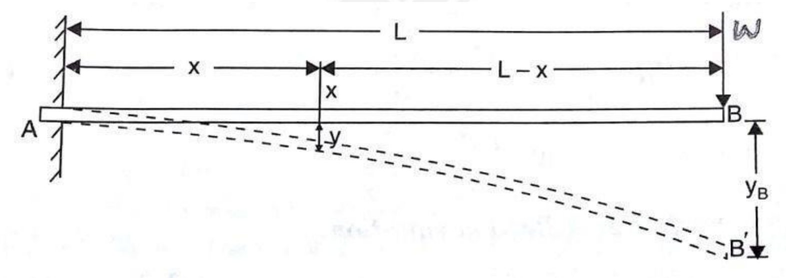
3.2 DOUBLE INTEGRATION METHOD

3.2.1. DEFLECTION OF CANTILEVER WITH A POINT LOAD AT THE FREE END BY DOUBLE INTEGRATION METHOD

A cantilever AB of length L fixed at the point A and free end at the point B and carrying a point load at the free end B as shown in fig. AB shows the position of cantilever before any load is applied whereas AB’ shows the position of cantilever after loading.



Consider a section X, at a distance x from the fixed end A. The B.M. at this section is given by,

M_x = -W(L-x)   (minus sign due to hogging)

But B.M at any section is also given by

M =EI$\frac{d^3y}{dx^2}$

Equating the two values of B.M., we get

EI$\frac{d^3y}{dx^2}$ = -W(L-x) = -WL + Wx

Integrating the above equation, we get

EI$\frac{dy}{dx}$ = -WLx + $\frac{Wx^2}{2}$ +C1   …(i)

Integrating again, we get

EIy = - $\frac{WLx^2}{2}$ + $\frac{W}{2}$ X $\frac{x^3}{3}$ + C1x + C2…(ii)

Where C1 and C2 are constant of integration. Their values are obtained from boundary conditions, which are: (i) at x=0, y=0 (ii) x=0, $\frac{dy}{dx} = 0$

(At the fixed end, deflection and slopes are zero)

(i) By substituting x=0, y=0 in equation (ii), we get 0=0+0+0+C2 ∴ C2 = 0

By substituting x=0, $\frac{dy}{dx}$ =0 in equation (i), we get

0=0+0+C1 ∴ C1 = 0

Substituting the value of C1 in equation (i), we get

$EI\frac{dy}{dx} = -WLx + \frac{Wx^2}{2}$

$= -W\left[Lx - \frac{x^2}{2}\right]$...(iii)

Equation (iii) is known as slope equation. We can find the slope at any point on the cantilever by substituting the value of x. The slope and deflection are maximum at the free end. These can be determined by substituting x=L in these equations. Substituting the values of C1 and C2 in equation (ii), we get

$EIy = -WL\frac{x^2}{2} + \frac{Wx^3}{6}$

$= -W\left[\frac{Lx^2}{2} - \frac{x^3}{6}\right]$...(iv)

Equation (iv) is known as deflection equation.

Let θ_B = slope at free end B y_B =Deflection at the free end B

Substituting θ_B for $\frac{dy}{dx}$ and x=L in equation (iii), we get

$EI\theta B = -W\left[L.L - \frac{L^2}{2}\right] = -W\frac{L^2}{2}$

$\therefore \theta_B = -\frac{WL^2}{2EI}$

Negative sign shows the tangent at B makes an angle in the anti-clockwise direction with AB.

$\therefore \boldsymbol{\theta_B = \frac{WL^2}{2EI}}$

Substituting y_B for y and x=L in equation (iv), we get

$EIy_B = -W\left[L.\frac{L^2}{2} - \frac{L^3}{6}\right]$

$= -W\left[\frac{L^3}{2} - \frac{L^3}{6}\right]$

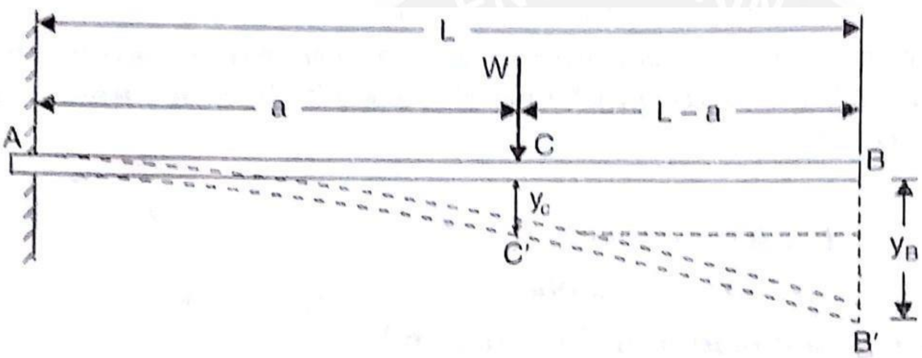
$= -W.\frac{L^3}{3}$

∴ $y_B = -\frac{WL^3}{3EI}$ (Negative sign shows that deflection is downwards)

$$\therefore y_B = \frac{WL^3}{3EI}$$

3.2.2 DFFLECTION OF A CANTILEVER WITH A POINT LOAD AT A DISTANCE 'a' FROM THE FIXED END

A cantilever AB of length L fixed at point B and carrying a point load W at a distance 'a' from the fixed end A, is shown in Fig.



