \int dx$$

$$= \frac{1}{D}\int\left(xy + 3x^2\right)dx = \frac{1}{D}\left(\frac{x^2 y}{2} + \frac{3x^3}{3}\right) = \int\left(\frac{x^2 y}{2} + x^3\right)dx$$

$$\boxed{P.I_1 = \frac{x^3 y}{6} + \frac{x^4}{4}}$$

To find P.I$_2$

$$P.I_2 = \frac{1}{D^2 - 6DD' + 5D'^2}\frac{e^{x+y}}{2}$$ Type:1

$$= \left(\frac{1}{2}\right)\frac{1}{1-6+5}e^{x+y} = \left(\frac{1}{2}\right)\frac{1}{0}e^{x+y}$$ Rule: replace $D = a = 1 \& D' = b = 1$

$$= \left(\frac{1}{2}\right)\frac{x}{2D - 6D' + 0}e^{x+y}$$

$$= \left(\frac{1}{2}\right)\frac{x}{2-6}e^{x+y}$$

$$= \left(\frac{1}{2}\right)\frac{-x}{4}e^{x+y}$$

$$\boxed{P.I_2 = \frac{-x}{8}e^{x+y}}$$

To find P.I₃

$$P.I_3 = \frac{1}{D^2 - 6DD' + 5D'^2}\frac{e^{x-y}}{2}$$ Type:1

$$= \left(\frac{1}{2}\right)\frac{1}{1+6+5}e^{x+y}$$ Rule: replace $D = a = 1 \& D' = b = -1$

$$\boxed{P.I_3 = \frac{1}{24}e^{x+y}}$$

$$(1) \Rightarrow \ P.I = P.I_1 + P.I_2 - P.I_3 = \boxed{\frac{x^3y}{6} + \frac{x^4}{4} - \frac{x}{8}e^{x+y} - \frac{1}{24}e^{x+y}}$$

The general solution is

$$z = C.F + P.I$$

$$\boxed{z = f_1(y+x) + f_2(y+5x) \ + \frac{x^3y}{6} + \frac{x^4}{4} - \frac{x}{8}e^{x+y} - \frac{1}{24}e^{x+y}}$$

6. **Solve** $\left(D^2 + 2DD' + D'^2\right)z = \sinh(x+y) + e^{x+2y}$

	Solution: same as previous problem **Hint:** $\left(D^2+2DD'+D'^2\right)z=\sinh(x+y)+e^{x+2y}$ $\left(D^2+2DD'+D'^2\right)z=\frac{e^{x+y}-e^{(x-y)}}{2}+e^{x+2y}=\frac{e^{x+y}}{2}-\frac{e^{-x-y}}{2}+e^{x+2y}$ $\left(D^2+2DD'+D'^2\right)z=\frac{e^{x+y}}{2}-\frac{e^{-x-y}}{2}+e^{x+2y}$ $m=-1,-1$ $C.F=f_1(y-x)+xf_2(y-x)$ $P.I=\frac{e^{x+y}}{8}-\frac{e^{-x-y}}{8}-\frac{e^{x+2y}}{9}$
7.	Solve $\left(D^2-4DD'+4D'^2\right)z=e^{x+2y}$ Solution: Hint: $m=2,2$ $C.F=f_1(y+x)+f_2(y+5x)$ $P.I=\frac{x^2}{2}e^{2x+y}$
8.	Solve $\left(2D^2-5DD'+2D'^2\right)z=5\sin(2x+y)$ **Solution:** Given $\left(2D^2-5DD'+2D'^2\right)z=5\sin(2x+y)$ **To find C.F** The auxiliary equation is $2m^2-5m+2=0$ $\because$ replace $D=m, D'=1, z=0$

$$m=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$ here $a=2, b=-5, c=2$

$$=\frac{-(-5)\pm\sqrt{25-4(2)(2)}}{2(2)}$$

$$=\frac{5\pm\sqrt{25-16}}{4}=\frac{5\pm\sqrt{9}}{4}$$

$$=\frac{5\pm\sqrt{25-16}}{4}=\frac{5\pm\sqrt{9}}{4}=\frac{5\pm 3}{4}$$

$$m=\frac{5+3}{4}\ ,\ m=\frac{5-3}{4}\ \Rightarrow \boxed{m=2\ ,\ m=\frac{1}{2}}$$

$$\therefore \boxed{C.F=f_1(y+2x)+f_2\left(y+\frac{x}{2}\right)}$$ $\because$ The roots are real and distinct $\Rightarrow C.F=f_1(y+m_1x)+f_2(y+m_2x)$

