PROPERTIES – HARMONIC CONJUGATES

Laplace equation

$\frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2} = 0$ is known as Laplace equation in two dimensions.

Laplacian Operator

$\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ is called the Laplacian operator and is denoted by ∇^2.

Note: (i) $\frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2} + \frac{\partial^2 \varphi}{\partial z^2} = 0$ is known as Laplace equation in three dimensions.

Note: (ii) The Laplace equation in polar coordinates is defined as

$$\frac{\partial^2 \varphi}{\partial r^2} + \frac{1}{r}\frac{\partial \varphi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \varphi}{\partial \theta^2} = 0$$

Properties of Analytic Functions

Property: 1 Prove that the real and imaginary parts of an analytic function are harmonic functions.

Proof:

Let $f(z) = u + iv$ be an analytic function

$u_x = v_y \ldots (1)$ and $u_y = -v_x \ldots (2)$ by C−R

Differentiate (1) & (2) p.w.r. to x, we get

$u_{xx} = v_{xy} \ldots (3)$ and $u_{xy} = -v_{xx} \ldots (4)$

Differentiate (1) & (2) p.w.r. to x, we get

$u_{yx} = v_{yy} \ldots (5)$ and $u_{yy} = -v_{yx} \ldots (6)$

(3) +(6) ⇒ $u_{xx} + u_{yy} = 0$ $[\because v_{xy} = v_{yx}]$

(5) – (4) ⇒ $v_{xx} + v_{yy} = 0$ $[\because u_{xy} = u_{yx}]$

$\therefore$ u and v satisfy the Laplace equation.

Harmonic function (or) [Potential function]

A real function of two real variables *x* and *y* that possesses continuous second order partial derivatives and that satisfies Laplace equation is called a harmonic function.

Note: A harmonic function is also known as a potential function.

Conjugate harmonic function

If *u* and *v* are harmonic functions such that $u + iv$ is analytic, then each is called the conjugate harmonic function of the other.

Property: 2 If w $= \boldsymbol{u}(\boldsymbol{x}, \boldsymbol{y}) + \boldsymbol{i}\boldsymbol{v}(\boldsymbol{x}, \boldsymbol{y})$ is an analytic function the curves of the family $\boldsymbol{u}(\boldsymbol{x}, \boldsymbol{y}) = \mathbf{c_1}$ and the curves of the family $\boldsymbol{v}(\boldsymbol{x}, \boldsymbol{y}) = \mathbf{c_2}$ cut orthogonally, where $\mathbf{c_1}$ and $\mathbf{c_2}$ are varying constants.

Proof: **[A.U D15/J16 R-13] [A.U N/D 2016 R-13] [A.U A/M 2017 R-08]**

Let $f(z) = u + iv$ be an analytic function

$\Rightarrow u_x = v_y \ldots(1)$ and $u_y = -v_x \ldots(2)$ by C−R

Given $u = \mathrm{c}_1$ and $v = \mathrm{c}_2$

Differentiate p.w.r. to *x*, we get

$$u_x + u_y \frac{dy}{dx} = 0 \text{ and } \quad v_x + v_y \frac{dy}{dx} = 0$$

$$\Rightarrow \frac{dy}{dx} = \frac{-u_x}{u_y} \text{ and } \frac{dy}{dx} = \frac{-v_x}{v_y}$$

$$\Rightarrow m_1 = \frac{-u_x}{u_y} \qquad \Rightarrow m_2 = \frac{-v_x}{v_y}$$

$$m_1 . m_2 = \left(\frac{-u_x}{u_y}\right)\left(\frac{-v_x}{v_y}\right) = \left(\frac{u_x}{u_y}\right)\left(\frac{u_y}{u_x}\right) = -1 \text{ by (1) and (2)}$$

Hence, the family of curves form an orthogonal system.

Property: 3 An analytic function with constant modulus is constant. [AU. A/M 2007] [A.U N/D 2010]

Proof:

Let $f(z) = u + iv$ be an analytic function.