Let θ_C= Slope at point C i.e., $\frac{dy}{dx}$ at C y_C= Deflection at point C

y_B= Deflection at point B

The portion AC of the cantilever may be taken as similar to a cantilever in Art. (i.e., load at the free end).

$$\therefore \theta_C = +\frac{Wa^2}{2EI}$$

$$\text{and } y_C = \frac{Wa^3}{3EI}$$

The beam will bend only between A and C, but from C to B, it will remain straight since B.M. between C and B is zero.

Since the portion CB of the cantilever is straight, therefore Slope at C = Slope at B

$$\theta_C = \theta_B = \frac{Wa^2}{2EI}$$

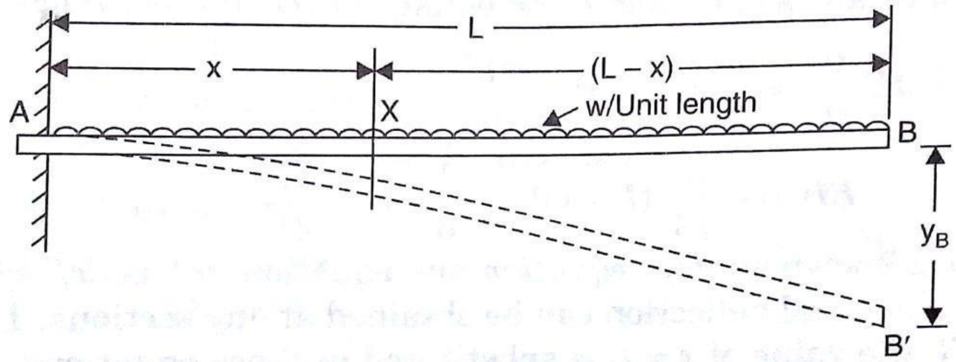
Now from Fig. we have

$$Y_B = y_C + \theta_C(L-a)$$

$$= \frac{Wa^3}{3EI} + \frac{Wa^2}{2EI}(L-a) \left[since, \quad \theta C = \frac{Wa^2}{2EI}\right]$$

3.2.3.DEFLECTION OF A CANTILEVER WITH A UNIFORMLY DISTRIBUTED LOAD

A cantilever AB of length L fixed at the point A and free at the point B and carrying a uniformly distributed load of w per unite length, is shown in Fig.



Consider a section X, at a distance x from the fixed end A. The B.M. at this section is given by,

$M_x = -\ w\ (L - x) \cdot \frac{(L-x)}{2}$ (Minus sign due to hogging)

But B.M. at any section is also given by equation as

$M = EI\ \frac{d^2y}{dx^2}$

Equating the two values of B.M., we get

$EI\ \frac{d^2y}{dx^2} = -\ \frac{w}{2}(L - x)^2$

Integrating the above equation, we get

$EI\ \frac{dy}{dx} = -\frac{w}{2}\frac{(L-x)^3}{3}\ (-1) + C_1$

$= \frac{w}{6}(L - x)^3 + C_1$ .... (i)

Integrating again, we get

$EIy = \frac{w}{6}\frac{(L-x)^4}{4}\ (-1) + C_1x + C_2$

$= -\ \frac{w}{24}(L - x)^4 + C_1x + C_2$..... (ii) where C_1 and C_2 are constant of integrations. There values are obtained from boundary conditions, which are : (i) at x = 0, y = 0 and (ii) at x = 0, $\frac{dy}{dx}$= 0 (as the deflection and slope at fixed end A are zero).

(i) By substituting x = 0, y = 0 in equation (ii), we get

$0= -\frac{w}{24}(L - 0)^4 + C_1 \times 0 + C_2 = -\frac{wL^4}{24} + C_2$

$\therefore C_2 = \frac{wL^4}{24}$

(ii) By substituting x = 0 and $\frac{dy}{dx}$ = 0 in equation (i), we get

$= \frac{w}{64}(L - 0)^3 + C_1 = -\frac{wL^3}{6} + C_1$

$\therefore C_1 = \frac{wL^3}{6}$

Substituting the values of C_1 and C_2in equations (i) and (ii), we get

EI $\frac{dy}{dx} = \frac{w}{6}(L - x)^3 - \frac{wL^3}{6}$ ….. (iii) andEIy = $-\frac{w}{24}(L - x)^4 - \frac{wL^3}{6}x + \frac{wL^4}{24}$ ….. (iv)

Equation (iii) is known as slope equation and equation (iv) as deflection equation. From these equations the slope and deflection can be obtained at any sections. To find the slope and deflection at point B, the value of x = L is substituted in these equations. Let θ_B = Slope at the free end B i.e., $\frac{dy}{dx}$ at B y_B = Deflection at the free end B.

