1.3 CAYLEY HAMILTON THEOREM

Cayley Hamilton Theorem

Statement: Every square matrix satisfies its own characteristic equation.

Uses of Cayley Hamilton Theorem:

To calculate (i) the positive integral power of A and

(ii) the inverse of a non-singular square matrix A.

Example: Show that the matrix $\begin{bmatrix} \mathbf{1} & \mathbf{-2} \\ \mathbf{2} & \mathbf{1} \end{bmatrix}$ satisfies its own characteristic equation.

Solution:

$$\text{Let } A = \begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix}$$

The characteristic equation of the given matrix is $|A - \lambda I| = 0$

$$\lambda^2 - S_1\lambda + S_1 = 0$$

Where $S_1 =$ sum of the main diagonal elements.

$$= 1 + 1 = 2$$

$$S_2 = |A| = \begin{vmatrix} 1 & -2 \\ 2 & 1 \end{vmatrix} = 1 + 4 = 5$$

∴ The characteristic equation is $\lambda^2 - 2\lambda + 5 = 0$

To prove: $A^2 - 2A + 5I = O$

$$A^2 = AA = \begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} -3 & -4 \\ 4 & -3 \end{pmatrix}$$

$$A^2 - 2A + 5I = \begin{pmatrix} -3 & -4 \\ 4 & -3 \end{pmatrix} - 2\begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix} + 5\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} -3 & -4 \\ 4 & -3 \end{pmatrix} + \begin{pmatrix} -2 & 4 \\ -4 & -2 \end{pmatrix} + \begin{pmatrix} 5 & 0 \\ 0 & 5 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = O$$

Therefore, the given matrix satisfies its own characteristic equation.

Example: Verify Cayley – Hamilton theorem find $\mathbf{A^4}$ and $\mathbf{A^{-1}}$ when $\mathbf{A} =$

$$\begin{bmatrix} \mathbf{-2} & \mathbf{-1} & \mathbf{2} \\ \mathbf{-1} & \mathbf{2} & \mathbf{-1} \\ \mathbf{1} & \mathbf{-1} & \mathbf{2} \end{bmatrix}$$

Solution:

The characteristic equation of A is $|A - \lambda I| = 0$

i.e., $\lambda^3 - S_1\lambda^2 + S_2\lambda = 0$ where

$S_1 =$ sum of its leading diagonal elements $= 2 + 2 + 2 = 6$

$S_2 =$ sum of the minors of its leading diagonal elements

$$= \begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix} + \begin{vmatrix} 2 & 2 \\ 1 & 2 \end{vmatrix} + \begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix}$$

$$= (4-1) + (4-2) + (4-1) = 3 + 2 + 3 = 8$$

$$S_3 = |A| = 2(4-1) + 1(-2+1) + 2(1-2)$$

$$= 2(3) + 1(-1) + 2(-1) = 6 - 1 - 2 = 3$$

∴ The characteristic equation of A is $\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$

$$i.e., \lambda^3 - 6\lambda^2 + 8\lambda - 3 = 0$$

By Cayley-Hamilton theorem

[Every square matrix satisfies its own characteristic equation]

$(i.e.)\ A^3 - 6A^2 + 8A - 3I = 0$ ... (1)

Verification:

$$A^2 = A \times A = \begin{bmatrix} -2 & -1 & 2 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}\begin{bmatrix} -2 & -1 & 2 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} = \begin{bmatrix} 7 & -6 & 9 \\ -5 & 6 & -6 \\ 5 & -5 & 7 \end{bmatrix}$$

$$A^3 = A \times A^2 = \begin{bmatrix} -2 & -1 & 2 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}\begin{bmatrix} 7 & -6 & 9 \\ -5 & 6 & -6 \\ 5 & -5 & 7 \end{bmatrix} = \begin{bmatrix} 29 & -28 & 38 \\ -22 & 23 & -28 \\ 22 & -22 & 29 \end{bmatrix}$$

