VECTOR SPACES

Definition :

Let F be a given field and let V be a non-empty set with addition and scalar multiplication rules applicable to any u, v $\in$V such as a sum u + v $\in$ V and to any u $\in$ V, α $\in$ F a product α u $\in$V . Then V is called a vector space over F if the following condition hold :

1. Closure : for all u, v $\in$ V => u + v $\in$ V
2. Associative : u + (v +w) = (u + v) + w $\forall$ u, v, w $\in$ V.
3. Identity : u + 0 = 0 + u = u for all u $\in$ V, there exist 0 $\in$ V
4. Inverse : (-u) + u = 0 = u + (-u) there exist -u $\in$V, for all u $\in$ V
5. Commutative : u + v = v + u for all u, v $\in$ V
6. For all α $\in$ F and for all u $\in$ V, α u $\in$ V.
7. α(u +v)= αu + αv, for all α $\in$ F for all u, v $\in$V
8. (α + β)v = αv + βv, for all α,β $\in$ F and for all v $\in$V
9. ($\alpha\beta$)v = α (βv), for all α, β $\in$F and for all u, v $\in$V
10. 1 . v = v for all v $\in$ V

Properties of vector space :

(i) α . 0 = 0, 0 $\in$ V, for all α $\in$ F

(ii) 0 . v = 0, for all v $\in$ V, 0 $\in$ F

(iii) (-α)v = -(αv)= α (-v) for all v ∈ F, v ∈ V

(iv) α v = 0, v ≠ 0, α = 0 where α ∈ F, α ∈ V

(v) α (u – v) = αv – αv for all α ∈ F and u, v ∈ V

Proof :

(i) since 0+0 =0 where 0 ∈ V

α (0 + 0) = α for all α ∈ F

⇨ α 0 + α 0 = α 0

⇨ α 0 + α 0 = α 0 + 0

Hence α 0 = 0 [by left cancellation law]

(ii) since 0 + 0 = 0 where 0 ∈ F

(0 + 0) v = 0 v for all v ∈V

⇨ 0v + 0v = 0v

⇨ 0v + 0v = 0v + 0

Hence 0v = 0 [by left cancellation law]

(iii) (-α)v = -(αv)= α (-v) for all v ∈ F, v ∈ V

Since α ∈ F => -α ∈ F and v ∈ V, - v∈V

=> α + (-α) = 0 ∈ F ; v + (-v) = 0 ∈ V

=> αv + (-α)v = [α + (-α)] v

For all v ∈ V ; αv + α(-v) = α[v + (-v)]

For all α ∈ F

=>αv + (-α)v = 0v for all v ∈ V; αv + α(-v) = α0 for all α ∈ F

=> $\alpha v + (-\alpha)v = 0$ for all $v \in V$; $\alpha v + \alpha(-v) = 0$ for all $\alpha \in F$

=> $(-\alpha)v$ is the additive inverse of αv in V ; $\alpha(-v)$ is the additive inverse of αv in V.

$$(-\alpha)v = -(\alpha v) \quad ; \alpha(-v) = -(\alpha v)$$

(iv) $\alpha v = 0, v \neq 0$

To prove $\alpha = 0$ where $\alpha \varepsilon F, v \varepsilon V$

Let $\alpha \neq 0$ then $\alpha^{-1} \varepsilon F$

Consider $\alpha v = 0$

$\therefore \alpha^{-1} (\alpha v) = \alpha^{-1} (0)$

⇨ $(\alpha^{-1}\alpha) v = 0$

⇨ $1 v = 0$

⇨ $v = 0$ which is a contradiction.

Hence $\alpha = 0$

Note : The vector space of V over the field F is denoted as V(F).

(i) C is a vector space over a field C and ℝ

(ii) R is a vector space over a field ℝ but not in a field C

(iii) Q is a vector space over a field Q.

(iv) Z is not a vector space over a field R.

(v) The set $R^n = \{(a_1,a_2,\ldots,a_n) \quad a_1 \varepsilon R\}$ is a vector space over R.

