5.5 Poynting's Theorem

This theorem states that the cross product of electric field vector, E and magnetic field vector, H at any point is a measure of the rate of flow of electromagnetic energy per unit area at that point, that is

P = E x H

Here P → Poynting vector and it is named after its discoverer, J.H. Poynting. The direction of P is perpendicular to E and H and in the direction of vector E x H

With Maxwell's Equations, we now have the tools necessary to derive Poynting's Theorem, which will allow us to perform many useful calculations involving the direction of power flow in electromagnetic fields. We will begin with Faraday's Law, and we will take the dot product of H with both sides:

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

$$\mathbf{H} \bullet (\nabla \times \mathbf{E}) = -\mathbf{H} \bullet \left(\frac{\partial \mathbf{B}}{\partial t}\right)$$

Next, we will start with Ampere's Law and will take the dot product of **E** with both sides:

$$\nabla \times \mathbf{H} = \frac{\partial \mathbf{D}}{\partial t} + \mathbf{J}$$

$$\mathbf{E} \bullet (\nabla \times \mathbf{H}) = \mathbf{E} \bullet \left(\frac{\partial \mathbf{D}}{\partial t} + \mathbf{J}\right)$$

$$\mathbf{H} \bullet (\nabla \times \mathbf{E}) - \mathbf{E} \bullet (\nabla \times \mathbf{H}) = -\mathbf{H} \bullet \left(\frac{\partial \mathbf{B}}{\partial t}\right) - \mathbf{E} \bullet \left(\frac{\partial \mathbf{D}}{\partial t} + \mathbf{J}\right)$$

We can now apply the following mathematical identity to the left side

$$\nabla \bullet (\mathbf{A} \times \mathbf{B}) = \mathbf{B} \bullet (\nabla \times \mathbf{A}) - \mathbf{A} \bullet (\nabla \times \mathbf{B})$$

$$\nabla \bullet (\mathbf{E} \times \mathbf{H}) = -\mathbf{H} \bullet \left(\frac{\partial \mathbf{B}}{\partial t}\right) - \mathbf{E} \bullet \left(\frac{\partial \mathbf{D}}{\partial t} + \mathbf{J}\right)$$

Distributing the E across the right side gives

$$\nabla \bullet (\mathbf{E} \times \mathbf{H}) = -\mathbf{H} \bullet \left(\frac{\partial \mathbf{B}}{\partial t}\right) - \mathbf{E} \bullet \frac{\partial \mathbf{D}}{\partial t} - \mathbf{E} \bullet \mathbf{J}$$

Now let's concentrate on the first time on the right side. Applying a constitutive equation,

$$\mathbf{H} \bullet \left(\frac{\partial \mathbf{B}}{\partial t}\right) = \mathbf{H} \bullet \mu \left(\frac{\partial \mathbf{H}}{\partial t}\right)$$

$$\frac{\partial \mathbf{H}^2}{\partial t} = \frac{\partial (\mathbf{H} \bullet \mathbf{H})}{\partial t} = 2\mathbf{H} \bullet \frac{\partial \mathbf{H}}{\partial t}$$

$$\mathbf{H} \bullet \frac{\partial \mathbf{H}}{\partial t} = \frac{1}{2}\frac{\partial (\mathbf{H} \bullet \mathbf{H})}{\partial t} = \frac{1}{2}\frac{\partial (H^2)}{\partial t}$$

Similarly,

$$\mathbf{E} \bullet \frac{\partial \mathbf{E}}{\partial t} = \frac{1}{2}\frac{\partial (\mathbf{E} \bullet \mathbf{E})}{\partial t} = \frac{1}{2}\frac{\partial (E^2)}{\partial t}$$

$$\nabla \bullet (\mathbf{E} \times \mathbf{H}) = -\frac{1}{2}\frac{\partial}{\partial t}\left(\mu H^2 + \epsilon E^2\right) - \mathbf{E} \bullet \mathbf{J}$$