$$\therefore \quad A^3 - 6A^2 + 8A - 3I = \begin{bmatrix} 29 & -28 & 38 \\ -22 & 23 & -28 \\ 22 & -22 & 29 \end{bmatrix} - 6\begin{bmatrix} 7 & -6 & 9 \\ -5 & 6 & -6 \\ 5 & -5 & 7 \end{bmatrix}$$

$$+8\begin{bmatrix} -2 & -1 & 2 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} - 3\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 29 & -28 & 38 \\ -22 & 23 & -28 \\ 22 & -22 & 29 \end{bmatrix} - \begin{bmatrix} 42 & -36 & 54 \\ -30 & 36 & -36 \\ 30 & -30 & 42 \end{bmatrix}$$

$$+ \begin{bmatrix} 16 & -8 & 16 \\ -8 & 16 & -8 \\ 8 & -8 & 16 \end{bmatrix} - \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = O$$

To find A^4:

$(1) \Rightarrow A^3 - 6A^2 - 8A + 3I \quad \dots (2)$

Multiply A on both sides, we get

$$A^4 = 6A^3 - 8A^2 + 3A = 6[6A^2 - 8A + 3I\,] - 8A^2 + 3A \text{ by } (2)$$

$$= 36A^2 - 48A + 18I - 8A^2 + 3A$$

$$A^4 = 28A^2 - 45A + 18I \qquad \dots (3)$$

$$(1) \Rightarrow \quad A^4 = 28\begin{bmatrix} 7 & -6 & 9 \\ -5 & 6 & -6 \\ 5 & -5 & 7 \end{bmatrix} - 45\begin{bmatrix} 2 & -1 & 2 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} + 18\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 196 & -168 & 252 \\ -140 & 168 & -168 \\ 140 & -140 & 196 \end{bmatrix} - \begin{bmatrix} 90 & -45 & 90 \\ -45 & 90 & -45 \\ 45 & -45 & 90 \end{bmatrix} +$$

$$\begin{bmatrix} 18 & 0 & 0 \\ 0 & 18 & 0 \\ 0 & 0 & 18 \end{bmatrix}$$

$$= \begin{bmatrix} 124 & -123 & 162 \\ -95 & 96 & -123 \\ 95 & -95 & 124 \end{bmatrix}$$

To find A^{-1}:

$(1) \times A^{-1} \Rightarrow A^2 - 6A + 8I - 3\,A^{-1} = O$

$3\,A^{-1} = A^2 - 6A + 8I$

$$3\,A^{-1} = \begin{bmatrix} 7 & -6 & 9 \\ -5 & 6 & -6 \\ 5 & -5 & 7 \end{bmatrix} - 6\begin{bmatrix} 2 & -1 & 2 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} + 8\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 7 & -6 & 9 \\ -5 & 6 & -6 \\ 5 & -5 & 7 \end{bmatrix} + \begin{bmatrix} -12 & 6 & -12 \\ 6 & -12 & 6 \\ -6 & 6 & -12 \end{bmatrix} + \begin{bmatrix} 8 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & 8 \end{bmatrix}$$

$$3\,A^{-1} = \begin{bmatrix} 3 & 0 & -3 \\ 1 & 2 & 0 \\ -1 & 1 & 3 \end{bmatrix} \Rightarrow A^{-1} = \frac{1}{3}\begin{bmatrix} 3 & 0 & -3 \\ 1 & 2 & 0 \\ -1 & 1 & 3 \end{bmatrix}$$

Example: Find A^{-1} if $A = \begin{bmatrix} 1 & -1 & 4 \\ 3 & 2 & -1 \\ 2 & 1 & -1 \end{bmatrix}$, using Cayley- Hamilton theorem.