$\Rightarrow u_x = v_y \ldots(1)$ and $u_y = -v_x \ldots(2)$ by C−R

Given $|f(z)| = \sqrt{u^2 + v^2} = c \neq 0$

$\Rightarrow |f(z)| = u^2 + v^2 = c^2$ (say)

$$(i.e)\ u^2 + v^2 = c^2 \quad \ldots(3)$$

Differentiate (3) p.w.r. to *x* and *y*; we get

$$2uu_x + 2vv_x = 0 \Rightarrow uu_x + vv_x = 0 \qquad \ldots(4)$$

$$2uu_y + 2vv_y = 0 \Rightarrow uu_y + vv_y = 0 \qquad \ldots(5)$$

$(4) \times u \quad \Rightarrow u^2 u_x + uv\, v_x = 0 \qquad \ldots(6)$

$(5) \times v \quad \Rightarrow uv\, u_y + v^2 v_y = 0 \qquad \ldots(7)$

(6)+(7) $\Rightarrow u^2u_x + v^2v_y + uv\,[\,v_x + u_y] = 0$

$\Rightarrow u^2u_x + v^2\,u_x + uv\,[\,-u_y + u_y] = 0$ by (1) & (2)

$\Rightarrow (u^2+v^2)\,u_x = 0$

$\Rightarrow u_x = 0$

Similarly, we get $v_x = 0$

We know that $f'(z) = u_x + v_x = 0 + i0 = 0$

Integrating w.r.to z, we get, $f(z) = c$ [Constant]

Property: 4 An analytic function whose real part is constant must itself be a constant. [A.U M/J 2016]

Proof :

Let $f(z) = u + iv$ be an analytic function.

$\Rightarrow u_x = v_y \ldots(1)$ and $u_y = -v_x \ldots(2)$ by C−R

Given $u = c$ [Constant]

$\Rightarrow u_x = 0,\quad u_y = 0$

$\Rightarrow u_x = 0,\quad v_x = 0$ by (2)

We know that $f'(z) = u_x + iv_x = 0 + i0 = 0$

Integrating w.r.to z, we get $f(z) = c$ [Constant]

Property: 5 Prove that an analytic function with constant imaginary part is constant. [A.U M/J 2005]

Proof:

Let $f(z) = u + iv$ be an analytic function.

$\Rightarrow u_x = v_y \ldots(1)$ and $u_y = -v_x \ldots(2)$ by C−R

Given $v = c$ [Constant]

$\Rightarrow v_x = 0, \quad v_y = 0$

We know that $f'(z) = u_x + iv_x$

$= v_y + iv_x$ by (1) $= 0 + i0$

$\Rightarrow f'(z) = 0$

Integrating w.r.to z, we get $f(z) = c$ [Constant]

Property: 6 If $\mathbf{f(z)}$ and $\overline{\mathbf{f(z)}}$ are analytic in a region D, then show that $\mathbf{f(z)}$ is constant in that region D.

Proof:

Let $f(z) = u(x,y) + iv\,(x,y)$ be an analytic function.

$\overline{f(z)} = u(x,y) - iv(x,y) = u(x,y) + i[-v(x,y)]$

Since, $f(z)$ is analytic in D, we get $u_x = v_y$ and $u_y = -v_x$

Since, $\overline{f(z)}$ is analytic in D, we have $u_x = -v_y$ and $u_y = v_x$

Adding, we get $u_x = 0$ and $u_y = 0$ and hence, $v_x = v_y = 0$

$\therefore\ f(z) = u_x + iv_x = 0 + i0 = 0$

$\therefore\ f(z)$ is constant in D.

