4.2 STATE MODELS FOR LINEAR AND TIME INVARIANT SYSTEMS

State model is given by state and output equation

State equation:

$$\dot{X}(t) = AX(t) + BU(t)$$

Output equation:

$$Y(t) = CX(t) + DU(t)$$

where,

A is state matrix of size (n x n)

B is the input matrix of size (n x m)

C is the output matrix of size (p x n)

D is the direct transmission matrix of size (p x m)

X(t) is the state vector of size (n x 1)

Y(t) is the output vector of size (p x 1)

U(t) is the input vector of size (m x 1)

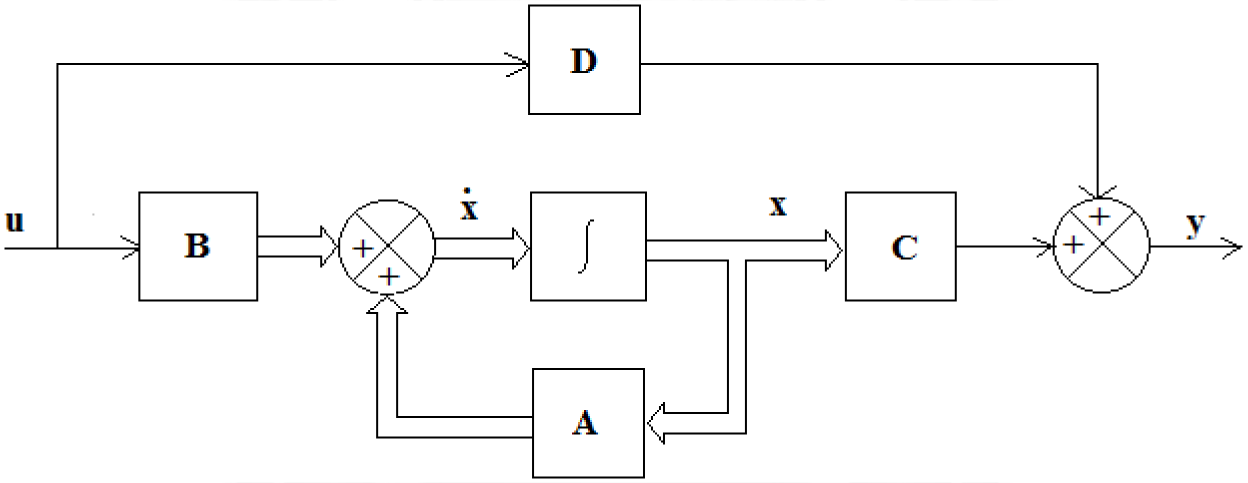

Figure 4.2.1 State space model diagram

[Source: "Modern Control Engineering" by Katsuhiko Ogata, Page: 828]

STATE SPACE REPRESENTATION USING PHYSICAL VARIABLES

In state-space modelling of systems, the choice of state variables is arbitrary. One of the possible choices of state variables. The physical variables of electrical systems are current or voltage in the R, L and C elements. The physical variables of mechanical systems are displacement, velocity and acceleration. The advantages of choosing the physical variables (or quantities) of the system as state variables are the following,

1. *The state variables can be utilized for the purpose of feedback*
2. *The implementation of design with state variable feedback becomes straight forward*
3. *The solution of state equation gives time variation of variables which have direct relevance to the physical system.*

The drawback in choosing the physical quantities as state variables is that the solution of state equation may be a difficult task. In state space modelling using physical variables, the state equations are obtained from a basic model of the system which is developed using the fundamental elements of the system.

Electrical System

The basic model of an electrical system can be obtained by using the fundamental elements Resistor, Capacitor and Inductor. Using these elements, the electrical network or equivalent circuit of the system is drawn. Then the differential equations governing the electrical systems can be formed by writing Kirchhoff's Current Law equations by choosing various nodes in the network or Kirchhoff's Voltage Law by choosing various closed path in the network. A minimal number of state variables are chosen for obtaining the state model of the system. The best choice of state variables in electrical system are currents and voltages in energy storage elements. The energy storage elements are inductance and capacitance. The physical variables in the differential equations are replaced by state variables and the equations are rearranged as first order differential equations. These set of first order equations constitutes the state equation of the system. The inputs to the system are voltage sources or current sources. The outputs in electrical system are usually voltages or currents in energy dissipating elements. The resistance is energy dissipating element in electrical network. In general, the output variables can be any voltage or current in the network.

Mechanical Translational System

The basic model of mechanical translational system can be obtained by using three basic elements; mass, spring and dash-pot. When a force is applied to a mechanical translational system, it is opposed by opposing forces due to mass, friction and elasticity of the system. The forces acting on a body are governed by Newton's second law of motion. The differential equations governing the system are obtaining by writing force balance equations at various nodes in the system. A node is a meeting points of elements.