Solution:

The characteristic equation of A is $|A - \lambda I| = 0$

$(i.e.)\ \lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$ where

$S_1 =$ sum of its leading diagonal elements

$= 1 + 2 + (-1) = 2$

$S_2 =$ sum of the minors of its leading diagonal elements

$$= \begin{vmatrix} 2 & -1 \\ 1 & -1 \end{vmatrix} + \begin{vmatrix} 1 & 4 \\ 2 & -1 \end{vmatrix} + \begin{vmatrix} 1 & -1 \\ 3 & 2 \end{vmatrix}$$

$$= (-2 + 1) + (-1 - 8) + (2 + 3)$$

$$= (-1) + (-9) + 5 = -5$$

$$S_3 = |A| = \begin{vmatrix} 1 & -1 & 4 \\ 3 & 2 & -1 \\ 2 & 1 & -1 \end{vmatrix}$$

$$= 1(-2 + 1) + 1(-3 + 2) + (3 - 4)$$

$$= 1(-1) + 1(-1) + 4(-1)$$

$$= -1 - 1 - 4 = -6$$

$\therefore$ The Characteristic equation is $\lambda^3 - 2\lambda^2 - 5\lambda + 6 = 0$

By Cayley Hamilton Theorem we get

[Every square matrix satisfies its own characteristic equation]

$\therefore A^3 - 2A^2 - 5A + 6I = O \qquad \ldots (1)$

To find A^{-1}

$(1) \times A^{-1} \Rightarrow A^2 - 2A - 5I + 6A^{-1} = O$

$A^2 - 2A - 5I + 6A^{-1} = O$

$6A^{-1} = -A^2 + 2A + 5I$

$$A^{-1} = \frac{1}{6}[-A^2 + 2A + 5I] \qquad \ldots (2)$$

$A^2 = A \times A$

$$= \begin{bmatrix} 1 & -1 & 4 \\ 3 & 2 & -1 \\ 2 & 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & -1 & 4 \\ 3 & 2 & -1 \\ 2 & 1 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 1-3+8 & -1-2+4 & 4+1-4 \\ 3+6-2 & -3+4-1 & 12-2+1 \\ 2+3-2 & -2+2-1 & 8-1+1 \end{bmatrix} = \begin{bmatrix} 6 & 1 & 1 \\ 7 & 0 & 11 \\ 3 & -1 & 8 \end{bmatrix}$$

$$-A^2 + 2A + 5I = \begin{bmatrix} -6 & -1 & -1 \\ -7 & 0 & -11 \\ -3 & 1 & -8 \end{bmatrix} + 2\begin{bmatrix} 1 & -1 & 4 \\ 3 & 2 & -1 \\ 2 & 1 & -1 \end{bmatrix} + 5\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -6 & -1 & -1 \\ -7 & 0 & -11 \\ -3 & 1 & -8 \end{bmatrix} + \begin{bmatrix} 2 & -2 & 8 \\ 6 & 4 & -2 \\ 4 & 2 & -2 \end{bmatrix} + \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -3 & 7 \\ -1 & 9 & -13 \\ 1 & 3 & -5 \end{bmatrix}$$

From (2) $\Rightarrow$ $A^{-1} = \frac{1}{6}\begin{bmatrix} 1 & -3 & 7 \\ -1 & 9 & -13 \\ 1 & 3 & -5 \end{bmatrix}$

Example: Use Cayley – Hamilton theorem to find the value of the matrix given by

(i) $\mathbf{f(A) = A^8 - 5A^7 + 7A^6 - 3A^5 + A^4 - 5A^3 + 8A^2 - 2A + I}$

(ii) $\mathbf{A^8 - 5A^7 + 7A^6 - 3A^5 + 8A^4 - 5A^3 + 8A^2 - 2A + I}$ **if the matrix**

$$\boldsymbol{A} = \begin{bmatrix} \mathbf{2} & \mathbf{1} & \mathbf{1} \\ \mathbf{0} & \mathbf{1} & \mathbf{0} \\ \mathbf{1} & \mathbf{1} & \mathbf{2} \end{bmatrix}$$