From equation (iii), we get slope at B as

$EI.\theta_B = \frac{w}{6}(L - L)^3 - \frac{wL^3}{6} = -\frac{wL^3}{6}$

$\theta_B = -\frac{wL^3}{6EI} = -\frac{wL^2}{6EI}$ (Since, W = Total load = w.L)

From equation (iv), we get the deflection at B as

$EI.y_B = -\frac{w}{24}(L - L)^4 - \frac{wL^3}{6} \times L + -\frac{wL^4}{24}$

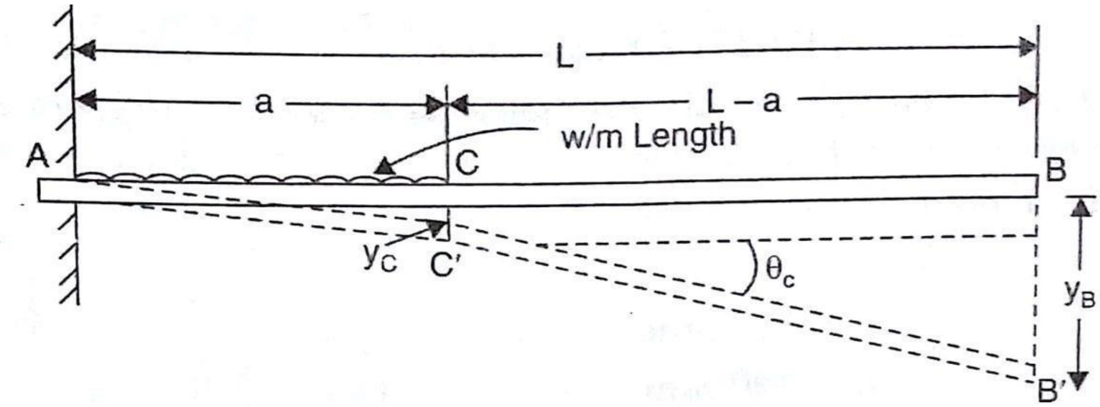
$= -\frac{wL^4}{6} + \frac{wL^4}{24} = -\frac{3}{24}wL^4 = -\frac{wL^4}{8}$

$\therefore \; y_B = -\frac{wL^4}{8EI} = -\frac{wL^3}{8EI}$ (Since, W = w.L)

$\therefore$Downward deflection at B, $\boldsymbol{y_B = -\frac{wL^4}{8EI}}$

3.2.4.DEFLECTION OF A CANTILEVER WITH A UNIFORMLY DISTRIBUTED LOAD FOR A DISTANCE 'a' FROM THE FIXED END

A cantilever AB of length L fixed at the point A and free at the point B and carrying a uniformly distributed load of w/m length for a distance 'a' from the fixed end, is shown in Fig.



The beam will bend only between A and C, but from C to B it will remain straight since B.M. between C and B is zero. The deflected shape of the cantilever is shown by AC'B' in which portion C'B' is straight.

Let θ_C= Slope at C, i.e.,$\frac{dy}{dx}$ at C y_C= Deflection at point C, and y_B= Deflection at point B.

The portion AC of the cantilever may be taken as similar to a cantilever in Art.

$\therefore \theta_C = \frac{w.a^2}{6EI}$

4 and $y_C = \frac{w.a}{8EI}$

Since the portion C'B' of the cantilever is straight, therefore slope at C = Slope at B

or $\theta_C = \theta_B = \frac{wa^3}{6EI}$ ….. (1)

