ANALYTIC FUNCTIONS – NECESSARY AND SUFFICIENT CONDITIONS FOR ANALYTICITY IN CARTESIAN AND POLAR CO-ORDINATES

Analytic [or] Holomorphic [or] Regular function

A function is said to be analytic at a point if its derivative exists not only at that point but also in some neighbourhood of that point.

Entire Function: [Integral function]

A function which is analytic everywhere in the finite plane is called an entire function.

An entire function is analytic everywhere except at $z = \infty$.

Example: $e^z, \sin z, \cos z, sinhz, \cosh z$

The necessary condition for $f = (z)$ to be analytic. [Cauchy – Riemann Equations]

The necessary conditions for a complex function $f = (\text{z}) = u(x, y) + iv(x, y)$ to be analytic in a region R are $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$ $i.e., u_x = v_y$ and $v_x = -u_y$

[OR]

Derive C – R equations as necessary conditions for a function $\mathbf{w = f(z)}$ to be analytic. [Anna, Oct. 1997] [Anna, May 1996]

Proof:

Let $f(z) = u(x, y) + iv(x, y)$ be an analytic function at the point z in a region R.

Since $f(z)$ is analytic, its derivative $f'(z)$ exists in R $f'(z) = \text{Lt} \frac{f(z+\Delta z)-f(z)}{\Delta_z}$

Let $z = x + iy$

$\Rightarrow \Delta z = \Delta_x + i\Delta_y$

$z + \Delta_z = (x + \Delta_x) + i(y + \Delta_y)$

$f(z) = u(x, y) + iv(x, y)$

$f(z + \Delta z) = u(x + \Delta x, y + \Delta y) + iv(x + \Delta x, y + \Delta y$

$f(z + \Delta z) - f(z) = u(x + \Delta x, y + \Delta y) + iv(x + \Delta x, y + \Delta y) - [u(x, y) + iv(x, y)]$

$$= [u(x + \Delta x, y + \Delta y) - u(x, y)] + i[v(x + \Delta x, y + \Delta y) - v(x, y)]$$

$$f'(z) = \underset{\Delta z \to 0}{\text{Lt}} \frac{f(z+\Delta z)-f(z)}{\Delta z}$$

$$= \underset{\Delta z \to 0}{\text{Lt}} \frac{u(x+\Delta x,y+\Delta y)-u(x,y)]+i[v(x+\Delta x,y+\Delta y)-v(x,y)]}{\Delta x+i\Delta y}$$

Case (i)

If $\Delta z \to 0$,firsts we assume that $\Delta y = 0$ and $\Delta x \to 0$.

$$\therefore \; f'(z) = \underset{\Delta x \to 0}{\text{Lt}} \frac{[u(x+\Delta x,y)-u(x,y)]+i[v(x+\Delta x,y)-v(x,y)]}{\Delta x}$$

$$= \underset{\Delta x \to 0}{\text{Lt}} \frac{u(x+\Delta x,y)-u(x,y)}{\Delta x} + \underset{\Delta x \to 0}{\text{Lt}} \frac{v(x+\Delta x,y)-v(x,y)}{\Delta x}$$

$$= \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} \qquad \ldots(1)$$

Case (ii)

If $\Delta z \to 0$ Now, we assume that $\Delta x = 0$ and $\Delta y \to 0$

$$\therefore\ f'(z) = \underset{\Delta y \to 0}{\text{Lt}} \frac{[u(x,y+\Delta y)-u(x,y)]+i[v(x,y+\Delta y)-v(x,y)]}{i\Delta y}$$

$$= \frac{1}{i} \underset{\Delta y \to 0}{\text{Lt}} \frac{u(x,y+\Delta y)-u(x,y)}{\Delta y} + \underset{\Delta y \to 0}{\text{Lt}} \frac{v(x,y+\Delta y)-v(x,y)}{\Delta y}$$

$$= \frac{1}{i}\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}$$

$$= -i\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y} \qquad \text{... (2)}$$

From (1) and (2), we get

$$\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = -\,i\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}$$

Equating the real and imaginary parts we get

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}\,, \qquad \frac{\partial v}{\partial x} = -\frac{\partial v}{\partial y}$$

$$(i.e.)\ u_x = v_y, \quad v_x = -u_y$$

The above equations are known as Cauchy – Riemann equations or C-R equations.