Solution:

The characteristic equation of A is $|A - \lambda I| = 0$

$\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$ where

S_1 = sum of the main diagonal elements

$= 2 + 1 + 2 = 5$

S_2 = sum of the minors of main diagonal elements

$$= \begin{vmatrix} 1 & 0 \\ 1 & 2 \end{vmatrix} + \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} + \begin{vmatrix} 2 & 1 \\ 0 & 1 \end{vmatrix}$$

$= (2 - 0) + (4 - 1) + (2 - 0) = 2 + 3 + 2 = 7$

$= (-1) + (-9) + 5 = -5$

$$S_3 = |A| = \begin{vmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{vmatrix}$$

$$= 2(2-0) - 1(0-0) + 1(0-1) = 4 - 1 = 3$$

Therefore, the characteristic equation is

$$\lambda^3 - 5\lambda^2 + 7\lambda - 3 = 0$$

By C – H theorem, we get

$$A^3 - 5A^2 + 7A - 3I = 0 \ldots (1)$$

Let i) $f(A) = A^8 - 5A^7 + 7A^6 - 3A^5 + A^4 - 5A^3 + 8A^2 - 2A + I$

ii) $g(A) = A^8 - 5A^7 + 7A^6 - 3A^5 + 8A^4 - 5A^3 + 8A^2 - 2A + I$

(i)

$$\begin{array}{r|l} & A^5 + A \\ \hline A^3 - 5A^2 + 7A - 3I & A^8 - 5A^7 + 7A^6 - 3A^5 + A^4 - 5A^3 + 8A^2 - 2A + I \\ & A^8 - 5A^7 + 7A^6 - 3A^5 \\ \hline (-) & \qquad\qquad A^4 - 5A^3 + 8A^2 - 2A \\ & \qquad\qquad A^4 - 5A^3 + 7A^2 - 3A \\ \hline & (-) \qquad\qquad A^2 + A + 1\,I \end{array}$$

$$f(A) = (A^3 - 5A^2 + 7A - 3I)(A^2 + A) + A^2 + A + I$$

$$= O + A^2 + A + I \quad \text{by (1)}$$

$$= A^2 + A + I \qquad \ldots (2)$$

Now,

$$A^2 = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix} \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 5 & 4 & 4 \\ 0 & 1 & 0 \\ 4 & 4 & 5 \end{bmatrix}$$

$$\therefore A^2 + A + I = \begin{bmatrix} 5 & 4 & 4 \\ 0 & 1 & 0 \\ 4 & 4 & 5 \end{bmatrix} + \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 8 & 5 & 5 \\ 0 & 3 & 0 \\ 5 & 5 & 8 \end{bmatrix}$$

(ii)

$$\begin{array}{r|l} & A^5 + 8A + 35I \\ \hline A^3 - 5A^2 + 7A - 3I & A^8 - 5A^7 + 7A^6 - 3A^5 + A^4 - 5A^3 + 8A^2 - 2A + 1\,I \\ & \underline{A^8 - 5A^7 + 7A^6 - 3A^5} \\ (-) & \qquad\qquad 8A^4 - 5A^3 + 8A^2 - 2A \\ & \qquad\qquad \underline{8A^4 - 40A^3 + 56A^2 - 24A} \\ & (-) \qquad\qquad 35A^3 - A^2 + 22A + 1\,I \\ & \qquad\qquad\qquad \underline{35A^3 - 175\,A^2 + 245A - 105I} \\ & \qquad (-) \qquad\qquad 127\,A^2 - 223A + 106\,I \end{array}$$

$$g(A) = (A^3 - 5A^2 + 7A - 3I)(A^4 + 8A + 35I) + 127A^2 - 223A + 106I$$

$$= O + 127A^2 - 223A + 106\,I$$

$$= 127A^2 - 223A + 106\,I$$

$$= 127\begin{bmatrix} 5 & 4 & 4 \\ 0 & 1 & 0 \\ 4 & 4 & 5 \end{bmatrix} - 223\begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix} + 106\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$g(A) = \begin{bmatrix} 295 & 285 & 285 \\ 0 & 10 & 0 \\ 285 & 285 & 295 \end{bmatrix}$$