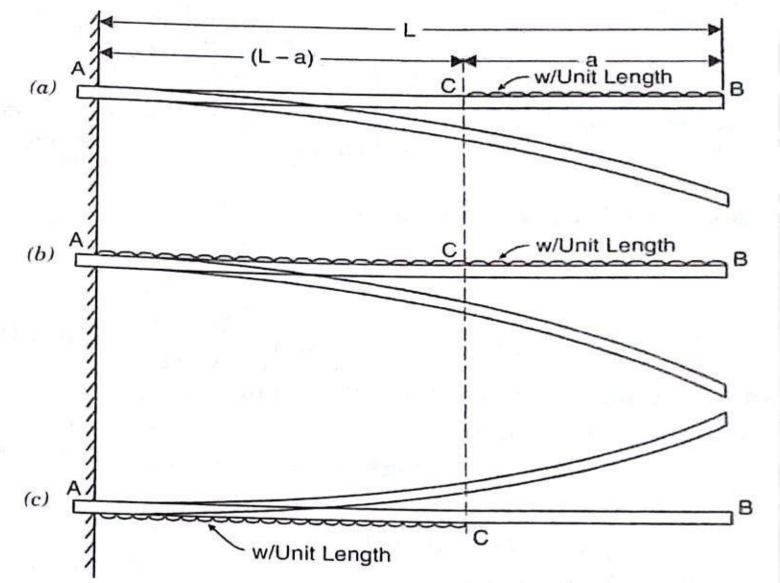
Now from Fig. we have

$y_B = y_C + \theta_C (L - a)$

$= \frac{wa^4}{8EI} + \frac{w.a^3}{6EI} (L - a)$ ….. (2)

3.2.5.DEFLECTION OF A CANTILEVER WITH A UNIFORMLY DISTRIBUTED LOAD FOR A DISTANCE 'a' FROM THE FREE END

A cantilever AB of length L fixed at the point A and free at the point B and carrying a uniformly distributed load of w/m length for a distance 'a' from the free end, is shown in Fig.



The slope and deflection at the point B is determined by considering :

(i) the wholecantilever AB loaded with a uniformly distributed load of w per unite length as shown in Fig.

(ii) a part of cantilever from A to C of length (L - a) loaded with an upward uniformly distributed load of w per unit length as shown in Fig.

Then slope at B = Slope due to downward uniform load over the whole length

- Slope due to upward uniform load from A to C

and deflection at B = Deflection due to downward uniform load over the whole length - deflection due to upward uniform load from A to C.

(a) Now slope at B due todownward uniformly distributed load over the whole length

$$= \frac{wL^3}{6EI}$$

(b) slope at B or at C due to upward uniformly distributed load over the length(L - a)

$=\frac{w(L-a)^3}{6EI}$

Hence net slope at B is given by,

$\theta_B = \frac{wL^3}{6EI} - \frac{w(L-a)^3}{6EI}$….. *(i)*

The downward deflection of point B due to downward distributed load over the whole length AB

$=\frac{wL^4}{8EI}$

The upward deflection of point B due to upward uniformly distributed load acting on the portionAC

= upward deflection of C + slope at C × CB

$=\frac{w(L-a)^4}{8EI} + \frac{w.(L-a)^3}{6EI} \times a$ (since CB = a)

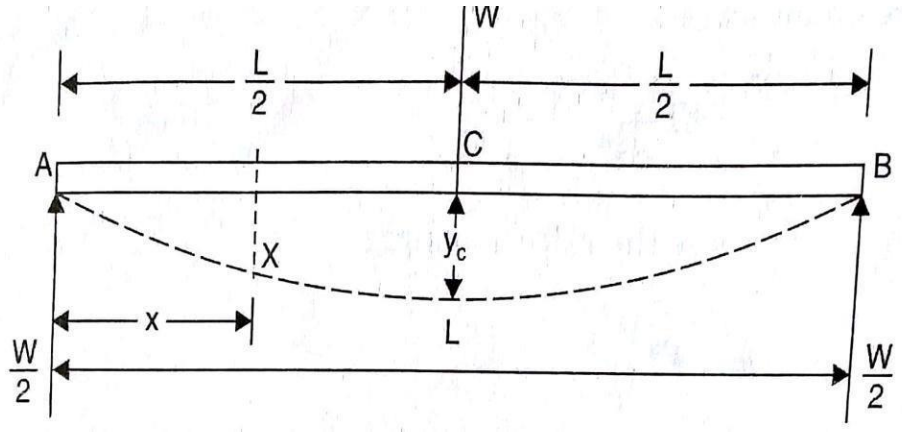
∴ Net downward deflection of the free end B is given by

$$\boldsymbol{y_B = \frac{wL^4}{8EI} - \left[\frac{w(L-a)^4}{8EI} + \frac{w(L-a)^3}{6EI} \times a\right]} \text{….. } (ii)$$

3.2.6. DEFLECTION OF A SIMPLY SUPPORTED BEAM CARRYING A POINT LOAD AT THE CENTRE

A simply supported beam AB of length L and carrying a point load W at the centre is shown in Fig.

As the load is symmetrically applied the reactions R_A and R_B will be equal. Also the maximum deflection will be at the centre.



