5.3 Triple Integrals- Volume of Solids

Triple integration in cartesian co-ordinates is defined over a region R is defined by

$\iiint_R f(x,y,z)dxdydz$ or $\iiint_R f(x,y,z)dV$ or $\iiint_R f(x,y,z)d(x,y,z)$.

Type I – Problems on Triple Integrals

Example:

Evaluate $\int_0^a \int_0^b \int_0^c (\mathbf{x}+\mathbf{y}+\mathbf{z})\mathbf{dzdydx}$

Solution:

$$\int_0^a \int_0^b \int_0^c (x+y+z)dzdydx = \int_0^a \int_0^b \left[xz+yz+\frac{z^2}{2}\right]_0^c dydx$$

$$= \int_0^a \int_0^b \left(xc+yc+\frac{c^2}{2}\right) dydx$$

$$= \int_0^a \left[xcy+\frac{y^2c}{2}+\frac{c^2y}{2}\right]_0^b dx$$

$$= \int_0^a \left(xbc+\frac{b^2c}{2}+\frac{bc^2}{2}\right) dx$$

$$= \left[\frac{x^2bc}{2}+\frac{b^2cx}{2}+\frac{bc^2x}{2}\right]_0^a$$

$$= \left[\frac{a^2bc}{2}+\frac{ab^2c}{2}+\frac{abc^2}{2}\right]$$

$$= \frac{abc}{2}(a+b+c)$$

Example:

Evaluate $\int_0^1 \int_{y^2}^1 \int_0^{1-x} \boldsymbol{xdzdxdy}$

Solution:

$$\int_0^1 \int_{y^2}^1 \int_0^{1-x} xdzdxdy = \int_0^1 \int_{y^2}^1 [xz]_0^{1-x} dxdy$$

$$= \int_0^1 \int_{y^2}^1 x(1-x)\, dxdy$$

$$= \int_0^1 \int_{y^2}^1 (x-x^2)\, dxdy$$

$$= \int_0^1 \left[\frac{x^2}{2}-\frac{x^3}{3}\right]_{y^2}^1 dy$$

$$= \int_0^1 \left(\frac{1}{2} - \frac{1}{3} - \frac{y^4}{2} + \frac{y^6}{3}\right) dy$$

$$= \left[\frac{y}{2} - \frac{y}{3} - \frac{y^5}{10} + \frac{y^7}{21}\right]_0^1$$

$$= \frac{1}{2} - \frac{1}{3} - \frac{1}{10} + \frac{1}{21}$$

$$= \frac{105-70-21+10}{210} = \frac{24}{210} = \frac{4}{35}$$

Example:

Evaluate $\int_0^1 \int_0^{1-x} \int_0^{x+y} e^z dxdydz$

Solution:

$$\int_0^1 \int_0^{1-x} \int_0^{x+y} e^z dxdydz = \int_0^1 \int_0^{1-x} \int_0^{x+y} e^z dzdydx$$

$$= \int_0^1 \int_0^{1-x} [e^z]_0^{x+y} dydx$$

$$= \int_0^1 \int_0^{1-x} (e^{x+y} - 1)\, dydx$$

$$= \int_0^1 [e^{x+y} - y]_0^{1-x}\, dx$$

$$= \int_0^1 (e^{x+1-x} - 1 + x - e^x)\, dx$$

$$= \int_0^1 (e - 1 + x - e^x)\, dx$$

$$= \left[ex - x + \frac{x^2}{2} - e^x\right]_0^1$$

$$= e - 1 + \frac{1}{2} - e - 0 + 0 - 0 + 1$$

$$= \frac{1}{2}$$

Example:

Evaluate $\int_0^a \int_0^{\sqrt{a^2-y^2}} \int_0^{\sqrt{a^2-x^2-y^2}} dzdxdy$

Solution:

$$\int_0^a \int_0^{\sqrt{a^2-y^2}} \int_0^{\sqrt{a^2-x^2-y^2}} dzdxdy = \int_0^a \int_0^{\sqrt{a^2-y^2}} [z]_0^{\sqrt{a^2-x^2-y^2}} dxdy$$

$$= \int_0^a \int_0^{\sqrt{a^2-y^2}} \sqrt{a^2 - x^2 - y^2}\, dxdy$$

$$= \int_0^a \int_0^{\sqrt{a^2-y^2}} \sqrt{\left(\sqrt{a^2-y^2}\right)^2 - x^2}\, dxdy$$

$$= \int_0^a \left[\frac{x}{2}\sqrt{(a^2-y^2)-x^2} + \frac{a^2-y^2}{2}\sin^{-1}\frac{x}{\sqrt{a^2-y^2}}\right]_0^{\sqrt{a^2-y^2}} dy$$

$$= \int_0^a \left(0 + \frac{a^2-y^2}{2}\sin^{-1}1 - 0 - 0\right) dy$$

$$= \int_0^a \left(\frac{a^2-y^2}{2}\right)\frac{\pi}{2}\, dy$$

$$= \frac{\pi}{4}\int_0^a (a^2-y^2)\, dy$$

$$= \frac{\pi}{4}\left[a^2y - \frac{y^3}{3}\right]_0^a$$

$$= \frac{\pi}{4}\left[a^3 - \frac{a^3}{3} - 0\right]$$

$$= \frac{\pi}{4}\left(\frac{2a^3}{3}\right)$$

$$= \frac{\pi a^3}{6}$$

Example:

Evaluate $\int_0^{\log a}\int_0^{x}\int_0^{x+y} e^{x+y+z}\, dzdydx$

Solution:

$$\int_0^{\log a}\int_0^{x}\int_0^{x+y} e^{x+y+z}\, dzdydx = \int_0^{\log a}\int_0^{x}\left[e^{x+y+z}\right]_0^{x+y} dydx$$

$$= \int_0^{\log a}\int_0^{x}\left(e^{2(x+y)} - e^{x+y}\right) dydx$$

$$= \int_0^{\log a}\left[\frac{e^{2(x+y)}}{2} - e^{x+y}\right]_0^{x} dx$$

$$= \int_0^{\log a}\left(\frac{e^{4x}}{2} - e^{2x} - \frac{e^{2x}}{2} + e^x\right) dx$$

$$= \int_0^{\log a}\left(\frac{e^{4x}}{2} - \frac{3}{2}\times\frac{e^{2x}}{2} + e^x\right) dx$$

$$= \left[\frac{e^{4x}}{8} - \frac{3e^{2x}}{8} + e^x\right]_0^{\log a}$$

$$= \frac{e^{4\log a}}{8} - \frac{3}{8}e^{2\log a} + e^{\log a} - \frac{1}{8} + \frac{3}{8} - 1$$

$$= \frac{e^{\log a^4}}{8} - \frac{3}{8}e^{\log a^2} + a + \left(\frac{-1+6-8}{8}\right)$$

$$= \frac{a^4}{8} - \frac{3a^2}{8} + a - \frac{6}{8} \qquad [\because e^{\log x} = x]$$

Type:II Problem on Triple Integral if region is given

Example:

Express the region $x \geq 0, y \geq 0, z \geq o, x^2 + y^2 + z^2 \leq 1$ by triple integration.

Solution:

For the given region, z varies from 0 to $\sqrt{1 - x^2 - y^2}$

y varies from 0 to $\sqrt{1 - x^2}$

x varies from 0 to 1

$$\therefore I = \int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2-y^2}} dzdydx$$

Example:

Evaluate $\iiint x^2yzdxdydz$ taken over the tetrahedron bounded by the planes $x = 0, y = 0, z = 0$ and $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$.

