2.8 POISSON'S AND LAPLACE EQUATIONS

POISSON'S EQUATIONS

Poisson's equations are derived from Gauss's law

According to Gauss's law in point form, the divergence of electric flux density is equal to the charge density

$$\nabla . \boldsymbol{D} = \rho_v$$

$$\boldsymbol{D} = \varepsilon \boldsymbol{E}$$

Substitute $\boldsymbol{D}$ in above equation

$$\nabla . \boldsymbol{D} = \rho_v$$

$$\nabla . \varepsilon \boldsymbol{E} = \rho_v$$

$$\varepsilon \, \nabla . \boldsymbol{E} = \rho_v$$

$$\nabla . \boldsymbol{E} = \frac{\rho_v}{\varepsilon}$$

But

$$\boldsymbol{E} = -\nabla V$$

Substitute $\boldsymbol{E}$ in above equation

$$\nabla . \boldsymbol{E} = \frac{\rho_v}{\varepsilon}$$

$$\nabla . -(\nabla V) = \frac{\rho_v}{\varepsilon}$$

$$\nabla . -(\nabla V) = \frac{\rho_v}{\varepsilon}$$

$$\nabla^2 V = -\frac{\rho_v}{\varepsilon}$$

This is the Poisson's Equation.

LAPLACE EQUATIONS

For Cartesian co-ordinate system.

$$\nabla . \nabla V = \frac{\partial}{\partial x}\left(\frac{\partial V}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial V}{\partial y}\right) + \frac{\partial}{\partial z}\left(\frac{\partial V}{\partial z}\right)$$

$$\nabla . \nabla V = \frac{\partial^2 V}{\partial x} + \frac{\partial^2 V}{\partial y} + \frac{\partial^2 V}{\partial z}$$

$$\nabla^2 V = \frac{\partial^2 V}{\partial x} + \frac{\partial^2 V}{\partial y} + \frac{\partial^2 V}{\partial z}$$

Equate both $\nabla^2 V$

$$\nabla^2 V = \frac{\partial^2 V}{\partial x} + \frac{\partial^2 V}{\partial y} + \frac{\partial^2 V}{\partial z} = -\frac{\rho_v}{\varepsilon}$$

For Cylindrical co-ordinate system.

$$\nabla . \nabla V = \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho \frac{\partial V}{\partial x}\right) + \frac{1}{\rho^2}\left(\frac{\partial^2 V}{\partial \varphi^2}\right) + \left(\frac{\partial^2 V}{\partial z^2}\right)$$

$$\nabla^2 V = \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho \frac{\partial V}{\partial x}\right) + \frac{1}{\rho^2}\left(\frac{\partial^2 V}{\partial \varphi^2}\right) + \left(\frac{\partial^2 V}{\partial z^2}\right)$$

Equate both $\nabla^2 V$

$$\nabla^2 V = \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho \frac{\partial V}{\partial x}\right) + \frac{1}{\rho^2}\left(\frac{\partial^2 V}{\partial \varphi^2}\right) + \left(\frac{\partial^2 V}{\partial z^2}\right) = -\frac{\rho_v}{\varepsilon}$$

For spherical co-ordinate system.

$$\nabla . \nabla V = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial V}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial V}{\partial \theta}\right) + \frac{1}{r^2 \sin^2\theta}\left(\frac{\partial^2 V}{\partial \varphi^2}\right)$$

$$\nabla^2 V = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial V}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial V}{\partial \theta}\right) + \frac{1}{r^2 \sin^2\theta}\left(\frac{\partial^2 V}{\partial \varphi^2}\right)$$

Equate both $\nabla^2 V$

$$\nabla^2 V = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial V}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial V}{\partial \theta}\right) + \frac{1}{r^2 \sin^2\theta}\left(\frac{\partial^2 V}{\partial \varphi^2}\right) = -\frac{\rho_v}{\varepsilon}$$

If the volume charge density (ρ_v) is zero, then

$$\nabla^2 V = -\frac{\rho_v}{\varepsilon}$$

$$\rho_v = 0$$

$$\nabla^2 V = 0$$

This is the Laplace Equations. This operator ∇^2 is called Laplacian operator.