(vi) The set $M_2(R)$ and $M_2(Q)$ of 2 X 2 matrices with entries from R and Q is a vector space over R.

(vii) The set $Z_p[R]$ of polynomials with coefficients from Z_p is a vector space over Z_p, where P is a prime.

(viii) Let E be a field and F be a subfield of E. Then E is a vector space over F.

(ix) Let $P_n(t)$ be the set of all polynomials P(t) over a field F, where the degree of P(t) is less than or equal to n. i.e.,

$P(t) = a_0+a_1 t +\ldots..+a_n t^n$.

PROBLEMS UNDER VECTOR SPACE

Example 1. Prove that $R \times R$ is a vector space aver R under addition and multiplication defined by $(x_1, x_2) + (y_1, y_2) = (x_1 + y_1, x_2 + y_2)$ and $a(x_1, x_2) = (\alpha x_1, \alpha x_2)$

Sol: Let $x, y \in V = R \times R$

Then $x = (x_1, x_2)$

$y = (y_1, y_2)$

Where $x_1, x_2 y_1 y_2 \in R$

$$x + y = (x_1, x_2) + (y_1, y_2)$$

$$= (x_1 + y_1, x_2 + y_2) \in R \times R$$

Let $\alpha \in F$ and $x \in \mathcal{Y}$

$$\alpha x = \alpha(x_1, x_2)$$

$$= (\alpha x_1, \alpha x_2) \in R \times R.$$

Therfore vector addition and scalar multiplications are true in $R \times R$.

1 Under addition

A_1: Commutativity: $x + y = x + y, \forall x, y \in R \times R$

$$x + y = (x_1, x_2) + (y_1, y_2)$$

$$= (x_1 + y_1, x_2 + y_2)$$

$$= (y_1 + x_1, y_2 + x_2)$$

$$= (y_1, y_2) + (x_1, x_2)$$

$$= x + y$$

$\therefore x + y = x + y, \forall\, x, y \in R \times R$

A_2: Associativity: $x + (y + z) = (x + y) + z, \forall x, y, z \in R \times R$

Let $x, y, z \in R \times R$. Then

$x = (x_1, x_2), y = (y_1, y_7)_2 z = (z_1, z_2)$

Where $x_1, x_2, y_1, y_2, z_1, z_2 \in R$

$x + (y + z) = (x_1, x_2) + [(y_1, y_z) + (z_1, z_2)]$

$= (x_1, x_2) + (y_1 + z_1, y_2 + z_2)$

$= \left(x_1 + (y_1 + z_1), x_2 + (y_2 + z_2)\right)$

$= \left((x_1 + y_1) + z_1, (x_2 + y_2) + z_2\right)$

$=(x_1 + y_1, x_2 + y_z) + (z_1, z_2)$

$= \left((x_1, x_2) + (y_1, y_2)\right) + (z_1, z_2)$

$= (x + y) + z$

$x + (y + z) = (x + y) + z, \forall x, y, z \in R \times R$

A_3: Existence of Identity: There exists $0 \in R \times R$ snch that

$x + 0 = x, \forall x \in R \times R$

Let $0 \in R$. Then $0 = (0,0) \in R \times R$.

$x + 0 \ = (x_1, x_2) + (0,0)$

$= (x_1 + 0, x_2 + 0)$

$= (x_1, x_2)$

$= x$

$0 = (0,0)$ is the zero clement of $R \times R$

A_4: Existence of Inverse: For all x in$R \times R$, there exists $-x \in R \times R$, such that

$x + (-x) = 0$

$$Let\ x \in R \times R$$

$\therefore x = (x_1, x_2)$, where $x_1, x_2 \in R$

Which implies $-x_1, -x_2 \in R$

$$\Rightarrow -x = (-x_1, -x_2) \in R \times R$$

$$x + (-x) = (x_1, x_2) + (-x_1 - x_2)$$

$$= (x_1 - x_1, x_2 - x_2)$$

$$= (0,0)$$

$$=0$$

$x + (-x) = 0$

=>lnverse of x is $-x$

ie, inverse of (x_1, x_2) is $(-x_1, -x_2)$

II Under scalar multiplicatiou:

M_1: $a(x + y) = arx + \alpha y; \forall a \in R$ and $\forall x, y \in R \times R$

$$\alpha(x + y) = \alpha(x_2 + y_1 x_2 + y_2)$$

$$= \left(a(x_1 + y_1), a(x_2 + y_2)\right)$$

$$= (\alpha x_1 + \alpha y_1 - ax_2 + ay_2)$$

$$= (\alpha x_1, \alpha x_2) + (\alpha y_1, \alpha y_2)$$

$$= \alpha(x_1, x_2) + \alpha(y_1, y_2)$$

$$= ax + ay$$

$\therefore \alpha(x + y) = \alpha x + \alpha y \forall a \in R$ and $\forall x, y \in R \times R$

$M_z: (\alpha + \beta)x = ax + \beta x,\ \forall \alpha, \beta \in R, \forall x \in R \times R$

$$(\alpha + \beta)x = (u + \beta)(x_1, x_2)$$

$$= ((\alpha + \beta)x_1(\alpha + \beta)x_2)$$

$$= (\alpha x_1 + \beta x_1, \alpha x_2 + \beta x_2)$$

$$= (\alpha x_1, \alpha x_2) + (\beta x_1, \beta x_2)$$

$$= \alpha(x_1, x_2) + \beta(x_1, x_2)$$

$$= \alpha x + \beta x$$

$(\boldsymbol{a} + \beta)x = ax + \beta x, \forall \alpha, \beta \in R, \forall x \in R \times R$

$M_3: \alpha(\beta x) = (\alpha\beta)(x), \forall a_v, \beta \in R, \forall x \in R \times R$

$$\alpha(\beta x) = \alpha\big(\beta(x_1, x_2)\big)$$

$$= \alpha(\beta x_1, \beta x_2)$$

$$= \big(\alpha(\beta x_1), \alpha(\beta x_2)\big)$$

$= ((\alpha\beta)x_1, (\alpha\beta)x_2)$

$= (\alpha\beta)(x_1, x_2)$

$= (\alpha\beta)(x)$

$\therefore \alpha(\beta x) = (\alpha\beta)(x) \forall \alpha, \beta \in R, \forall x \in R \times R$

M_4: $1. x = x, \forall x \in R \times R$ and $1 \in R$

$1. x = 1(x_1, x_2)$

$= (1. x_1, 1. x_2)$

$= (x_1, x_2) = x$

$1. x = x, \forall x \in R \times R$ and $1 \in R$

Therefore $V = R \times R$ is a vector space over R.

Example 2. Prove that F^n is a vector space over a field F under addition and muitiplication defined by $(x_1, x_2, \dots, x_n) + (y_1, y_2, \dots, y_n) = (x_1 + y_1, x_2 + y_2, \dots, x_n + y_n)$ and $\alpha(x_1, x_2, \dots, x_n) = (\alpha x_1, \alpha x_2, \dots, \alpha x_n)$

Let $x, y \in V = F^n$

Then $x = (x_1, x_2, \dots, x_n)$

$y = (y_1, y_2, \dots, y_n)$

where $x_1, x_2, \dots, x_n, y_1, y_2, \dots, y_n \in F$

$x + y = (x_1 + y_1, x_2 + y_2 \dots., x_n + y_n) \in F^n$

Let $\alpha \in F$ and $x \in F^n$

$\alpha x = \alpha(x_1, x_2, \dots, x_n)$

$= (\alpha x_1, \alpha x_2, \dots, \alpha x_2) \in F^n$

Therefore vector addition and scalar multiplications are true in F^n.