Solution:

Given $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1.$

Limits are , z varies from 0 to $c\left(1 - \frac{x}{a} - \frac{y}{b}\right)$

y varies from 0 to $b\left(1 - \frac{x}{a}\right)$

x varies from 0 to a

$$\iiint x^2yzdxdydz = \int_0^a \int_0^{b\left(1-\frac{x}{a}\right)} \int_0^{c\left(1-\frac{x}{a}-\frac{y}{b}\right)} x^2 dzdydx$$

$$= \int_0^a \int_0^{b\left(1-\frac{x}{a}\right)} \left[x^2y\frac{z^2}{2}\right]_0^{c\left(1-\frac{x}{a}-\frac{y}{b}\right)} dydx$$

$$= \int_0^a \int_0^{b\left(1-\frac{x}{a}\right)} \left(\frac{x^2yc^2\left(1-\frac{x}{a}-\frac{y}{b}\right)^2}{2}\right) dydx$$

$$= \frac{c^2}{2}\int_0^a \int_0^{bk} x^2 y\left(k - \frac{y}{b}\right)^2 dydx \qquad \left[\because k = 1 - \frac{x}{a}\right]$$

$$= \frac{c^2}{2}\int_0^a \int_0^{bk} x^2 y\left(k^2 + \frac{y^2}{b^2} - \frac{2ky}{b}\right) dydx$$

$$= \frac{c^2}{2}\int_0^a \int_0^{bk} x^2\left(yk^2 + \frac{y^3}{b^2} - \frac{2ky^2}{b}\right) dydx$$

$$= \frac{c^2}{2}\int_0^a x^2\left[\frac{k^2y^2}{2} + \frac{y^4}{4b^2} - \frac{2ky^3}{3b}\right]_0^{bk} dx$$

$$= \frac{c^2}{2}\int_0^a x^2\left(\frac{b^2k^4}{2} + \frac{b^4k^4}{4b^2} - \frac{2b^3k^4}{3b}\right) dx$$

$$= \frac{c^2}{2}\int_0^a x^2\left(\frac{b^2k^4}{2} + \frac{b^2k^4}{4} - \frac{2b^2k^4}{3}\right) dx$$

$$= \frac{b^2c^2}{2}\int_0^a k^4x^2\left(\frac{1}{2} + \frac{1}{4} - \frac{2}{3}\right) dx$$

$$= \frac{b^2c^2}{24}\int_0^a x^2\left(1 - \frac{x}{a}\right)^4 dx$$

$$\left[\because (1-x)^n = 1 - nx + \frac{n(n-1)}{2!}x^2 - \frac{n(n-1)(n-2)}{3!}x^3 + \cdots\right]$$

$$= \frac{b^2c^2}{24}\int_0^a x^2\left(1 - \frac{4x}{a} + \frac{4\times3}{2!}\times\frac{x^2}{a^2} - \frac{4\times3\times2}{3!}\times\frac{x^3}{a^3} + \frac{4\times3\times2\times\times1}{4!}\times\frac{x^4}{a^4}\right) dx$$

$$= \frac{b^2c^2}{24}\int_0^a \left(x^2 - \frac{4x^3}{a} + \frac{6x^4}{a^2} - \frac{4x^5}{a^3} + \frac{x^6}{a^4}\right) dx$$

$$= \frac{b^2c^2}{24}\left[\frac{x^3}{3} - \frac{4x^4}{4a} + \frac{6x^5}{5a^2} - \frac{4x^6}{6a^3} + \frac{x^7}{7a^4}\right]_0^a$$

$$= \frac{b^2c^2}{24}\left[\frac{a^3}{3} - \frac{a^4}{a} + \frac{6a^5}{5a^2} - \frac{2a^6}{3a^3} + \frac{a^7}{7a^4}\right]$$

$$= \frac{b^2c^2}{24}\left[\frac{a^3}{3} - a^3 + \frac{6a^3}{5} - \frac{2a^3}{3} + \frac{a^3}{7}\right]$$

$$= \frac{a^3b^2c^2}{24}\left[\frac{1}{3} - 1 + \frac{6}{5} - \frac{2}{3} + \frac{1}{7}\right]$$

$$= \frac{a^3b^2c^2}{24}\left(\frac{35-105+126-70+15}{105}\right)$$

$$= \frac{a^3b^2c^2}{24}\left(\frac{1}{105}\right)$$

$$= \frac{a^3b^2c^2}{2520}$$

Example:

Find the value of $\iiint xyzdxdydz$ through the positive spherical octant for which $x^2+y^2+z^2 \le a^2$

Solution:

In the positive octant, the limits are

z varies from 0 to $\sqrt{a^2-x^2-y^2}$

y varies from 0 to $\sqrt{a^2-x^2}$

x varies from 0 to a

$$I= \int_0^a \int_0^{\sqrt{a^2-x^2}} \int_0^{\sqrt{a^2-x^2-y^2}} xyzdzdydx$$

$$= \int_0^a \int_0^{\sqrt{a^2-x^2}} \left[\frac{xyz^2}{2}\right]_0^{\sqrt{a^2-x^2-y^2}} dydx$$

$$= \int_0^a \int_0^{\sqrt{a^2-x^2}} xy(a^2-x^2-y^2)\, dydx$$

$$= \frac{1}{2}\int_0^a \int_0^{\sqrt{a^2-x^2}} (a^2xy-x^3y-xy^3)\, dydx$$

$$= \frac{1}{2}\int_0^a \left[\frac{a^2xy^2}{2} - \frac{x^3y^2}{2} - \frac{xy^4}{4}\right]_0^{\sqrt{a^2-x^2}} dx$$

$$= \frac{1}{2}\int_0^a \left(\frac{a^2x(a^2-x^2)}{2} - \frac{x^3(a^2-x^2)}{2} - \frac{x(a^2-x^2)^2}{4}\right) dx$$

$$= \frac{1}{2}\int_0^a \frac{x}{2}(a^2-x^2)\left[a^2-x^2-\frac{(a^2-x^2)}{2}\right] dx$$

$$= \frac{1}{2}\int_0^a \frac{x(a^2-x^2)(a^2-x^2)}{4} dx$$

$$= \frac{1}{8}\int_0^a x(a^2-x^2)^2\, dx$$

$Put\ a^2-x^2=t$ $\qquad\qquad$ $x=0 \to t=a^2$

$-2xdx = dt$ $\qquad\qquad$ $x=a \to t=0$

$$\Rightarrow \quad I = \frac{1}{8}\int_{a^2}^0 t^2\left(-\frac{dt}{2}\right)$$

$$= -\frac{1}{16}\int_{a^2}^{0} t^2 dt$$

$$= \frac{1}{16}\int_{0}^{a^2} t^2 dt$$

$$= \frac{1}{16}\left[\frac{t^3}{3}\right]_0^{a^2}$$

$$= \frac{1}{16}\left(\frac{a^6}{3}\right)$$

$$= \frac{a^6}{48}$$

Example:

Evaluate $\iiint_D (x+y+z)\,dxdydz$ where $D: 1 \le x \le 2, 2 \le y \le 3, 1 \le z \le 3$

Solution:

$$\iiint_D (x+y+z)\,dxdydz = \int_1^2\int_2^3\int_1^3 (x+y+z)\,dzdydx$$

$$= \int_1^2\int_2^3 \left[xz + yz + \frac{z^2}{2}\right]_1^3 dydx$$

$$= \int_1^2\int_2^3 (3x + 3y + \frac{9}{2} - x - y - \frac{1}{2})\,dydx$$

$$= \int_1^2\int_2^3 (2x + 2y + 4)\,dydx$$

$$= \int_1^2 \left[2xy + \frac{2y^2}{2} + 4y\right]_2^3 dx$$

$$= \int_1^2 (6x + 9 + 12 - 4x - 4 - 8)\,dx$$

$$= \int_1^2 (2x + 9)\,dx$$

$$= \left[\frac{2x^2}{2} + 9x\right]_1^2$$

$$= 4 + 18 - 1 - 9$$

$$= 12$$

Example:

Evaluate $\iiint \frac{dxdydz}{\sqrt{a^2-x^2-y^2-z^2}}$ over the first octant of the sphere $x^2+y^2+z^2=a^2$