Example: Using Cayley Hamilton theorem find A^{-1} when $A = \begin{bmatrix} 1 & 0 & 3 \\ 2 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix}$

Solution:

The Characteristic equation of A is $|A - \lambda I| = 0$

$\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$ where

$S_1 =$ sum of the main diagonal elements $= 1 + 1 + 1 = 3$

$S_2 =$ Sum of the minors of the main diagonal elements.

$$= \begin{vmatrix} 1 & -1 \\ -1 & 1 \end{vmatrix} + \begin{vmatrix} 1 & 3 \\ 1 & 1 \end{vmatrix} + \begin{vmatrix} 1 & 0 \\ 2 & 1 \end{vmatrix}$$

$$= (1-1) + (1-3) + (1-0)$$

$$= 0 - 2 + 1 = -1$$

$$S_3 = |A| = \begin{vmatrix} 1 & 0 & 3 \\ 2 & 1 & -1 \\ 1 & -1 & 1 \end{vmatrix}$$

$$= 1(1-1) - 0(2+1) + 3(-2-1)$$

$$= 0 - 0 + 3(-3) = -9$$

∴ The characteristic equation A is $\lambda^3 - 3\lambda^2 - \lambda + 9 = 0$

By Cayley - Hamilton Theorem every square matrix satisfies its own Characteristic equation

$$\therefore A^3 - 3A^2 - A + 9\,I = O$$

$$A^{-1} = \frac{-1}{9}[A^2 - 3A - \ I] \qquad \ldots (1)$$

$$A^2 = \begin{bmatrix} 1 & 0 & 3 \\ 2 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 3 \\ 2 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1+0+3 & 0+0-3 & 3+0+3 \\ 2+2-1 & 0+1+1 & 6-1-1 \\ 1-2+1 & 0-1-1 & 3+1+1 \end{bmatrix} = \begin{bmatrix} 4 & -3 & 6 \\ 3 & 2 & 4 \\ 0 & -2 & 5 \end{bmatrix}$$

$$-3A = \begin{bmatrix} -3 & 0 & -9 \\ -6 & -3 & 3 \\ -3 & 3 & -3 \end{bmatrix}$$

$$(1) \Rightarrow A^{-1} = \frac{-1}{9}\left[\begin{pmatrix} 4 & -3 & 6 \\ 3 & 2 & 4 \\ 0 & -2 & 5 \end{pmatrix} + \begin{pmatrix} -3 & 0 & -9 \\ -6 & -3 & -3 \\ -3 & 3 & -3 \end{pmatrix} - \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\right]$$

$$= \frac{-1}{9}\begin{bmatrix} 0 & -3 & -3 \\ -3 & -2 & 7 \\ -3 & 1 & 1 \end{bmatrix}$$

$$= \frac{1}{9}\begin{bmatrix} 0 & 3 & 3 \\ 3 & 2 & -7 \\ 3 & -1 & -1 \end{bmatrix}$$

Example: Verify Cayley- Hamilton for the matrix $A = \begin{bmatrix} \mathbf{1} & \mathbf{3} & \mathbf{7} \\ \mathbf{4} & \mathbf{2} & \mathbf{3} \\ \mathbf{1} & \mathbf{2} & \mathbf{1} \end{bmatrix}$

Solution :