To find P.I

$$P.I=\frac{1}{2D^2-5DD'+2D'^2}5\sin(2x+y)$$ Type:2

$$P.I=\frac{1}{2D^2-5DD'+2D'^2}5\sin(2x+y)$$ here $a=2\ \&\ b=1$

Rule: replace $D^2=-(a^2)=-4; D'^2=-(b^2)=-1\ \&\ DD'=-(ab)=-2$

$$P.I=\frac{1}{2(-4)-5(-2)+2(-1)}5\sin(2x+y)$$

$$P.I=\frac{1}{-8+10-2}5\sin(2x+y)=\frac{1}{0}5\sin(2x+y)$$

$$=\frac{x}{4D-5D'+0}5\sin(2x+y)$$ {Introducing x in Nr. & Diff Dr. partially w.r.to D}

$$=\frac{x}{4D-5D'}\times\frac{D}{D}5\sin(2x+y)$$

$$=5\frac{xD}{4D^2-5DD'}\sin(2x+y)$$

	$= 5\frac{xD}{4(-4)-5(-2)}\sin(2x+y) = 5\frac{xD\left[\sin(2x+y)\right]}{-16+10}$ $= \frac{5}{-6}x\left[2\cos(2x+y)\right]$ $\boxed{P.I = \frac{-5}{3}x\cos(2x+y)}$ The general solution is $z = C.F + P.I$ $\boxed{z = f_1(y+2x) + f_2\left(y+\frac{x}{2}\right) - \frac{5}{3}x\cos(2x+y)}$
9.	Solve the equation $\left(D^3 + D^2D' - 4DD'^2 - 4D'^3\right)z = \cos(2x+y)$ **Solution:** Given $\left(D^3 + D^2D' - 4DD'^2 - 4D'^3\right)z = \cos(2x+y)$ **To find C.F** The auxiliary equation is $m^3 + m^2 - 4m - 4 = 0$ $\quad\quad \because$ replace $D = m, D' = 1, z = 0$ $m^2(m+1) - 4(m+1) = 0$ $(m+1)\left(m^2 - 4\right) = 0$ $\boxed{m = -1, -2, 2}$ $\therefore \boxed{C.F = f_1(y-x) + f_2(y-2x) + f_3(y+2x)}$ **To find P.I** $P.I = \frac{1}{D^3 + D^2D' - 4DD'^2 - 4D'^3}\cos(2x+y) \quad here\, a = 2, b = 1 \quad \left(\text{Type:2}\right)$ Rule: replace $D^2 = -(a^2) = -4; D'^2 = -(b^2) = -1 \,\&\, DD' = -(ab) = -2$

$$P.I = \frac{1}{-4D-4D'-4D(-1)-4(-1)D'}\cos(2x+y)$$

$$= \frac{1}{-4D-4D'+4D+4D'}\cos(2x+y) == \frac{1}{0}\cos(2x+y)$$

$$P.I = \frac{x}{3D^2+2DD'-4D'^2-0}\cos(2x+y) \quad \{\text{Introducing } x \text{ in Nr.} \,\&\, \text{Diff } D\text{r. partially w.r.to } D\}$$

$$= \frac{x}{3(-4)+2(-2)-4(-1)}\cos(2x+y) = \frac{x}{-12-4+4}\cos(2x+y)$$

$$\boxed{P.I = \frac{-x}{12}\cos(2x+y)}$$

The general solution is

$$z = C.F + P.I$$

$$\boxed{z = f_1(y-x) + f_2(y-2x) + f_3(y+2x) - \frac{x}{12}\cos(2x+y)}$$

10. **Solve** $\left(D^3 - 7DD'^2 - 6D'^3\right)z = \cos(x+2y)+4$

Solution:

Given $\left(D^3 - 7DD'^2 - 6D'^3\right)z = \cos(x+2y)+4$

To find C.F

The auxiliary equation is

$m^3 - 7m - 6 = 0 \qquad \because$ replace $D = m, D' = 1, z = 0$

m=-1	1	0	-7	6
	0	-1	1	6
	1	-1	-6	0

$m=-1, m^2-m-6=0$

$m=-1, (m-3)(m+2)=0$

$\boxed{m=-1,-2,3}$

$\therefore \boxed{C.F = f_1(y-x)+f_2(y-2x)+f_3(y+3x)}$

To find P.I

$$P.I = \frac{1}{D^3-7DD'^2-6D'^3}\left[\cos(x+2y)+4\right]$$

$$P.I = P.I_1 + P.I_2$$

$$P.I_1 = \frac{1}{D^3-7DD'^2-6D'^3}\cos(x+2y) \quad \textit{here}\; a=1, b=2 \quad (\text{Type:}2)$$