Problems based on properties

Theorem: 1 If $\mathrm{f}(\mathrm{z}) = \boldsymbol{u} + \boldsymbol{i}\boldsymbol{v}$ is a regular function of z in a domain D, then

$$\boldsymbol{\nabla^2 |f(z)|^2 = 4|f'(z)|^2}$$

Solution:

$$\text{Given } f(z) = u + iv$$

$$\Rightarrow |f(z)| = \sqrt{u^2 + v^2}$$

$$\Rightarrow |f(z)|^2 = u^2 + v^2$$

$$\Rightarrow \nabla^2|f(z)|^2 = \nabla^2(u^2 + v^2)$$

$$= \nabla^2(u^2) + \nabla^2(v^2) \qquad \ldots(1)$$

$$\nabla^2(u^2) = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)u^2 += \frac{\partial^2(u^2)}{\partial x^2} + \frac{\partial^2(u^2)}{\partial y^2} \qquad \ldots(2)$$

$$\frac{\partial^2}{\partial x^2}(u^2) = \frac{\partial}{\partial x}\left[2u\frac{\partial u}{\partial x}\right] = 2\left[u\frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial x}\frac{\partial u}{\partial x}\right] = 2u\frac{\partial^2 u}{\partial x^2} + 2\left(\frac{\partial u}{\partial x}\right)^2$$

Similarly, $\frac{\partial^2}{\partial y^2}(u^2) = 2u\frac{\partial^2 u}{\partial y^2} + 2\left(\frac{\partial u}{\partial y}\right)^2$

$$(2) \Rightarrow \nabla^2(u^2) = 2u\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) + 2\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2\right]$$

$$= 0 + 2\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2\right] \qquad [\therefore u \text{ is harmonic}]$$

$$\nabla^2(u^2) = 2u_x^2 + 2u_y^2$$

Similarly, $\nabla^2(v^2) = 2v_x^2 + 2v_y^2$

$$(1) \Rightarrow \nabla^2|f(z)|^2 = 2\left[u_x^2 + u_y^2 + v_x^2 + v_y^2\right]$$

$$= 2[u_x^2 + (-v_x)^2 + v_x^2 + u_x^2] \qquad [\because u_x = v_y; u_y = -v_x]$$

$$= 4[u_x^2 + v_x^2]$$

$$(i.e.)\nabla^2|f(z)|^2 = 4|f'(z)|^2$$

Note : $f(z) = u + iv; f'(z) = u_x + iv_x$;

(or) $f'(z) = v_y + iu_y$; $|f'(z)| = \sqrt{u_x^2 + v_x^2}$;

$|f'(z)|^2 = u_x^2 + v_x^2$

Theorem: 2 If $\mathbf{f(z) = u + iv}$ is a regular function of z in a domain D, then ∇^2 log $|\mathbf{f(z)}| = 0$ if $\mathbf{f(z)\,f'(z) \neq 0}$ in D. i.e., log $|\mathbf{f(z)}|$ is harmonic in D. [A.U A/M 2017 R-13]

Solution:

Given $f(z) = u + iv$

$$|f(z)| = \sqrt{u^2 + v^2}$$

$$\log|f(z)| = \frac{1}{2}\log(u^2 + v^2)$$

$$\nabla^2\log|f(z)| = \frac{1}{2}\nabla^2\log(u^2 + v^2) = \frac{1}{2}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\log(u^2 + v^2)$$

$$= \frac{1}{2}\frac{\partial^2}{\partial x^2}[log(u^2 + v^2)] + \frac{1}{2}\frac{\partial^2}{\partial y^2}[log(u^2 + v^2)] \quad \ldots(1)$$

$$\frac{1}{2}\frac{\partial^2}{\partial x^2}[log(u^2 + v^2)] = \frac{1}{2}\frac{\partial^2}{\partial x}\left[\frac{1}{u^2+v^2}\left(2u\frac{\partial u}{\partial x} + 2v\frac{\partial v}{\partial x}\right)\right] = \frac{\partial}{\partial x}\left[\frac{uu_x+vv_x}{u^2+v^2}\right]$$

$$= \frac{(u^2+v^2)[uu_{xx}+u_xu_x+vv_{xx}+v_xv_x]-(uu_x+vv_x)(2uu_x+2vv_x)}{(u^2+v^2)^2}$$

$$= \frac{(u^2+v^2)[uu_{xx}+vv_{xx}+u_x^2+v_x^2]-2(uu_x+vv_x)^2}{(u^2+v^2)^2}$$

Similarly, $\frac{1}{2}\frac{\partial^2}{\partial y^2}[log(u^2+v^2)] = \frac{(u^2+v^2)[uu_{yy}+vv_{yy}+u_y^2+v_y^2]-2(uu_y+vv_y)^2}{(u^2+v^2)^2}$