Now $\qquad R_A = R_B = \frac{W}{2}$

Consider a section X at a distance x from A. The bending moment at this section is given by,

$M_x = R_A \times x$

$= \frac{W}{2} \times x$

(Plus sign is as B.M. for left portion at X is clockwise) But B.M. at any section is also given by equation as

$$M = EI\frac{d^2y}{dx^2}$$

Equation the two values of B.M., we get

$$EI\frac{d^2y}{dx^2} = \frac{W}{2} \times x \qquad \text{..... (i)}$$

On integration, we get

$$EI\frac{dy}{dx} = \frac{W}{2} \times \frac{x^2}{2} + C_1 \qquad \text{..... (ii)}$$

Where C_1 is the constant of integration. And its value is obtained from boundary conditions. The boundary condition is that at $x = \frac{L}{2}$, slope $\frac{dy}{dx} = 0$ (As the deflection is at the centre, hence slope at the centre will be zero). Substituting this boundary condition in equation (ii), we get

$$O = \frac{W}{4} \times \left(\frac{L}{2}\right)^2 + C_1$$

or $C_1 = \frac{WL^2}{16}$

Substituting the value of C_1 in equation (ii), we get

2

EI….. $\frac{dy}{dx} = \frac{Wx^2}{4} - \frac{WL}{16}$ (iii)

The above equation is known the slope equation. We can find the slope at any point on the beam by substituting the values of x. Slope is maximum at A. At A , x = 0 and hence slope at A will be obtained by substituting x = 0 in equation (iii).

$\therefore$ EI$\frac{dy}{dx} = \frac{W}{4} \times 0 - \frac{WL^2}{16}$

EI × θA= - $\frac{WL^2}{16}$

$\therefore \theta_A = -\frac{WL^2}{16EI}$

The slope at point B will be equal to $\theta_{A,}$ sincetheload is symmetrically applied.

$\therefore \boldsymbol{\theta_B = \theta_A = -\frac{WL^2}{16EI}}$

The above equation gives the slope in radians.

Deflection at any point

Deflection at any point is obtained by integrating the slope equation (iii). Hence integrating equation (iii), we get

2

EI × y = ….. $\frac{W}{} \cdot \frac{x^3}{} - \frac{WL}{}x + C_2$ (iv)

Where C_2 is another constant of integration. At A, x = 0 and the deflection (y) is zero.

Hence substituting these values in equation (iv), we get

EI × O = O – 0 + C_2

Or $C_2 = 0$

Substituting the values of C_2 in equation (iv), we get ….. (v)

EI × y = $\frac{Wx^3}{12} - \frac{WL^2.x}{16}$

The above equation is known as the deflection equation. We can find the deflection at any point on the beam by substituting the values of x. The deflection is maximum at centre point C, where x = $\frac{L}{2}$. Let y_c representsthe deflection at C. Then substituting x $=\frac{L}{2}$ and y = y_c in equation (iv), we get

$EI \times y_c = \frac{W}{12}\left(\frac{L}{2}\right)^3 - \frac{WL^2}{16} \times \left(\frac{L}{2}\right)$

$= \frac{WL^3}{96} - \frac{WL^3}{32} = \frac{WL^3 - 3WL^3}{96}$

$= -\frac{2WL^3}{96} = \frac{WL^3}{48}$

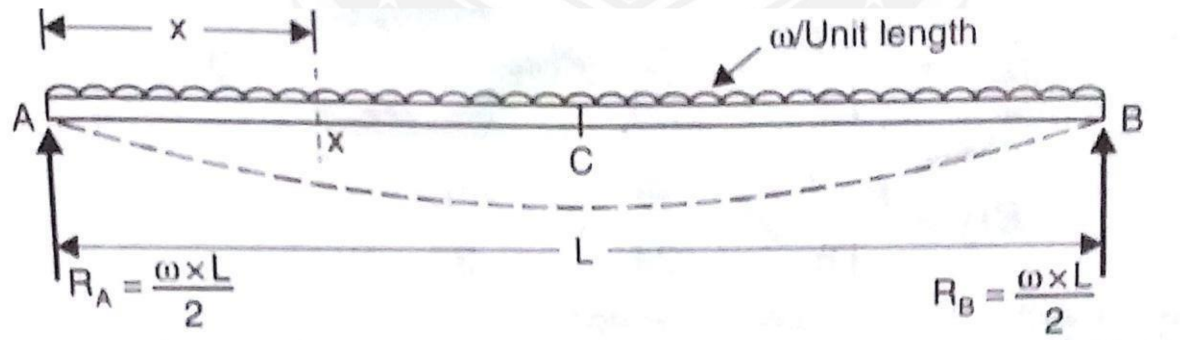
$\therefore y_c = -\frac{WL^3}{48EI}$

(Negative sign shows that deflection is downwards)

$\therefore$ Maximum deflection, $\mathbf{y_c} = \frac{\boldsymbol{WL^3}}{\boldsymbol{48EI}}$

3.2.7. DEFLECTION OF A SIMPLY SUPPORTED BEAM WITH A UNIFORMLY DISTRIBUTED LOAD

A simply supported beam AB of length L and carrying a uniformly distributed load of w per unite length is shown in Fig. The reactions at A and B will be equal. Also the maximum deflection will be at the centre. Each vertical reaction = $\frac{w \times L}{2}$.