I. Under addition

A_1 : Commutativity: $x + y = y + x, \forall x, y \in F^n$

$x + y = (x_1 + y_1, x_2 + y_2, \dots, x_n + y_n)$

$= (y_1 + x_1, y_2 + x_2, \dots, y_n + x_n)$

$= (y_1, y_2, \dots, y_n) + (x_1, x_2, \dots, x_n)$

$= y + x$

$x + y = y + x, \forall x, y \in F^n$

A_2: Associativity: $x + (y + z) = (x + y) + z, \forall x, y, z \in F^n$

Let $x, y, z \in F^n$. Then

$x = (x_1, x_2, \dots, x_n), y = (y_1, y_2, \dots, y_n), z = (z_1, z_2, \dots, z_n)$

Where $x_1, x_2, \dots, x_n, y_1, y_2, \dots, y_n, z_1, z_2, \dots z_n \in F$

$x + (y + z) = (x_1, x_2, \dots, x_n) + [(y_1, y_2, \dots, y_n) + (z_1, z_2, \dots z_n)]$

$= (x_1, x_2, \dots, x_n) + (y_1 + z_1, y_2 + z_2, \dots, y_n + z_n)$

$= \big(x_1 + (y_1 + z_1), x_2 + (y_2 + z_2), \dots, x_n + (y_n + z_n)\big)$

$$= \big((x_1 + y_1) + z_1, (x_2 + y_2) + z_2, \dots, (x_n + y_n) + z_n\big)$$

$$= (x_1 + y_1, x_2 + y_2, \dots, x_n + y_n) + (z_1, z_2, \dots, z_n)$$

$$= \big((x_1, x_2, \dots, x_n) + (y_1, y_2, \dots, y_n)\big) + (z_1, z_2, \dots, z_n)$$

$$= (x + y) + z$$

$\therefore x + (y + z) = (x + y) + z, \forall x, y, z \in F^n$

A_3 : Existence of Identity: There exists $0 \in F^n$ such that

$$x + 0 = 0 + x = x, \forall x \in F^n$$

Let $0 \in F$. Then $0 = (0,0, \dots ,0)) \in F^n$

$x + 0 = (x_1, x_2, \dots, x_n) + (0,0, \dots ,0)$

$= (x_1 + 0, x_2 + 0, \dots, x_n + 0)$

$= (x_1, x_2, \dots, x_0)$

$= x$

$0 = (0,0, \dots 0)$) is the zero element of F^x

A_4 : Fxistence of Inverse: For all x in F^n, there exists $-x$ in F^n such that

$$(-x) + x = 0$$

Let $x \in F^n$.

$\therefore x = (x_1, x_2, \dots, x_n)$; where $x_1 x_2, \dots, x_n \in F_1$

Which implies $-x_1, -x_{2i} \dots, -x_n \in F$

$$-x = (-x_1 - x_2, \dots, -x_n) \in k^n$$

$$x + (-x) = (x_1, x_2, \dots, x_n) + (-x_{1+} - x_2 \dots \dots, -x_n)$$

$$= (x_1 - x_1, x_2 - x_2 + \cdots, x_n - x_n)$$

$$= (0,0, \dots ,0)$$

$$= 0$$

=>Inverse of x is - x

ie, inverse of $(x_1, x_2, \dots, x_n)$ is $(-x_1 - x_2, \dots, -x_n)$

II Under scalar multiplication:

$M_1 = \alpha(x + y) = \alpha x + \alpha y, \forall a \in F$ and $\forall x, y \in F^a$

$\alpha(x + y) = \alpha(x_1 + y_1 x_2 + y_2 \dots, x_n + y_n)$

$= \big(\alpha(x_1 + y_1), \alpha(x_2 + y_2), \dots, \alpha(x_n + y_n)\big)$

$= (\alpha x_1 + \alpha y_1, \alpha x_2 + \alpha y_2 \dots \dots, \alpha x_n + \alpha y_n)$

$= (\alpha x_1, a x_2) + (\alpha y_1, \alpha y_2), \dots, (\alpha x_n + \alpha y_n)$