Rule: replace $D^2=-(a^2)=-1; D'^2=-(b^2)=-4 \;\&\; DD'=-(ab)=-2$

$$P.I_1 = \frac{1}{(-1)D-7D(-2)-6(-2)D'}\cos(x+2y) = \frac{1}{-D+14D+12D'}\cos(x+2y)$$

$$= \frac{1}{13D+12D'}\times\frac{D}{D}\cos(x+2y)$$

$$= \frac{D}{13D^2+12D'D}\cos(x+2y)$$

$$= \frac{D}{13(-1)+12(-2)}\cos(x+2y) = \frac{D\cos(x+2y)}{-13-24}$$

$$\boxed{P.I_1 = \frac{-\sin(x+2y)}{37}}$$

To find P.I₂

	$P.I_2 = \frac{1}{D^3 - 7DD'^2 - 6D'^3} 4e^{0x+0y}$ $\quad \because e^0 = 1$ Type:1 $= \frac{1}{D^3 - 7DD'^2 - 6D'^3} 4e^{0x+0y} = \frac{1}{0} 4e^{0x+0y}$ Rule: Replace $D = 0, D' = 0$ $= \frac{x}{3D^2 - 7D'^2 - 0} 4e^{0x+0y} = \frac{x}{0} 4e^{0x+0y}$ $= \frac{x}{6D - 0} 4e^{0x+0y} = \frac{x}{0} 4e^{0x+0y}$ $= \frac{x^2}{6} 4$ $\boxed{P.I_2 = \frac{2x^2}{3}}$ $\boxed{P.I = \frac{-1}{37}\sin(x+2y) + \frac{2x^2}{3}}$ **The general solution is** $z = C.F + P.I$ $\boxed{z = f_1(y-x) + f_2(y-2x) + f_3(y+3x) - \frac{1}{37}\sin(x+2y) + \frac{2x^2}{3}}$
11.	**Solve** $\left(D^2 + 3DD' - 4D'^2\right)z = xy + \cos(2x+y)$ **Solution: same as previous problem** **Hint:** $m = -4, 1$ $C.F = f_1(y-4x) + xf_2(y+x)$ $P.I = \frac{-1}{6}\cos(2x+y) + \frac{x^3 y}{6} - \frac{x^4}{8}$
12.	**Solve** $\left(D^2 + 3DD' - 4D'^2\right)z = x + \sin y$

	Hint: $m=-4,1$ $C.F = f_1(y-4x)+xf_2(y+x)$ $P.I = \frac{1}{D^2+3DD'-4D'^2}\left[x+\sin(0x+y)\right]=...=\frac{x^3}{6}+\frac{\sin y}{4}$
13.	**Solve** $\left(D^2-DD'-2D'^2\right)z=\left(2x+3y\right)+e^{3x+4y}$ **Hint:** $m=-1,2$ $C.F = f_1(y-x)+f_2(y+2x)$ $P.I = \frac{5x^3}{6}+\frac{3x^2y}{2}+\frac{1}{35}e^{3x+4y}$
14.	Solve $\left(D^2-2DD'+D'^2\right)z=(2+4x)e^{x+2y}$ **Solution:** Given $\left(D^2-2DD'+D'^2\right)z=x^2y^2e^{x+2y}$ **To find C.F** The auxiliary equation is $m^2-2m+1=0 \qquad \because$ replace $D=m, D'=1, z=0$ $(m-1)(m-1)=0$ $\boxed{m=1,1}$ $\therefore \boxed{C.F = f_1(y+x)+xf_2(y+x)}$ **To find P.I** $P.I = \frac{1}{D^2-2DD'+D'^2}(2+4x)e^{x+2y} \quad here\, a=2, b=1 \quad \left(\text{Type:4}\right)$