$\therefore R_A = R_B = \frac{w \times L}{2}$

Consider a section X at a distance x from A. The bending moment at this section is given by,

$w{\cdot}x$2

$M_x = R_A \times x - w \times \frac{x}{2} = \frac{w\,.\,L}{2}\ x \times\ \ .x -$ ___

2

But B.M. at any section is also given by equation (), as

$M = EI \frac{d^2y}{dx^2}$

Equation the two values of B.N., we get

$EI \times \frac{d^2y}{dx^2} = \frac{w \cdot L}{2} - \frac{w \cdot x^2}{2}$

Integrating the above equation, we get

$EI \frac{dy}{dx} = \frac{w \cdot L}{2} \cdot \frac{x^2}{2} - \frac{w}{2} \cdot \frac{x^3}{3} + C_1$ .....(*i*)

where C_1 is a constant of integration.

Integrating the above equation again, we get

$EI.y = \frac{w \cdot L}{4} \cdot \frac{x^3}{3} - \frac{w}{6} \cdot \frac{x^4}{4} + C_1 x + C_2$ .....(ii)

where C_2 is a another constant of integration. Thus two constant of integration (i.e., C_1 and C_2) are obtained from boundary conditions. The boundary conditions are:

(i) at x = 0, y= 0 and (ii) at x = L, y = 0

Substituting first boundary condition i.e., x = 0, y = 0 in equation (ii), we get

$0 = 0 – 0 + 0 + C_2$ or $C_2 = 0$

Substituting the secondary boundary condition i.e., x = L, y = 0 in equation (ii), we get

$0 = \frac{w \cdot L}{4} \cdot \frac{L^3}{3} - \frac{w}{6} \cdot \frac{L^4}{4} + C_1 . L$ (C_2 is already zero)

$= \frac{w.L^4}{12} - \frac{w.L^4}{24} + C_1 . L$

or $C_1 = -\frac{wL^3}{12} + \frac{wL^3}{24} = -\frac{wL^3}{12}$

Substituting the value of C_1 in equation (i) and (ii), we get

$EI \frac{dy}{dx} = \frac{w.L}{4} . x^2 - \frac{w}{6} x^2 - \frac{w}{6} x^3 - \frac{wL^3}{24}$ .....(*iii*)

$EI.y = \frac{w.L}{12} x^3 - \frac{w}{24} x^4 + \left[-\frac{wL^3}{24}\right] x + 0$ (since, $C_2 = 0$)

$EIy = \frac{w.L}{12} x^3 - \frac{w}{24} x^4 - \frac{wL^3}{24} x$ --- (iv)

Equation (iii) is known as slope equation. We can find the slope $\left[i.e., the\ value\ of\ \frac{dy}{dx}\right]$ at any point on the beam by substituting the different values of x in this equation.

Equation (iv) is known as deflection equation. We can find the deflection[$i.e., the\ value\ of\ y$] at any point on the beam by substituting the different values of x in this equation.

Slope at the supports

Let θ_A = Slope at support A.

and θ_B = Slope at support B

At A, x = 0 and $\frac{dy}{dx} = \theta_A$,

Substituting these value in equation (iii), we get,

$$EI\theta A = \frac{WL}{4} \times 0 - \frac{w}{6} \times 0 - \frac{WL^3}{24}$$

$$EI \times \theta A = -\frac{WL}{24}$$

$$\therefore \theta_A = -\frac{WL^3}{24EI}$$

The slope at point B will be equal to θ_A, sincetheload is symmetrically applied.

$$\boldsymbol{\therefore \theta_B = \theta_A = -\frac{WL^3}{24EI}}$$

The above equation gives the slope in radians.

Deflection at any point

The deflection is maximum at centre point C, where x = $\frac{L}{2}$. Let y_c represents the deflection at C. Then substituting x = $\frac{L}{2}$ and y = y_c in equation (iv), we get

$$EI \times y_c = \frac{WL}{12}\left(\frac{L}{2}\right)^3 - -\frac{W}{24}\left(\frac{L}{2}\right)^4 - \frac{WL^3}{24} \times \left(\frac{L}{2}\right)$$

$$= \frac{WL^4}{96} - \frac{WL^4}{384} - \frac{WL^4}{48} = -\frac{5WL^4}{384}$$

$$\therefore y_c = -\frac{5WL^4}{384EI}$$

(Negative sign shows that deflection is downwards

∴ Maximum deflection,

$$\boldsymbol{y_c = \frac{5WL^4}{384EI}}$$

Example.3.2.1. A cantilever of length 3 m is carrying a point load of 25 kN at the free end. If the moment of inertia of the beam = 10^8 mm^4 and value of E = 2.1 × 1O^5 N/mm^2, find (i) slope of the cantilever at the free end and (ii) deflection at the free end.