(1)⇒ $\nabla^2\log|f(z)| =$

$$\frac{(u^2+v^2)[u(u_{xx}+u_{yy})+v(v_{xx}+v_{yy})+(u_x^2+u_y^2)+(v_x^2+v_y^2)]-2[uu_x+vv_x]^2-2[uu_y+vv_y]^2}{(u^2+v^2)^2}$$

$$= \frac{(u^2+v^2)[u(0)+(u_x^2+v_x^2)+u_y^2+v_y^2)--2[u^2u_x^2+v^2v_x^2+2uv\,u_xv_x+u^2\,u_y^2+v^2v_y^2+2uv\,u_y\,v_y]}{(u^2+v^2)^2}$$

$[\because u_{xx}+u_{yy} = 0, v_{xx}+vv_{yy} = 0]$

$$= \frac{(u^2+v^2)[|f'(z)^2|+|f'(z)^2|-2[u^2(u_x^2+u_y^2)+v^2(v_x^2+v_y^2)+2uv(u_x\,v_x+u_y\,v_y]}{(u^2+v^2)^2}$$

$[\because f'(z) = u+iv, |f'(z)| = u_x + iv_x$ (or) $f'(z) = v_y - iu_y, |f'(z)|^2 = u_x^2 + v_x^2$

(or) $|f'(z)|^2 = u_y^2 + v_y^2$

$$= \frac{2(u^2+v^2)[|f'(z)^2|-2[u^2|f'(z)|^2+v^2|f'(z)^2|+2uv(0)]}{(u^2+v^2)^2}$$

$[\because u_x = v_y, u_y = -v_x]$

$$\Rightarrow u_x\ v_x + u_y v_y\ = 0$$

$$\Rightarrow u_x^2 + u_y^2 = u_x^2 +\ v_x^2 = |f'(z)|^2$$

$$\Rightarrow v_x^2\ + v_y^2 =\ u_y^2 +\ v_y^2 = |f'(z)|^2$$

$$= \frac{2(u^2+v^2)|f(z)|^2 - 2(u^2+v^2)|f'(z)|^2}{(u^2+v^2)^2}$$

$$(i.e.)\ \nabla^2 \log|f(z)| = 0$$

Theorem: 3 If f(z) $= u + iv$ is a regular function of z in a domain D, then $\nabla^2(u^p) = p(p-1)\ u^{p-2}|f'(z)|^2$

Solution:

$$\nabla^2(u^p) = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)(u^p)$$

$$= \frac{\partial^2}{\partial x^2}(u^p) + \frac{\partial^2}{\partial y^2}(u^p)$$

$$\frac{\partial^2}{\partial x^2}(u^p) = \frac{\partial}{\partial x}\left[p u^{p-1}\frac{\partial u}{\partial x}\right] = p u^{p-1} u_{xx}\ + p(p-1)u^{p-2}(u_x)^2$$

Similarly, $\frac{\partial^2}{\partial y^2}(u^p) =\ p u^{p-1} u_{yy}\ + p(p-1)u^{p-2}(u_y)^2$

$$(1) \Rightarrow\ \nabla^2(u^p) =\ p u^{p-1}(u_{xx}\ + u_{yy}) + p(p-1)u^{p-2}[u_x^2 + u_y^2]$$

$$=\ p u^{p-1}(0)\ + p(p-1)u^{p-2}|f'(z)|^2$$

$$[\because u_{xx} + u_{yy} = 0, f(z) = u + iv, f'(z)\ = u_x + i v_x,\ |f'(z)|^2 = u_x^2 + u_y^2$$

$$\therefore\ \ \nabla^2(u^p) = p(p-1)\, u^{p-2}|f'(z)|^2$$

Theorem: 4 If f(z) $= u + iv$ is a regular function of z, then $\nabla^2|f(z)|^p = p^2|f(z)|^{p-2}|f'(z)|^2$.