Note: (i) The above conditions are not sufficient for $f(z)$ to be analytic. The sufficient conditions are given in the next theorem.

(ii) Sufficient conditions for $f(z)$ to be analytic.

If the partial derivatives u_x, u_y, v_x and v_y are all continuous in D and $u_x = v_y$ and $u_y = -v_x$, then the function $f(z)$ is analytic in a domain D.

(ii) Polar form of C-R equations

In Cartesian co-ordinates any point z is $z = x + iy$.

In polar co-ordinates, $z = re^{i\theta}$ where r is the modulus and θ is the argument.

Theorem: If $f(z) = u(r,\theta) + iv(r,\theta)$ is differentiable at $z = re^{i\theta}$, then $u_r = \frac{1}{r} v_\theta, v_r = -\frac{1}{r} u_\theta$

(OR) $\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta},\ \frac{\partial v}{\partial r} = \frac{-1}{r}\frac{\partial u}{\partial \theta}$

Proof:

Let $z = re^{i\theta}$ and $w = f(z) = u + iv$

$$(i.e.)\ u + iv = f(z) = f(re^{i\theta})$$

Diff. p.w. r. to r, we get

$$\frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r} = f'(re^{i\theta})\ e^{i\theta} \qquad \ldots(1)$$

Diff. p.w. r. to θ, we get

$$\frac{\partial u}{\partial \theta} + i\frac{\partial v}{\partial \theta} = f'(re^{i\theta})\ e^{i\theta} \qquad \ldots(2)$$

$$= ri\left[f'(re^{i\theta})\ e^{i\theta}\right]$$

$$= ri\left[\frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r}\right] \quad \text{by} \quad (1)$$

$$= ri\frac{\partial u}{\partial r} - r\frac{\partial v}{\partial r}$$

Equating the real and imaginary parts, we get

$$\frac{\partial u}{\partial \theta} = -\,i\frac{\partial v}{\partial r},\quad \frac{\partial v}{\partial \theta} = r\frac{\partial u}{\partial r}$$

$$(i.e.)\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}, \quad \frac{\partial v}{\partial r} = \frac{-1}{r}\frac{\partial v}{\partial \theta}$$

Problems based on Analytic functions – necessary conditions Cauchy – Riemann equations

Example: 1 Show that the function $f(z) = xy + iy$ is continuous everywhere but not differentiable anywhere.

Solution:

Given $f(z) = xy + iy$

$(i.e.)\ u = xy, v = y$

x and y are continuous everywhere and consequently $u(x, y) = xy$ and $v(x, y) = y$ are continuous everywhere.

Thus $f(z)$ is continuous everywhere.

But

$u = xy$	$v = y$
$u_x = y$	$v_x = 0$
$u_y = x$	$v_y = 1$
$u_x \neq v_y$	$u_y \neq -v_x$

C−R equations are not satisfied.

Hence, $f(z)$ is not differentiable anywhere though it is continuous everywhere .

Example: 2 Show that the function $f(z) = \bar{z}$ is nowhere differentiable. [A.U N/D 2012]

Solution:

Given $f(z) = \bar{z} = x - iy$

i.e.,

$u = x$	$v = -y$
$\frac{\partial u}{\partial x} = 1$ $\frac{\partial u}{\partial y} = 0$	$\frac{\partial v}{\partial x} = 0$ $\frac{\partial v}{\partial y} = -1$

$\therefore u_x \neq v_y$

C−R equations are not satisfied anywhere.

Hence, $f(z) = \bar{z}$ is not differentiable anywhere (or) nowhere differentiable.

Example: 3 Show that $f(z) = |z|^2$ is differentiable at $z = 0$ but not analytic at $z = 0$.