Sol. Given:

Length, $L = 3m = 3000$ mm

Point load, $W = 25kN = 25000$ N

M.O.I., $I = 10^8$ mm^4

Value of $E = 2.1 \times 10^5$ N/mm^2

(i) Slope at the free end is given by equation.

$$\therefore \theta_B = \frac{WL^2}{2EI} = \frac{25000 \times 3000^2}{2 \times 2.5 \times 10^5 \times 10^8} = \textbf{0.005357 rad}. \quad \textbf{Ans.}$$

(ii) Deflection at the free end is given by equation

$$y_B = \frac{WL^3}{3EI} = \frac{25000 \times 3000^3}{3 \times 2.1 \times 10^5 \times 10^8} = \textbf{10.71 mm}. \quad \textbf{Ans.}$$

Example.3.2.2 A cantilever of length 3 m is carrying a point load of 50 kN at a distance of 2 m from the fixed end. If $I = 10^5$ mm^4 and value of $E = 2 \times 1O^5$ N/mm^2, find (i) slope at the free end and (ii) deflection at the free end.

Sol. Given:

Length, $L = 3m = 3000$ mm

Point load, $W = 50$ kN $= 50000$ N

Distance between the load and the fixed end,

$a = 2$ m $=2000$ mm

M.O.I. , $I = 10^8$ mm^4 Value of $E = 2 \times 10^5$ N/mm^2

(i) Slope at the free end is given by equation as

$$\therefore \theta_B = \frac{Wa^2}{2EI} = \frac{50000 \times 2000^2}{2 \times 2 \times 10^5 \times 10^8} = \textbf{0.005 rad}. \quad \textbf{Ans.}$$

(ii) Deflection at the free end is given by equation as

$$y_B = \frac{Wa^3}{3EI} + \frac{Wa^2}{2EI}(L - a)$$

$$= \frac{50000 \times 2000^3}{3 \times 2 \times 10^5 \times 10^8} + \frac{50000 \times 2000^2}{2 \times 2 \times 10^5 \times 10^8}(3000 - 2000)$$

$= 6.67 + 5.0 =$ **11.67 mm**. **Ans.**

Example 3.2.3. A cantilever of length 2m carries a uniformly distributed load of 2.5kN/m run for a length of 1.25m from the fixed end and a point load of 1 kN at the free end. Find the deflection at the free end if the section is rectangular 12 cm wide and 24 cm deep and $E = 1 \times 10^4$ N/mm^2

Given Data:

Length, L = 2m =2000mm

u.d.l,w = 2.5 kN/m = 2.5 X $\frac{1000}{1000}$ N/mm = 2.5 N/mm run for a

'length of 'a

= 1.25m = 1250mm from the fixed end.

Point load at the free end

W = 1 kN = 1000N

Width, b = 12cm = 120mm

Depth, d = 24cm = 240mm

E = 1 X 10^4N/mm^2 **To Find:** The deflection at the free end **Solution:**

Moment of inertia of the rectangular section

$$I = \frac{bd^3}{12} = \frac{120 X 240^3}{12} = 13824 X 10^4 mm^4$$

Downward deflection at the free end due to point load

$$y^1 = \frac{WL^3}{3EI} = \frac{1000 X 2000^3}{3 X10^4 \ X 13824 X 10^4} = 1.929 \text{ mm.}$$

Downward deflection at the free end due to uniformly distributed load run over 1.25m from the fixed end.

$$y^2 = \frac{Wa^4}{8EI} + \frac{Wa^3}{6EI}(L-a)$$

$$= \frac{2.5 X 1250^4}{8 X 10^4 X 13824 X 10^4} + \frac{2.5 X 1250^3}{6 X 10^4 X 13824 X 10^4} (2000\text{-}1250)$$

= 0.9934 mm

∴ Total deflection at the free end due to point load and u.d.l

= $y_1 + y_2$ =1.929 + 0.9934 **= 2.9224 mm**

Example.3.2.4. A cantilever of length 2m carries a uniformly distributed load 2 kN/m over a length of 1m from the free end, and a point load of 1 kN at the free end. Find the slope and deflection at the free end if E = 2.1 X 10^5N/mm^2 and I = 6.667 X $10^7$$mm^4$.

Given Data

Length, L = 2m =2000mm

u.d.l,w= 2 kN/m = 2 X $\frac{1000}{1000}$ N/mm = 2 N/mm run for a length of . 1m =

1000mm from the free end

Point load at the free end

W = 1 kN = 1000N

E = $2.1 \times 10^5 N/mm^2$

I = $6.667 \times 10^7 mm^4$

To Find: The slope and deflection at the free end Solution:

Slope at the free end.

The slope at the free end due to a point load

$$\theta_1 = \frac{WL^2}{2EI}$$

$$= \frac{1000 \, X \, 2000^2}{2 \, X \, 2.1 \, X \, 10^5 X 6.667 \, X \, 10^7}$$

= 0.0001428 radians.