[A.U N/D 2015 R-13]

Solution:

Let $f(z) = u + iv$

$$|f(z)| = \sqrt{u^2 + v^2} \qquad \ldots (a)$$

$$|f(z)|^p = (u^2 + v^2)^{p/2} \qquad \ldots (b)$$

$$\nabla^2 |f(z)|^p = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)(u^2 + v^2)^{p/2}$$

$$= \frac{\partial^2}{\partial x^2}(u^2 + v^2)^{p/2} + \frac{\partial^2}{\partial y^2}(u^2 + v^2)^{p/2}$$

$$\frac{\partial^2}{\partial x^2}(u^2 + v^2)^{p/2} = \frac{\partial}{\partial x}\left[\frac{p}{2}(u^2 + v^2)^{\frac{p}{2}-1}\left[2u\frac{\partial u}{\partial x} + 2v\frac{\partial v}{\partial x}\right]\right]$$

$$= p(u^2 + v^2)^{\frac{p}{2}-1}[uu_{xx} + u_x u_x + vv_{xx} + v_x v_x]$$

$$+p\left(\frac{p}{2} - 1\right)(u^2 + v^2)^{\frac{p}{2}-2}(uu_x + vv_x)(2uu_x + 2vv_x)$$

$$= p(u^2 + v^2)^{\frac{p}{2}-1}[uu_{xx} + u_x^2 + vv_{xx} + v_x^2]$$

$$+2p\left(\frac{p}{2} - 1\right)(u^2 + v^2)^{\frac{p}{2}-2}(uu_x + vv_x)^2$$

Similarly, $\frac{\partial^2}{\partial y^2}(u^2 + v^2)^{p/2} = p(u^2 + v^2)^{\frac{p}{2}-1}[uu_{yy} + u_y^2 + vv_{yy} + v_y^2]$

$$+2p\left(\frac{p}{2} - 1\right)(u^2 + v^2)^{\frac{p}{2}-2}(uu_y + vv_y)^2$$

$\Rightarrow \ \nabla^2|f(z)|^p = p(u^2+v^2)^{\frac{p}{2}-1}\left[u(u_{xx}+u_{yy}) + v(v_{vx}+v_{yy}) + u_x^2 + u_y^2 + v_x^2 + v_y^2\right] +$

$2p\left(\frac{p}{2}-1\right)(u^2+v^2)^{\frac{p}{2}-2}\left[u^2u_x^2 + v^2v_x^2 + 2uv\, u_xv_x + u^2u_y^2 + v^2v_y^2 + 2uv\, u_yv_y\right]$

$= p(u^2+v^2)^{\frac{p}{2}-1}\left[u(0)+v(0)+2(u_x^2+u_y^2)\right]$

$+2p\left(\frac{p}{2}-1\right)(u^2+v^2)^{\frac{p}{2}-2}\left[u^2(u_x^2+u_y^2)+v^2(v_x^2+v_y^2)+2uv(u_xv_x+u_yv_y)\right]$

$= 2p(u^2+v^2)^{\frac{p}{2}-1}|f'(z)|^2 + 2p\left(\frac{p}{2}-1\right)(u^2+v^2)^{\frac{p}{2}-2}[u^2|f'(z)|^2+v^2|f'(z)|^2+2uv(0)]$

$= 2p(u^2+v^2)^{\frac{p}{2}-1}|f'(z)|^2 + 2p\left(\frac{p}{2}-1\right)(u^2+v^2)^{\frac{p}{2}-2}(u^2+v^2)|f'(z)|^2$

$= 2p(u^2+v^2)^{\frac{p}{2}-1}|f'(z)|^2 + 2p\left(\frac{p}{2}-1\right)(u^2+v^2)^{\frac{p}{2}-1}|f'(z)|^2$

$= 2p(u^2+v^2)^{\frac{p}{2}-1}|f'(z)|^2\left[1+\frac{p}{2}-1\right]$

$= 2p(u^2+v^2)^{\frac{p}{2}-1}|f'(z)|^2 \ = p^2(u^2+v^2)^{\frac{p-2}{2}}|f'(z)|^2$