$= \alpha(x_1, x_2) + \alpha(y_1, y_2), \dots, (\alpha x_n + \alpha y_n)$

$= \alpha x + \alpha y$

$\therefore \alpha(x+y)=\alpha x+\alpha y, V\alpha \in F$ and $Vx, y \in F^{n}$

$\mathbf{M}_{2}: (\alpha+\beta)x = ax+\beta x\, \alpha, \beta \in F, \forall x \in F''$

$$(\alpha+\beta)x = (\alpha+\beta)(x_1, x_2, \dots, x_n)$$

$$= ((\alpha+\beta)x_1(\alpha+\beta)x_2, \dots, (\alpha+\beta)x_n)$$

$$= (\alpha x_1+\beta x_1, \alpha x_2+\beta x_2, \dots, \alpha x_n+\beta x_n)$$

$$= (\alpha x_1, \alpha x_2, \dots, \alpha x_n) + (\beta x_1, \beta x_2, \dots, \beta x_n)$$

$$= \alpha(x_1, x_2, \dots, x_n) + \beta(x_1, x_2, \dots, x_n)$$

$$= \alpha x + \beta x$$

$\therefore (\alpha+\beta)x = \alpha x+\beta x, \forall \alpha, \beta \in F, \forall x \in F^{n}$

$M_{3}: \alpha(\beta x) = (\alpha\beta)\,(x), a, \beta \in F, \forall x \in F^{n}$

$$\alpha(\beta x) = \alpha\big(\beta(x_1, x_2, \dots, x_n)\big)$$

$$= \alpha(\beta x_1, \beta x_2, \dots, \beta x_n)$$

$$= \big(\alpha(\beta x_1), \alpha(\beta x_2), \dots, \alpha(\beta x_n)\big)$$

$$= ((\alpha\beta)x_1, (\alpha\beta)x_2, \dots, (\alpha\beta)\hat{x}_n)$$

$$= (\alpha\beta)(x_1, x_2, \dots, x_n)$$

$$= (\alpha\beta)(x)$$

$\therefore \alpha(\beta x) = (\alpha\beta)(x), \forall \alpha, \beta \in F, \forall x \in F^n$

M_4: $1. x = x_1 \forall x \in F^n$ and $1 \in F$

$1. x \ = 1. (x_1, x_2, \ldots, x_n)$

$= (1. x_1, 1. x_2, \ldots, 1 \cdot x_n)$

$= (x_1, x_2, \ldots, x_n) = x$

$\therefore 1. x = x, \forall x \in F^n$ and $1 \in F$

$\therefore F^n$ is a vector space over F.

Example 3. Prove that set of complex numbers is a vector space over field

R.

Sol: $V = C = \{(x + iy)/x, y \in R\}$

Let $x, y \in C$

Then $x = x_1 + iy_1, y = x_2 + iy_2$

Where $x_1, y_1, x_2, y_2 \in R$

Addition of vectors is defined by

$$x + y = (x_1 + iy_1) + (x_2 + iy_2)$$

$$= x_1 + x_2 + i(y_1 + y_2) \in C$$

Scalar multiplication is defined by

For $\alpha \in R$ and $x \in C$

$\alpha x = a(x_1 + iy_1)$

$= \alpha x_1 + i\alpha x_2 \in C$

Therefore vector addition and scalar multiplications are true in C.