The slope at the free end due to u.d.l of 2 kN/m over a length of 1m from the free end.

$$\theta_2 = \frac{WL^3}{6EI} - \frac{W(L-a)^3}{6EI}$$

$$= \frac{2 \, X \, 2000^3}{6 \, X \, 2.1 \, X \, 10^5 X 6.667 \, X \, 10^7} - \frac{2 \, X \, (2000-1000)^3}{6 \, X \, 2.1 \, X \, 10^5 X 6.667 \, X \, 10^7}$$

= 0.0001666 radians.

∴ Total slope at the free end = $\theta 1 + \theta 2$

= 0.0001428 + 0.0001666

= **0.0003094 radians**

Deflection at the free end.

The Deflection at the free end due to a point load

$$y^1 = \frac{WL^3}{3EI}$$

$$= \frac{1000 \, X \, 2000^3}{3 \, X \, 2.1 \, X \, 10^5 X 6.667 \, X \, 10^7}$$

= 0.1904mm.

The Deflection at the free end due to u.d.l of 2 kN/m over a length of 1m from the free end.

$$y^2 = \frac{WL^4}{8EI} - \left[\frac{w(L-a)^4}{8EI} + \frac{w(L-a)^3}{6EI} \, X \, a\right]$$

$$= \frac{2 \, X \, 2000^4}{8 \, X \, 2.1 \, X \, 10^5 X 6.667 \, X \, 10^7} - \frac{2 \, X \, (2000-1000)^4}{8 \, X \, 2.1 \, X \, 10^5 X 6.667 \, X \, 10^7} - \frac{2 \, X \, (2000-1000)^3 \, X \, 1000}{6 \, X \, 2.1 \, X \, 10^5 X 6.667 \, X \, 10^7}$$

= 0.244mm

∴ Total deflection at the free end = $y_1 + y_2$

= 0.1904 + 0.244mm **= 0.4344mm**

Example.3.2.5. A beam 6m long, simply supported at its ends, is carrying a point load of 50 kN at its centre. The moment of inertia of the beam is equal to 78 X 10^6 mm^4. If E for the material of the beam = 2.1 X 10^5N/mm^2, Calculate the slope at the supports and the deflection at the centre of the beam.

Given Data:

Length, L = 6m =6000mm

Point load, W = 50kN = 50000N

M.O.I I = 78 X 10^6 mm^4

Value of E= 2.1 X 10^5N/mm^2 **To Find:**

The maximum slope and Deflection.

Solution:

Maximum slope at supports

$$\theta_B = \theta_A = -\frac{WL^2}{16EI}$$

$$= \frac{WL^2}{16EI}$$

$$= \frac{50000 \, X \, 6000^2}{16 \, X \, 2.1 \, X \, 10^5 X 78 \, X \, 10^6}$$

= 0.06868 radians.

Maximum deflection at centre

$$y_c = \frac{WL^3}{48EI} = \frac{50000 \, X \, 6000^3}{48 \, X \, 2.1 \, X \, 10^5 X 78 \, X \, 10^6} = \mathbf{13.736mm.}$$

Example.3.2.6. A beam 4m long, simply supported at its ends, carries a point load W at its centre. If the slope at the ends of the beam is not to exceed 1 degree. Find the deflection at the centre of the beam **Given Data:**

Length, L = 4m =4000mm Point load at centre, = W

Slope at supports, $\theta_B = \theta_A = 1^o = \frac{1 \, X \, \pi}{180}$ = 0.01745 radians.

We know that slope at supports, $\theta_A = \frac{WL^2}{16EI}$= 0.01745 radians. Maximum deflection at centre

$y_c = \frac{WL^3}{48EI} = \frac{WL^2}{16EI} X \frac{L}{3}$

$= 0.01745 X \frac{4000}{3}$

= **23.26mm**.

Example.3.2.7. A beam of uniform rectangular section 200mm wide and 300mm deep is simply supported at its ends. It carries a uniformly distributed load of 9kN/m run over the entire span of 5m. If the value of E for the material is 1 X 10^4N/mm^2, Find the slope at the supports and maximum deflection.

Solution.

Moment of inertia of the rectangular section

$$I = \frac{bd^3}{12} = \frac{200 X 300^3}{12} = 4.5 X 10^8 mm^4$$

Maximum slope at supports,

$$\theta_B = \theta_A = \frac{WL^3}{24EI} = \frac{9 X 5000^3}{24 X 1 X 10^4 X 4.5 X 10^8} = \mathbf{0.0104\ radians}$$

Maximum Deflection at centre

$$y_c = \frac{5WL^4}{384EI} = \frac{5 X 9 X 5000^4}{384 X 1 X 10^4 X 4.5 X 10^8} = \mathbf{16.27mm.}$$