$= p^2(\sqrt{u^2+v^2})^{p-2}|f'(z)|^2$

$= p^2|f(z)|^{p-2}|f'(z)|^2$ by (a) & (b)

Theorem: 5 If $f(z) = u + iv$ is a regular function of z, in a domain D, then

$\left[\frac{\partial}{\partial x}|f(z)|\right]^2+\left[\frac{\partial}{\partial y}|f(z)|\right]^2=|f'(z)|^2$ **[A.U A/M 2015 R8]**

Solution:

Given $f(z)=u+iv$

$$|f(z)|=\sqrt{u^2+v^2}$$

$$\frac{\partial}{\partial x}|f(z)|=\frac{\partial}{\partial x}\left[\sqrt{u^2+v^2}\right]$$

$$=\frac{1}{2\sqrt{u^2+v^2}}[2uu_x+2vv_x]=\frac{uu_x+vv_x}{\sqrt{u^2+v^2}}$$

$$\left[\frac{\partial}{\partial x}|f(z)|\right]^2=\frac{(uu_x+vv_x)^2}{u^2+v^2}=\frac{u^2u_x^2+v^2v_x^2+2uv\,u_xv_x}{u^2+v^2}$$

Similarly, $\left[\frac{\partial}{\partial y}|f(z)|\right]^2=\frac{u^2u_y^2+v^2v_y^2+2uv\,u_yv_y}{u^2+v^2}$

$$\left[\frac{\partial}{\partial x}|f(z)|\right]^2+\left[\frac{\partial}{\partial y}|f(z)|\right]^2=\frac{u^2[u_x^2+u_y^2]+v^2[v_x^2+v_y^2]+2uv\,[u_xv_x+u_yv_y]}{u^2+v^2}$$

$$=\frac{u^2|f'(z)|^2+v^2|f'(z)|^2+2u\,v\,(0)}{u^2+v^2}\quad[\because u_x=v_y;\ u_y=-v_x]$$

$$=\frac{(u^2+v^2)|f(z)|^2}{u^2+v^2}=|f'(z)|^2\quad[\because u_xv_x+u_yv_y=0]$$

Theorem: 6 If $f(z)=u+iv$ is a regular function of z, then $\nabla^2|\text{Re } f(z)|^2=2|f'(z)|^2$

Solution:

Let $f(z) = u + iv$

$\text{Re}f(z) = u$

$|\text{Re}\, f'(z)|^2 = u^2$

$\nabla^2 |\text{Re}\, f'(z)|^2 = \nabla^2 u^2$

$$= \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)(u^2)$$

$$= \left(\frac{\partial^2}{\partial x^2}\right)(u^2) + \left(\frac{\partial^2}{\partial y^2}\right)(u^2)$$

$$= 2[u_x^2 + u_y^2]$$

$$= 2\,|f'(z)|^2$$

Theorem: 7 If $f(z) = u + iv$ is a regular function of z, then prove that $\nabla^2 |\text{Im}\, f(z)|^2 = 2|f'(z)|^2$

Proof:

Let $f(z) = u + iv$

$\text{I m}f(z) = v$

$|Im\, f(z)|^2 = v^2$

$$\frac{\partial}{\partial x}(v^2) = 2vv_x$$

$$\frac{\partial^2}{\partial x^2}(v^2) = 2[vv_{xx} + v_x v_x] = 2[vv_{xx} + v_x^2]$$

Similarly, $\frac{\partial^2}{\partial y^2}(v^2) = 2[vv_{yy} + v_y^2]$

$$\therefore \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)|Im\, f(z)|^2 = 2[v(v_{xx} + v_{yy}) + v_x^2 + v_y^2]$$

$$= 2[v(0) + u_x^2 + v_x^2] \quad \text{by C−R equation}$$

$$= 2|f'(z)|^2$$

Theorem: 8 Show that $\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = 4\frac{\partial^2}{\partial z \partial \bar{z}}$ (or) S T $\nabla^2 = 4\frac{\partial^2}{\partial z \partial \bar{z}}$

Proof:

Let x & y are functions of z and $\bar{z}$

that is $x = \frac{z+\bar{z}}{2}, y = \frac{z-\bar{z}}{2i}$

$$\frac{\partial}{\partial z} = \frac{\partial}{\partial x}\frac{\partial x}{\partial z} + \frac{\partial}{\partial y}\frac{\partial y}{\partial z}$$

$$= \frac{\partial}{\partial x}\left(\frac{1}{2}\right) + \frac{\partial}{\partial y}\left[\frac{1}{2i}\right] = \frac{1}{2}\left[\frac{\partial}{\partial x} + \frac{1}{i}\frac{\partial}{\partial y}\right]$$

$$2\frac{\partial}{\partial z} = \frac{\partial}{\partial x} + \frac{1}{i}\frac{\partial}{\partial y} \qquad \ldots(1)$$

$$\frac{\partial}{\partial \bar{z}} = \frac{\partial}{\partial x}\frac{\partial x}{\partial z} + \frac{\partial}{\partial y}\frac{\partial y}{\partial \bar{z}}$$

$$= \frac{\partial}{\partial x}\left(\frac{1}{2}\right) + \frac{\partial}{\partial y}\left[\frac{-1}{2i}\right] = \frac{1}{2}\left[\frac{\partial}{\partial x} - \frac{1}{i}\frac{\partial}{\partial y}\right]$$

$$2\frac{\partial}{\partial \bar{z}} = \left(\frac{\partial}{\partial x} - \frac{1}{i}\frac{\partial}{\partial y}\right) \qquad \ldots(2)$$

$$\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = \left(\frac{\partial}{\partial x} + \frac{1}{i}\frac{\partial}{\partial y}\right)\left(\frac{\partial}{\partial x} - \frac{1}{i}\frac{\partial}{\partial y}\right) [\because (a+b)(a-b) = a^2 - b^2]$$

$$= \left(2\frac{\partial}{\partial z}\right)\left(2\frac{\partial}{\partial \bar{z}}\right) \text{ by (1) \& (2)}$$

$$= 4\frac{\partial^2}{\partial z \partial \bar{z}}$$

Theorem: 9 If $f(z)$ is analytic, show that $\nabla^2 |f(z)|^2 = 4|f'(z)|^2$

Solution:

We know that, $\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}= 4\frac{\partial^2}{\partial z\partial \bar{z}}$

$$|f(z)|^2 = f(z)\overline{f(z)}$$

$$\nabla^2|\text{f(z)}|^2 = 4\frac{\partial}{\partial z}\frac{\partial}{\partial \bar{z}}[f(z)\overline{f(z)}]$$

$$= 4\left[\frac{\partial}{\partial z}f(z)\right]\left[\frac{\partial}{\partial \bar{z}}\overline{f(z)}\right]$$

[$\because f(z)$is independent of $\bar{z}$and $\overline{f(z)}$ is independent of z]

$$\therefore\ \nabla^2|f(z)|^2 = 4[f'(z)\left[\overline{\frac{\partial}{\partial z}f(z)}\right] = 4f'(z)\overline{f'(z)}$$

$$= 4|f'(z)|^2 \quad [\because z\bar{z} = |(z)|^2]$$

Example: 3.20 Give an example such that u and v are harmonic but $u+iv$ is not analytic. [A.U. N/D 2005]

Solution:

$u = x^2 - y^2,\ \ v = \frac{-y}{x^2+y^2}$

Example: 3.21 Find the value of m if $u = 2x^2 - my^2 + 3x$ is harmonic. [A.U N/D 2016 R-13]

Solution:

Given $u = 2x^2 - my^2 + 3x$

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2} = 0 \text{ [} \because \text{ u is harmonic]} \qquad \ldots(1)$$

$$\frac{\partial u}{\partial x} = 4x + 3 \qquad \frac{\partial u}{\partial y} = -2my$$

$$\frac{\partial^2 u}{\partial x^2} = 4 \qquad \frac{\partial^2 u}{\partial y^2} = -2m$$

$$\therefore (1) \Rightarrow (4) + (-2m) = 0$$

$$\Rightarrow m = 2$$