Solution:

Let $z = x + iy$

$\bar{z} = x - iy$

$|z|^2 = z\,\bar{z} = x^2 + y^2$

$(i.e.)\ f(z) = |z|^2 = (x^2 + y^2) + i0$

$u = x^2 + y^2$	v= 0
$u_x = 2x$ $u_y = 2y$	$v_x = 0$ $v_y = 0$

So, the C−R equations $u_x = v_y$ and $u_y = -v_x$ are not satisfied everywhere except at $z = 0$.

So, f(z) may be differentiable only at $z = 0$.

Now, $u_x = 2x$, $u_y = 2y$,$v_x = 0$ and $v_y = 0$ are continuous everywhere and in particular at (0,0).

Hence, the sufficient conditions for differentiability are satisfied by $f(z)$ at $z = 0$.

So, $f(z)$ is differentiable at $z = 0$ only and is not analytic there.

Inverse function

Let $w = f(z)$ be a function of z and $z = F(w)$ be its inverse function.

Then the function $w = f(z)$ will cease to be analytic at $\frac{dz}{dw} = 0$ and $z = F(w)$ will be so, at point where $\frac{dw}{dz} = 0$.

Example: 4 Show that $f(z) = log\ z$ analytic everywhere except at the origin and find its derivatives.

Solution:

Let $z = re^{i\theta}$

$$f(z) = log\ z$$

$$= \log\left(re^{i\theta}\right) = logr + \log\left(e^{i\theta}\right) = logr + i\theta$$

But, at the origin, $r = 0$. Thus, at the origin,

$$f(z) = log0 + i\theta = -\infty + i\theta$$

So, $f(z)$is not defined at the origin and hence is not differentiable there.

Note : $e^{-\infty} = 0$

$\log e^{-\infty} = \log 0\,; -\infty = log0$

At points other than the origin, we have

$u(r,\theta) = log\ r$	$v(r,\theta) = \theta$
$u_r = \frac{1}{r}$ $u_\theta = 0$	$v_r = 0$ $v_\theta = 1$

So, $logz$ satisfies the C−R equations.

Further $\frac{1}{r}$ is not continuous at $z = 0$.

So, $u_r, u_\theta, v_r, v_\theta$ are continuous everywhere except at $z = 0$. Thus log z satisfies all the sufficient conditions for the existence of the derivative except at the origin.

The derivative is

$$f'(z) = \frac{u_r + iv_r}{e^{i\theta}} = \frac{\left(\frac{1}{r}\right) + i(0)}{e^{i\theta}} = \frac{1}{re^{i\theta}} = \frac{1}{z}$$

Note: $f(z) = u + iv \Rightarrow f(re^{i\theta}) = u + iv$

Differentiate w.r.to ‘r’, we get

$$(i.e.)\ e^{i\theta} f'(re^{i\theta}) = \frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r}$$

Example: 5 Check whether $w = \bar{z}$ is analytics everywhere. [Anna, Nov 2001]

[A.U M/J 2014]

Solution:

Let $w = f(z) = \bar{z}$

u+$iv = x - iy$

$u = x$	$v = -y$
$u_x = 1$ $u_y = 0$	$v_x = 0$ $v_y = -1$

$u_x \neq v_y$ at any point p(x,y)

Hence, C−R equations are not satisfied.

∴The function $f(z)$ is nowhere analytic.

Example: 6 Test the analyticity of the function $w = sin\ z$.

Solution:

Let $w = f(z) = sinz$

$$u + iv = \sin(x + iy)$$

$$u + iv = \sin x \cos iy + \cos x \sin iy$$

$$u + iv = \sin x \cosh y + i \cos x \sin hy$$

Equating real and imaginary parts, we get

$u = \sin x \cosh y$	$v = \cos x \sinh y$
u_x $= \cos x \cosh y$ u_y $= \sin x \sinh y$	v_x $= -\sin x \sinh y$ $v_y = \cos x \cosh y$

$$\therefore u_x = v_y \text{ and } u_y = -v_x$$

C −R equations are satisfied.

Also the four partial derivatives are continuous.

Hence, the function is analytic.