	$P.I = \frac{1}{(D-D')^2}(2+4x)e^{x+2y}$ Rule: replace $D = D+a = D+1; D' = D'+b = D'+2$ $P.I = e^{x+2y}\frac{1}{(D+1-D'-2)^2}(2+4x)$ $= e^{x+2y}\frac{1}{(D-D'-1)^2}(2+4x)$ $= e^{x+2y}\frac{1}{[-(1-D+D')]^2}(2+4x) = e^{x+2y}\frac{1}{\left[1-\left(D-D'\right)\right]^2}(2+4x)$ $= e^{x+2y}\left[1-\left(D-D'\right)\right]^{-2}(2+4x)$ $= e^{x+2y}\left[1+2\left(D-D'\right)+3\left(D-D'\right)^2+...\right](2+4x) \quad \because (1-x)^{-2} = 1+2x+3x^2+4x^3+...$ $= e^{x+2y}\left[1+2D-2D'+3\left(D^2-2DD'+D'^2\right)+...\right](2+4x)$ $= e^{x+2y}\left[1+2D-2D'+3D^2-6DD'+3D'^2\right](2+4x)$ $= e^{x+2y}\left[1+2D+3D^2\right](2+4x)$ $\because$ there is no y term in RHS, neglect the term D' $= e^{x+2y}\left[\left(2+4x\right)+2D(2+4x)+3D^2(2+4x)\right]$ $= e^{x+2y}\left[2+4x+2(4)+0\right]$ $\boxed{P.I = e^{x+2y}\left[4x+10\right]}$ **The general solution is** $z = C.F + P.I$ $\boxed{z = f_1(y+x)+xf_2(y+x)+e^{x+2y}\left[4x+10\right]}$
15.	**Solve** $\frac{\partial^2 z}{\partial x^2}-\frac{\partial^2 z}{\partial y^2} = e^{x-y}\sin(2x+3y)$

Solution:

Given $\frac{\partial^2 z}{\partial x^2} - \frac{\partial^2 z}{\partial y^2} = e^{x-y}\sin(2x+3y)$

$\left(D^2 - D'^2\right)z = e^{x-y}\sin(2x+3y)$

To find C.F

The auxiliary equation is

$m^2 - 1 = 0 \qquad \because$ replace $D = m, D' = 1, z = 0$

$m^2 = 1 \Rightarrow \boxed{m = -1, 1}$

$\therefore \boxed{C.F = f_1(y-x) + f_2(y+x)}$

To find P.I

$P.I = \frac{1}{D^2 - D'^2} e^{x-y}\sin(2x+3y) \quad here\, a = 1, b = -1 \quad (\text{Type:4})$

Rule: replace $D = D + a = D + 1; D' = D' + b = D' - 1$

$$P.I = e^{x-y}\frac{1}{(D+1)^2 - (D'-1)^2}\sin(2x+3y)$$

$$= e^{x-y}\frac{1}{D^2 + 2D + 1 - D'^2 + 2D' - 1}\sin(2x+3y)$$

$$= e^{x-y}\frac{1}{D^2 + 2D - D'^2 + 2D'}\sin(2x+3y) \quad \text{Here } a = 2, b = 3$$

Rule: replace $D^2 = -(a^2) = -4; D'^2 = -(b^2) = -9$ & $DD' = -(ab) = -6$

$$= e^{x-y}\frac{1}{-4 + 2D - (-9) + 2D'}\sin(2x+3y) = e^{x-y}\frac{1}{2D + 2D' + 5}\sin(2x+3y)$$