Given A $= \begin{bmatrix} 1 & 3 & 7 \\ 4 & 2 & 3 \\ 1 & 2 & 1 \end{bmatrix}$

The characteristic equation A is $|A - \lambda I| = 0$

$$\lambda^3 - S_1\lambda^2 + S_2\lambda S_3 = 0 \cdots (1) \quad \text{where}$$

$$S_1 = \text{Sum of the main diagonal elements}$$

$$= 1 + 2 + 1 = 4$$

$$S_2 = \text{Sum of the minors of its leading diagonal elements}$$

$$= \begin{vmatrix} 2 & 3 \\ 2 & 1 \end{vmatrix} + \begin{vmatrix} 1 & 7 \\ 1 & 1 \end{vmatrix} + \begin{vmatrix} 1 & 3 \\ 4 & 2 \end{vmatrix}$$

$$= (2 - 6) + (1 - 7) + (2 - 12)$$

$$= -4 - 6 - 10 = -20$$

$$S_3 = |A| = \begin{vmatrix} 1 & 3 & 7 \\ 4 & 2 & 3 \\ 1 & 2 & 1 \end{vmatrix}$$

$$= 1(2 - 6) - 3(4 - 3) + 7(8 - 2)$$

$$= -4 - 3(1) + 7(6)$$

$$= -4 - 3 + 42 = 35$$

$$\therefore (1) \Rightarrow \lambda^3 - 4\lambda^2 - 20\lambda - 35 = 0$$

By Cayley –Hamilton theorem

$(2) \Rightarrow \quad A^3 - 4A^2 - 20A - 351 = O$

To find A^2 *and* A^3:

$$A^2 = \begin{bmatrix} 1 & 3 & 7 \\ 4 & 2 & 3 \\ 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 3 & 7 \\ 4 & 2 & 3 \\ 1 & 2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 + 12 + 7 & 3 + 6 + 14 & 7 + 9 + 7 \\ 4 + 8 + 3 & 12 + 4 + 6 & 28 + 6 + 3 \\ 1 + 8 + 1 & 3 + 4 + 2 & 7 + 6 + 1 \end{bmatrix}$$

$$= \begin{bmatrix} 20 & 23 & 23 \\ 15 & 22 & 37 \\ 10 & 9 & 14 \end{bmatrix}$$

$$A^3 = \begin{bmatrix} 20 & 23 & 23 \\ 15 & 22 & 37 \\ 10 & 9 & 14 \end{bmatrix} \begin{bmatrix} 1 & 3 & 7 \\ 4 & 2 & 3 \\ 1 & 2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 20+92+23 & 60+46+46 & 140+69+23 \\ 15+88+37 & 45+44+74 & 105+66+37 \\ 10+36+14 & 30+18+28+ & 70+27+14 \end{bmatrix}$$

$$= \begin{bmatrix} 135 & 152 & 232 \\ 140 & 163 & 208 \\ 60 & 76 & 111 \end{bmatrix}$$

$A^3 - 4A^2 - 20A - 35\,I$

$$= \begin{bmatrix} 135 & 152 & 232 \\ 140 & 163 & 208 \\ 60 & 76 & 111 \end{bmatrix} - 4\begin{bmatrix} 20 & 23 & 23 \\ 15 & 22 & 37 \\ 10 & 9 & 14 \end{bmatrix} - 20\begin{bmatrix} 1 & 3 & 7 \\ 4 & 2 & 3 \\ 1 & 2 & 1 \end{bmatrix} -$$

$$35\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 135 & 152 & 232 \\ 140 & 163 & 208 \\ 60 & 76 & 111 \end{bmatrix} + \begin{bmatrix} -80 & -92 & -92 \\ -60 & -88 & -148 \\ -40 & -36 & -56 \end{bmatrix} + \begin{bmatrix} -20 & -60 & -140 \\ -80 & -40 & -60 \\ -20 & -40 & -20 \end{bmatrix}$$

$$+ \begin{bmatrix} -35 & 0 & 0 \\ 0 & -35 & 0 \\ 0 & 0 & -35 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

∴ The given matrix A satisfies its own characteristic equation.

Hence, cayley Hamilton theorem is verified.