1. Under Addition

A_1: Commutativity: $x + y = y + x, \forall x, y \in C$

$x + y = (x_1 + x_2) + i(y_1 + y_2)$

$= (x_2 + x_1) + i(y_2 + y_1)$

$= (x_2 + iy_2) + (x_1 + iy_1)$

$= y + x$

$\therefore x + y = y + x, \forall x, y \in C$

A_2: Associativity: $x + (y + z) = (x + y) + z, \forall x, y, z \in C$

Let $x, y, z \in$ C

$\therefore x = x_1 + iy_1 . y = x_2 + iy_2, z = x_3 + iy_3$

$x + (y + z) = (x_1 + iy_1) + [(x_2 + iy_2) + (x_3 + iy_3)]$

$=(x_1 + iy_1) + [(x_2 + x_3) + i(y_2 + y_3)]$

$=\left(x_1 + (x_2 + x_3)\right) + i\left(y_1 + (y_2 + y_3)\right)$

$=\big((x_1 + x_2) + x_3\big) + i\big((y_1 + y_2)y_3\big)$

$=[(x_1 + x_2) + i(y_1 + y_2)] + (x_3 + iy_3)$

$=[(x_1 + iy_1) + (x_3 + iy_2)] + (x_3 + iy_3)$

$=(x + y) + z$

$x + (y + z) = (x + y) + z, \forall x, y, z \in C$

A_3 : Existence of Identity: There exists $0 \in C$ such that

$x + 0 = x, \forall\in C$

Let $0 \in R$. Then $0 = 0 + i0 \in C$

$x + 0 = (x_1 + iy_1) + (0 + i0)$

$= x_1 + 0 + i(y_1 + 0)$

$= x_1 + iy_1$

$= x$

$0 = 0 + i0$ is the zero element of C

A_4 : Existence of Inverse: For all x in C, there exists $-x$ in C such that

$(-x) + x = 0$

Let $x \in C$. Then

$x = x_1 + iy_1$, where $x_1, y_1 \in R$

Which implies $-x_1, y - {}_1 \in R$

$\therefore -x = -x_1 + i(-y_1) \in C$

$$x + (-x) = (x_1 + iy_1) + \left(-x_1 + i(-y_1)\right)$$

$$= x_1 - x_1 + i(y_1 - y_1)$$

$$= 0 + i0$$

$$= 0$$

$\therefore$ Inverse of x is $-x$

i.e. inverse of $x_1 + iy_1$ is $-x_1 + i(-y_1)$

II. Under scalar multiplication

$\boldsymbol{M_1}$: $\alpha(x + y) = \alpha x + \alpha y, \forall \alpha \in R$ and $\forall x, y \in C$

$$\alpha(x + y) = \alpha[(x_1 + x_2) + i(y_1 + y_2)]$$

$$= \alpha(x_1 + x_2) + i\alpha(y_1 + y_2)$$

$$= (\alpha x_1 + \alpha x_2) + i(\alpha y_1 + \alpha y_2)$$

$$= (\alpha x_1 + i\alpha y_1) + (\alpha x_2 + i\alpha y_2)$$

$= \alpha(x_1 + iy_1) + \alpha(x_2 + iy_2)$

$= \alpha x + \alpha y$

$\therefore \alpha(x + y) = \alpha x + \alpha x, \forall \alpha \in R$ and $\forall x, y \in C$

$M_2: (\alpha + \beta)x = \alpha x + \beta x, \forall \alpha, \beta \in R, \forall x \in C$

$(\alpha + \beta)x = (\alpha + \beta)(x_1 + iy_1)$

$= (\alpha + \beta)x_1 + i(\alpha + \beta)y_1$

$= \alpha x_1 + \beta x_1 + i(\alpha y_1 + \beta y_1)$

$= (\alpha x_1 + i a y_1) + (\beta x_1 + i\beta y_1)$

$= \alpha(x_1 + iy_1) + \beta(x_1 + iy_1)$

$= \alpha x + \beta x$

$\therefore (\alpha + \beta)\, x = \alpha x + \beta x, \forall \alpha, \beta \in R, \forall x \in C$

$M_3 d\alpha(\beta x) = (\alpha\beta)(x), \forall \alpha, \beta \in R, \forall x \in C$

$\alpha(\beta x)\& = \alpha\big(\beta(x_1 + iy_1)\big)$

$= \alpha(\beta x_1 + i\beta y_1)$

$= \alpha(\beta x_1) + i\alpha(\beta y_1)$

$= (\alpha\beta)x_1 + l(\alpha\beta)y_1$

$= (\alpha\beta)(x_1 + iy_1)$

$= (\alpha\beta)x$

$\therefore \alpha(\beta x) = (\alpha\beta)x, \forall \alpha, \beta \in R, \forall x \in C$

M_4: $1. x = x, \forall x \in C$ and $1 \in R$

$1. x = (1 + i0)(x_1 + iy_1)$

$= x_1 + iy_1$

$= x$

$\therefore 1. x^n = x, \forall x \in C$ and $1 \in R$

$\therefore C$ is a vector space over R.