Solution:

$$\iiint \frac{dxdydz}{\sqrt{a^2-x^2-y^2-z^2}} = \int_0^a \int_0^{\sqrt{a^2-x^2}} \int_0^{\sqrt{a^2-x^2-y^2}} \frac{dzdydx}{\sqrt{a^2-x^2-y^2-z^2}}$$

$$= \int_0^a \int_0^{\sqrt{a^2-x^2}} \int_0^{\sqrt{a^2-x^2-y^2}} \frac{dzdydx}{\sqrt{\sqrt{(a^2-x^2-y^2)^2}-z^2}}$$

$$= \int_0^a \int_0^{\sqrt{a^2-x^2}} \left[\sin^{-1}\frac{z}{\sqrt{a^2-x^2-y^2}}\right]_0^{\sqrt{a^2-x^2-y^2}} dydx$$

$$= \int_0^a \int_0^{\sqrt{a^2-x^2}} (\sin^{-1}1 - 0)\, dydx$$

$$= \int_0^a \int_0^{\sqrt{a^2-x^2}} \frac{\pi}{2}\, dydx$$

$$= \frac{\pi}{2}\int_0^a [y]_0^{\sqrt{a^2-x^2}}\, dx$$

$$= \frac{\pi}{2}\int_0^a \sqrt{a^2-x^2}\, dx$$

$$= \frac{\pi}{2}\left[\frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a}\right]_0^a$$

$$= \frac{\pi}{2}\left[0 + \frac{a^2}{2}\sin^{-1}1 - 0 - 0\right]$$

$$= \frac{\pi}{2}\frac{a^2}{2}\frac{\pi}{2}$$

$$= \frac{\pi^2 a^2}{8}$$

Exercise:

1. Evaluate $\int_0^4 \int_0^1 \int_0^1 (x+y+z)\, dzdydx$ **Ans:** 12
2. Evaluate $\int_0^1 \int_0^{1-x} \int_0^{x+y} x\, dzdydx$ **Ans:** $\frac{1}{8}$
3. Evaluate $\int_1^3 \int_{\frac{1}{x}}^1 \int_0^{\sqrt{xy}} x\, yz dzdydx$ **Ans:** $\frac{2}{5}\left[\frac{2}{5}\left(9\sqrt{3}-1\right) - log3\right]$
4. Evaluate $\int_0^{\log 2} \int_0^x \int_0^{x+\log y} e^{x+y+z} dzdydx$ **Ans:** $\frac{5}{8}$
5. Evaluate $\int_0^a \int_0^x \int_0^{x+y} e^{x+y+z} dzdydx$ **Ans:** $\frac{1}{8}[e^{4a} - 6e^{2a} + 8e^a - 3]$
6. Evaluate $\iiint_V (x+y+z)\, dxdydz$ where the region V is bounded by

$x + y + z = a\ (a > 0), x = 0, y = 0, z = 0$ **Ans:** $\frac{a^4}{8}$

7. Evaluate $\int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2-y^2}} \frac{dzdydx}{\sqrt{1-x^2-y^2-z^2}}$ **Ans:** $\frac{\pi^2}{8}$

8. Evaluate $\int_{-1}^1 \int_0^z \int_{x-z}^{x+z} (x + y + z)\ dxdydz$ **Ans:** 0

Triple Integrals – Volume of Solids

Volume $= \iiint_V$ dzdydx where V is the volume of the given surface.

Example:

Find the volume of the sphere $x^2 + y^2 + z^2 = a^2$

Solution:

Volume = 8 X volume of the first octant

z varies from 0 to $\sqrt{\mathbf{a^2 - x^2 - y^2}}$

y varies from 0 to $\sqrt{\mathbf{a^2 - x^2}}$

x varies from 0 to a

$$= 8 \int_0^a \int_0^{\sqrt{a^2-x^2}} \int_0^{\sqrt{a^2-x^2-y^2}} dzdydx$$

$$= 8 \int_0^a \int_0^{\sqrt{a^2-x^2}} [Z]_0^{\sqrt{a^2-x^2-y^2}}\ dydx$$

$$= 8 \int_0^a \int_0^{\sqrt{a^2-x^2}} \sqrt{a^2 - x^2 - y^2}\, dydx$$

$$= 8 \int_0^a \int_0^{\sqrt{a^2-x^2}} \sqrt{\left(\sqrt{a^2 - x^2}\right)^2 - y^2}\, dydx$$

$$= 8 \int_0^a \left[\frac{y}{2}\sqrt{a^2 - x^2 - y^2} + \frac{a^2-x^2}{2} \sin^{-1} \frac{y}{\sqrt{a^2-x^2}}\right]_0^{\sqrt{a^2-x^2}} dx$$