$$= e^{x-y}\frac{1}{2D + 2D' + 5}\times\frac{D}{D}\sin(2x+3y)$$

$$= e^{x-y}\frac{D}{2D^2 + 2DD' + 5D}\sin(2x+3y)$$

	$= e^{x-y} \dfrac{D}{-8-12+5D} \sin(2x+3y) = e^{x-y} \dfrac{D}{5D-20} \sin(2x+3y)$ If we multiply and divide by D, we can not get the term D^2, D'^2 term, so we take conjugate for constent term and multiplied with both Nr. & Dr. $= e^{x-y} \dfrac{D}{5D-20} \times \dfrac{5D+20}{5D+20} \sin(2x+3y)$ $= e^{x-y} \dfrac{5D^2+20D}{25D^2-400} \sin(2x+3y)$ $= e^{x-y} \dfrac{5D^2 \sin(2x+3y) + 20D \sin(2x+3y)}{25(-4)-400}$ $= e^{x-y} \dfrac{5D \cos(2x+3y) \times 2 + 20 \cos(2x+3y) \times 2}{-100-400}$ $= e^{x-y} \dfrac{-20 \sin(2x+3y) + 40 \cos(2x+3y)}{-500}$ $\boxed{P.I = \dfrac{e^{x-y}}{25}\left[\sin(2x+3y) - 2\cos(2x+3y)\right]}$ **The general solution is** $z = C.F + P.I$ $\boxed{z = f_1(y-x) + f_2(y+x) + \dfrac{e^{x-y}}{25}\left[\sin(2x+3y) - 2\cos(2x+3y)\right]}$
16.	**Solve** $\left(D^2 + DD' - 6D'^2\right) z = y \cos x$ **Solution:** Given $\left(D^2 + DD' - 6D'^2\right) z = y \cos x$ **To find C.F** The auxiliary equation is $m^2 + m - 6 = 0 \qquad \because$ replace $D = m, D' = 1, z = 0$ $(m-2)(m+3) = 0$

	$\boxed{m=2,-3}$ $\therefore \boxed{C.F = f_1(y-3x)+f_2(y+2x)}$ $\because$ The roots are real and distinct $\Rightarrow C.F = f_1(y+m_1x)+f_2(y+m_2x)$ **To find P.I** $P.I = \dfrac{1}{D^2+DD'-6D'^2} y\cos x$ Type: 5 $= \dfrac{1}{(D+3D')(D-2D')} y\cos x$ $= \dfrac{1}{(D+3D')}\int (c-2x)\cos x\, dx$ $\because$ *Rule*: $y=c-mx$ here $m=2$ $= \dfrac{1}{(D+3D')}\left[(c-2x)(\sin x)-(-2)(-\cos x)\right]$ $\because \int uvdx = uv_1 - u'v_2 + \ldots$ $= \dfrac{1}{(D+3D')}\left[y\sin x - 2\cos x\right]$ $= \dfrac{1}{(D+3D')}\int\left[(c+3x)\sin x - 2\cos x\right]dx$ $\because$ *Rule*: $y=c+mx$ here $m=3$ $=(c+3x)(-\cos x)-(3)(-\sin x)-2\sin x$ $\because$ $y=c+3x$ $=-y\cos x+3\sin x-2\sin x$ $\boxed{P.I=\sin x - y\cos x}$ **The general solution is** $z = C.F + P.I$ $\boxed{z = f_1(y-3x)+f_2(y+2x)+\sin x - y\cos x}$
17.	**Solve** $\left(D^2-5DD'+6D'^2\right)z = y\sin x$ **Solution:** Given $\left(D^2-5DD'+6D'^2\right)z = y\sin x$

To find C.F

The auxiliary equation is

$m^2 - 5m + 6 = 0$ $\quad\because$ replace $D = m, D' = 1, z = 0$

$(m-2)(m-3) = 0$

$\boxed{m = 2, 3}$

$\therefore \boxed{C.F = f_1(y+3x) + f_2(y+2x)}$ $\quad\because$ The roots are real and distinct $\Rightarrow C.F = f_1(y+m_1x) + f_2(y+m_2x)$

To find P.I

$P.I = \dfrac{1}{D^2 - 5DD' + 6D'^2} y \sin x$ Type: 5

$= \dfrac{1}{(D-3D')(D-2D')} y \sin x$

$= \dfrac{1}{(D-3D')} \int (c-2x) \sin x\, dx$ $\quad\because$ *Rule*: $y = c - mx$ here $m = 2$

$= \dfrac{1}{(D-3D')} \left[(c-2x)(-\cos x) - (-2)(-\sin x)\right]$ $\quad\because \int uv dx = uv_1 - u'v_2 + \ldots$

$= \dfrac{1}{(D-3D')} \left[-y \cos x - 2 \sin x\right] = \dfrac{-1}{(D-3D')} \left[y \cos x + 2 \sin x\right]$

$= -\int \left[(c-3x) \cos x + 2 \sin x\right] dx$ $\quad\because$ *Rule*: $y = c - mx$ here $m = 3$

$= -\left[(c-3x)(\sin x) - (-3)(-\cos x) + 2(-\cos x)\right]$ $\quad\because \; y = c - 3x$

$= -\left[y \sin x - 3\cos x - 2\cos x\right]$

$\boxed{P.I = 5\cos x - y \sin x}$

The general solution is

$z = C.F + P.I$

$\boxed{z = f_1(y+3x) + f_2(y+2x) + 5\cos x - y \sin x}$