Example: 7 Determine whether the function $2xy + i(x^2 - y^2)$ is analytic or not. [Anna, May 2001]

Solution:

Let $f(z) = 2xy + i(x^2 - y^2)$

$u = 2xy$	$v = x^2 - y^2$

$(i.e.)$

$\frac{\partial u}{\partial x} = 2y$ $\frac{\partial u}{\partial y} = 2x$	$\frac{\partial v}{\partial x} = 2x$ $\frac{\partial v}{\partial y} = -2y$

$u_x \neq v_y$ and $u_y \neq -v_x$

C−R equations are not satisfied.

Hence, $f(z)$ is not an analytic function.

Example: 8 Prove that $f(z) = \cosh z$ is an analytic function and find its derivative.

Solution:

Given $f(z) = \cosh z = \cos(iz) = \cos[i(x + iy)]$

$$= \cos(ix - y) = \cos ix \cos y + \sin(ix) \sin y$$

$$u + iv = \cosh x \cos y + i \sinh x \sin y$$

$u = \cosh x \cos y$	$v = \sinh x \sin y$
u_x =sinh x cos y u_y $= -\cosh x \sin y$	v_x =cosh x sin y v_y =sinh x cos y

$\therefore u_x, u_y, v_x$ and v_y exist and are continuous.

$u_x = v_y$ and $u_y = -v_x$

C−R equations are satisfied.

$\therefore f(z)$ is analytic everywhere.

Now, $f'(z) = u_x + iv_x$

$= \sinh x \cos y + i \cosh x \sin y$

$= \sinh(x + iy) = \sinh z$

Example: 9 If $w = f(z)$ is analytic, prove that $\frac{dw}{dz} = \frac{\partial w}{\partial x} = -i\frac{\partial w}{\partial y}$ where $z = x + iy$, and prove that $\frac{\partial^2 w}{\partial z \partial \bar{z}} = 0$. [Anna, Nov 2001]

Solution:

Let $w = u(x, y) + iv(x, y)$

As $f(z)$ is analytic, we have $u_x = v_y, u_y = -v_x$

Now, $\frac{dw}{dz} = f'(z) = u_x + iv_x = v_y - iu_y = i(u_y + iv_y)$

$$= \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = -i\left[\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}\right]$$

$$= \frac{\partial}{\partial x}(u + iv) = -i\frac{\partial}{\partial y}(u + iv)$$

$$= \frac{\partial w}{\partial x} = -i\frac{\partial w}{\partial y}$$

We know that, $\frac{\partial w}{\partial z} = 0$

$$\therefore \frac{\partial^2 w}{\partial z \partial \bar{z}} = 0$$

Also $\qquad \frac{\partial^2 w}{\partial \bar{z}\, \partial z} = 0$

Example: 10 Prove that every analytic function $\boldsymbol{w = u(x, y) + iv(x, y)}$can be expressed as a function of z alone. [A.U. M/J 2010, M/J 2012]

Proof:

Let $z = x + iy$ and $\quad \bar{z} = x - iy$

$$x = \frac{z+\bar{z}}{2} \quad \text{and} \quad y = \frac{z+\bar{z}}{2i}$$

Hence, u and v and also w may be considered as a function of z and $\bar{z}$

Consider $\frac{\partial w}{\partial \bar{z}} = \frac{\partial u}{\partial \bar{z}} + i\frac{\partial v}{\partial \bar{z}}$

$$= \left(\frac{\partial u}{\partial x} \cdot \frac{\partial x}{\partial \bar{z}} + \frac{\partial u}{\partial y} \cdot \frac{\partial y}{\partial \bar{z}}\right) + \left(\frac{\partial v}{\partial x}\frac{\partial x}{\partial \bar{z}} + \frac{\partial v}{\partial y}\frac{\partial y}{\partial \bar{z}}\right)$$

$$= \left(\frac{1}{2}u_x - \frac{1}{2i}u_y\right) + i\left(\frac{1}{2}v_x - \frac{1}{2i}v_y\right)$$

$$= \frac{1}{2}(u_x - v_y) + \frac{i}{2}(u_y + v_x)$$

$= 0$ by C−R equations as w is analytic.

This means that w is independent of $\bar{z}$

$(i.e.)$ w is a function of z alone.