$$= 8 \int_0^a \left(0 + \frac{(a^2-x^2)}{2} \sin^{-1} 1 - 0\right) dx$$

$$= 4 \int_0^a (a^2 - x^2) \frac{\pi}{2}\, dx$$

$$= 2\pi \int_0^a (a^2 - x^2)\, dx$$

$$= 2\pi \left[a^2 x - \frac{x^3}{3}\right]_0^a$$

$$= 2\pi \times \frac{2a^3}{3}$$

$$= \frac{4\pi a^3}{3} \text{ cu. units.}$$

Example:

Find the volume of the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$

Solution:

Volume =8 X volume of the first octant

z varies from 0 to $c\sqrt{1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}}$

y varies from 0 to $\sqrt{1 - \frac{x^2}{a^2}}$

x varies from 0 to a

$$V = 8\int_0^a \int_0^{\sqrt{1-\frac{x^2}{a^2}}} \int_0^{c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}} dzdydx$$

$$= 8\int_0^a \int_0^{b\sqrt{1-\frac{x^2}{a^2}}} [Z]_0^{c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}} dydx$$

$$= 8\int_0^a \int_0^{b\sqrt{1-\frac{x^2}{a^2}}} c\sqrt{\left(1-\frac{x^2}{a^2}\right) - \frac{y^2}{b^2}}\, dydx$$

$$= 8c\int_0^a \int_0^{b\sqrt{1-\frac{x^2}{a^2}}} \sqrt{\frac{b^2\left(1-\frac{x^2}{a^2}\right)-y^2}{b^2}}\, dydx$$

$$= \frac{8c}{b}\int_0^a \int_0^{b\sqrt{1-\frac{x^2}{a^2}}} \sqrt{b^2\left(1-\frac{x^2}{a^2}\right) - y^2}\, dydx$$

$$= \frac{8c}{b}\int_0^a \int_0^k \sqrt{k^2 - y^2}\, dydx \qquad where\ k^2 = b^2\left(1 - \frac{x^2}{a^2}\right)$$

$$= \frac{8c}{b}\int_0^a \left[\frac{y}{2}\sqrt{k^2-y^2} + \frac{k^2}{2}sin^{-1}\frac{y}{k}\right]_0^k dx$$

$$= \frac{8c}{b}\int_0^a \left(0 + \frac{k^2}{2}sin^{-1}1 - 0\right) dx$$

$$= \frac{8c}{b}\int_0^a \left(\frac{k^2}{2}\right)\frac{\pi}{2}\, dx$$

$$= \frac{2c\pi}{b}\int_0^a k^2\,dx$$

$$= \frac{2c\pi}{b}\int_0^a b^2\left(1-\frac{x^2}{a^2}\right)dx$$

$$= 2bc\pi\int_0^a \left(1-\frac{x^2}{a^2}\right)dx$$

$$= 2bc\pi\left[x-\frac{x^3}{3a^2}\right]_0^a$$

$$= 2bc\pi\left[a-\frac{a^3}{3a^2}\right]$$

$$= 2bc\pi\left(a-\frac{a}{3}\right)$$

$$= 2bc\pi\times\frac{2a}{3}$$

$$= \frac{4\pi abc}{3}\ cu.units.$$

Example:

Find the volume of the tetrahedron bounded by the coordinate planes and $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$

Solution:

Volume = $\iiint_V \, dzdydx$

z varies from 0 to $c\left(1-\frac{x}{a}-\frac{y}{b}\right)$

y varies from 0 to $b\left(1-\frac{x}{a}\right)$

x varies from 0 to a

$$V = \int_0^a\int_0^{b\left(1-\frac{x}{a}\right)}\int_0^{c\left(1-\frac{x}{a}-\frac{y}{b}\right)} dzdydx$$

$$= \int_0^a\int_0^{b\left(1-\frac{x}{a}\right)}[Z]_0^{c\left(1-\frac{x}{a}-\frac{y}{b}\right)} dydx$$

$$= \int_0^a\int_0^{b\left(1-\frac{x}{a}\right)} c\left(1-\frac{x}{a}-\frac{y}{b}\right) dydx$$

$$= c\int_0^a\left[y-\frac{xy}{a}-\frac{y^2}{2b}\right]_0^{b\left(1-\frac{x}{a}\right)} dx$$

$$= c\int_0^a b\left[\left(1-\frac{x}{a}\right)-\frac{x}{a}\left(1-\frac{x}{a}\right)-\frac{b^2\left(1-\frac{x}{a}\right)^2}{2b}\right]dx$$

$$= bc\int_0^a \left[\left(1-\frac{x}{a}\right)^2-\frac{1}{2}\left(1-\frac{x}{a}\right)^2\right]dx$$

$$= \frac{bc}{2}\int_0^a \left(1-\frac{x}{a}\right)^2 dx$$

$$= \frac{bc}{2}\left[\frac{\left(1-\frac{x}{a}\right)^3}{3\left(-\frac{1}{a}\right)}\right]_0^a$$

$$= -\frac{abc}{6}\left[\left(1-\frac{x}{a}\right)^3\right]_0^a$$

$$= -\frac{abc}{6}(0-1)$$

$$= \frac{abc}{6}\ \text{cu. units.}$$

Example:

Evaluate $\iiint_V dxdydz$ where V is the volume enclosed by the cylinder $x^2+y^2=1$ and the planes $z=0$ and $z=2-x$.

Solution:

In the positive octant, the limits are

z varies from 0 to $2-x$

x varies from 0 to $\sqrt{1-y^2}$

y varies from -1 to 1

$$\iiint dxdydz = 2\int_{-1}^{1}\int_0^{\sqrt{1-y^2}}\int_0^{2-x} dzdxdy$$

$$= 2\int_{-1}^{1}\int_0^{\sqrt{1-y^2}}[z]_0^{2-x}\,dxdy$$

$$= 2\int_{-1}^{1}\int_0^{\sqrt{1-y^2}}(2-x)\,dxdy$$

$$= 2\int_{-1}^{1}\left[2x-\frac{x^2}{2}\right]_0^{\sqrt{1-y^2}} dy$$

$$= 2\int_{-1}^{1}\left[2\sqrt{1-y^2}-\left(\frac{1-y^2}{2}\right)\right]dy$$

$= 4\int_{-1}^{1}\left[\sqrt{1-y^2}\right]dy - \int_{-1}^{1}[1-y^2]\,dy$

$= 4\left[\frac{y}{2}\sqrt{1-y^2} + \frac{1}{2}\sin^{-1}y\right]_{-1}^{1} - \left[y - \frac{y^3}{3}\right]_{-1}^{1}$

$= 4\left[0 + \frac{1}{2}\sin^{-1}1 - 0 - \frac{1}{2}\sin^{-1}(-1)\right] - \left[1 - \frac{1}{3} + 1 - \frac{1}{3}\right]$

$= 4\left[\frac{1}{2}\frac{\pi}{2} + \frac{1}{2}\frac{\pi}{2}\right] - \left[2 - \frac{2}{3}\right]$

$= 4\left(\frac{2\pi}{4}\right) - \frac{4}{3}$

$= 2\pi - \frac{4}{3}$

Exercise:

1. Find the volume of the tetrahedron whose vertices are (0,0,0), (0,1,0), (1,0,0) and (0,0,1)

Ans: $\frac{1}{6}$ cu. units.

2. Evaluate $\iiint_V$ dzdxdy, where V is the volume enclosed by the cylinder $x^2 + y^2 = 4$ and the planes $y + z = 4$ and $z = 0$. **Ans:** 1 6π cu. units

3. Find the volume of the region bounded by the paraboloid $z = x^2 + y^2$ and the plane $z = 4$

Ans: 8π cu. units

4. Find the volume of the ellipsoid $\frac{x^2}{4} + \frac{y^2}{9} + \frac{z^2}{16} = 1$ by using triple integration.

Ans: 3 2π cu. units

5. Find the volume of the tetrahedron bounded by coordinate planes and the plane $\frac{x}{2} + \frac{y}{3} + \frac{z}{4} = 1$ **Ans:** 4 cu. units