This means that if $w = u(x, y) + iv(x, y)$ is analytic, it can be rewritten as a function of $(x + iy)$.

Equivalently a function of $\bar{z}$ cannot be an analytic function of z.

Example: 11 Find the constants a, b, c if $\mathbf{f(z) = (x + ay) + i(bx + cy)}$ is analytic.

Solution:

$$f(z) = u(x, y) + iv(x, y)$$

$$= (x + ay) + i(bx + cy)$$

$u = x + ay$	$v = bx + cy$
$u_x = 1$ $u_y = a$	$v_x = b$ $v_y = c$

Given $f(z)$ is analytic

$\Rightarrow u_x = v_y$ and $u_y = -v_x$

$1 = c$ and $a = -b$

Example: 12 Examine whether the following function is analytic or not $\mathbf{f(z) = e^{-x}(\cos y - i \sin y)}$.

Solution:

Given $f(z) = e^{-x}(\cos y - i \sin y)$

$$\Rightarrow u + iv = e^{-x} \cos y - ie^{-x} \sin y$$

$u = e^{-x} \cos y$	$v = -e^{-x} \sin y$
$u_x = -e^{-x} \cos y$	$v_x = e^{-x} \sin y$

$u_y = -e^{-x} \sin y$	$v_y = -e^{-x} \cos y$

Here, $u_x = v_y$ and $u_y = -v_x$

$\Rightarrow$ C−R equations are satisfied

$\Rightarrow f(z)$ is analytic.

Example: 13 Test whether the function $\mathbf{f(z) = \frac{1}{2}\log(x^2 + y^2 + \tan^{-1}\left(\frac{y}{x}\right)}$ is analytic or not.

Solution:

Given $f(z) = \frac{1}{2}\log(x^2 + y^2 + i\tan^{-1}\left(\frac{y}{x}\right)$

$(i.e.)u + iv = \frac{1}{2}\log(x^2 + y^2 + i\tan^{-1}\left(\frac{y}{x}\right)$

$u = \frac{1}{2}\log(x^2 + y^2)$	$v = \tan^{-1}\left(\frac{y}{x}\right)$
u_x $= \frac{1}{2}\frac{1}{x^2 + y^2}(2x)$ $= \frac{x}{x^2 + y^2}$ u_y $= \frac{1}{2}\frac{1}{x^2 + y^2}(2y)$	$v_x = \frac{1}{1 + \frac{y^2}{x^2}}\left[-\frac{y}{x^2}\right]$ $= \frac{-y}{x^2 + y^2}$ $v_y = \frac{1}{1 + \frac{y^2}{x^2}}\left[\frac{1}{x}\right]$

$= \frac{y}{x^2+y^2}$	$= \frac{x}{x^2+y^2}$

Here, $u_x = v_y$ and $u_y = -v_x$

$\Rightarrow$ C−R equations are satisfied

$\Rightarrow f(z)$ is analytic.

Example: 14 Find where each of the following functions ceases to be analytic.

(i) $\frac{z}{(z^2-1)}$ **(ii)** $\frac{z+i}{(z-i)^2}$

Solution:

(i) Let $f(z) = \frac{z}{(z^2-1)}$

$$f'(z) = \frac{(z^2-1)(1)-z(2z)}{(z^2-1)^2} = \frac{-(z^2+1)}{(z^2-1)^2}$$

$f(z)$ is not analytic, where $f'(z)$ does not exist.

$(i.e.)\ f'(z) \to \infty$

$(i.e.)(z^2-1)^2 = 0$

$(i.e.)\quad z^2 - 1 = 0$

$z = 1$

$z = \pm 1$

$\therefore f(z)$ is not analytic at the points $z = \pm 1$

(ii) Let $f(z) = \frac{z+i}{(z-i)^2}$

$$f'(z) = \frac{(z-i)^2(1)(z+i)[2(z-i)]}{(z-i)^4} = \frac{(z+3i)}{(z-i)^3}$$

$$f'(z) \rightarrow \infty, at\ z = i$$

$\therefore f(z)$ is not analytic at $z = i$.