Guidelines to form the state model of mechanical translational systems

1. For each node in the system one differential equation can be framed by equating the sum of applied forces to the sum of opposing forces. Generally, the nodes are mass elements of the system, but in some cases the nodes may be without mass element.
2. Assign a displacement to each node and draw a free body diagram for each node. The free body diagram is obtained by drawing each mass of node separately and them marking all the forces acting on it.
3. In the free body diagram, the opposing forces due to mass, spring and dash –pot are always act in a direction opposite to applied force. The displacement, velocity and acceleration will be in the direction of applied force or in the direction opposite to that of opposing force.
4. For each free body diagram write one differential equation by equating the sum of applied forces to the sum of opposing forces.
5. Choose a minimum number of state variables. The choice of state variables are displacement, velocity or acceleration.
6. The physical variables in differential equations are replaced by state variables and the equations are rearranged as first order differential equations. These set of first order equations constitute the state equation of the system.
7. The inputs are the applied forces and the outputs are the displacement, velocity or acceleration of the desired nodes.

Mechanical Rotational System

The basic model of mechanical rotational system can be obtained by using three basic elements moment of inertia of mass, rotational dash-pot and rotational spring. When a torque is applied to a mechanical rotational system, it is opposed by opposing torques due to moment of inertia, friction and elasticity of the system. The torque acting on a body are governed by Newton's second law of motion. The differential equations governing the system are obtained by writing torque balance equations at various nodes in the system. A node is a meeting point of elements.

Guidelines to form the state model of mechanical rotational systems

1. For each node in the system one differential equation can be framed by equating the sum of applied torques to the sum of opposing torques. Generally, the nodes are mass elements of the system, but in some cases the nodes may be without mass element.
2. Assign an angular displacement to each node and draw a free body diagram for each node. The free body diagram is obtained by drawing each mass of node separately and them marking all the torques acting on it.
3. In the free body diagram, the opposing torques due to mass of inertia, spring and dashpot are always act in a direction opposite to applied force. The angular displacement, velocity and acceleration will be in the direction of applied torque or in the direction opposite to that of opposing torque.
4. For each free body diagram write one differential equation by equating the sum of applied torques to the sum of opposing torques.
5. Choose a minimum number of state variables. The choice of state variables are angular displacement, velocity or acceleration.
6. The physical variables in differential equations are replaced by state variables and the equations are rearranged as first order differential equations. These set of first order equations constitute the state equation of the system.
7. The inputs are the applied torques and the outputs are the angular displacement, velocity or acceleration of the desired nodes.

STATE SPACE REPRESENTATION USING PHASE VARIABLES

The phase variables are defined as those particular state variables which are obtained from one of the system variables and its derivatives. There are three methods of modelling a system using phase variables. They are,

METHOD 1

Consider the following n[th] order linear differential equation relating the output y(t) to the input u(t) of a system,

$$\dot{y}^n + a_1\dot{y}^{n-1} + a_2\dot{y}^{n-2} + \cdots + a_{n-2}\ddot{y} + a_{n-1}\dot{y} + a_n y = bu$$

By choosing the output, y and their derivatives as state variables, we get,

$$x_1 = y$$
$$x_2 = \dot{y}$$
$$x_3 = \ddot{y}$$
$$\vdots$$
$$x_n = \dot{y}^{n-1}$$
$$\dot{x_n} = \dot{y}^n$$
$$\dot{x_n} + a_1 x_n + a_2 x_{n-1} + \cdots + a_{n-2}x_3 + a_{n-1}x_2 + a_n x_1 = bu$$
$$\dot{x_n} = -a_1 x_n - a_2 x_{n-1} - \cdots - a_{n-2}x_3 - a_{n-1}x_2 - a_n x_1 + bu$$

The state equations of the system are

$$\dot{x_1} = x_2$$
$$\dot{x_2} = x_3$$
$$\dot{x_3} = x_4$$
$$\vdots$$
$$\dot{x}_{n-1} = x_n$$
$$\dot{x_n} = -a_1 x_n - a_2 x_{n-1} - \cdots - a_{n-2}x_3 - a_{n-1}x_2 - a_n x_1 + bu$$

On arranging the above equations in the matrix form, we get,

$$\begin{bmatrix} \dot{x_1} \\ \dot{x_2} \\ \dot{x_3} \\ \vdots \\ \dot{x}_{n-1} \\ \dot{x_n} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 & \vdots & 0 \\ 0 & 0 & 1 & 0 & \vdots & 0 \\ 0 & 0 & 0 & 1 & \vdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \vdots & 1 \\ -a_n & -a_{n-1} & -a_{n-2} & -a_{n-3} & \vdots & -a_1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ b \end{bmatrix} u$$

$$\dot{X} = AX + BU$$

This form of matrix A is known as ***Bush form (or) Companion form***.

$$y = [1 \quad 0 \quad 0 \quad \cdots \quad 0 \quad 0]\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix}$$

$$Y = CX$$

METHOD 2

Consider the following nth order linear differential equation relating the output y(t) to the input u(t) of a system,

$$\dot{y}^n + a_1\dot{y}^{n-1} + a_2\dot{y}^{n-2} + \cdots + a_{n-2}\ddot{y} + a_{n-1}\dot{y} + a_n y = bu$$

Let n = m = 3

$$\dddot{y} + a_1\ddot{y} + a_2\dot{y} + a_3 y = b_0\dddot{u} + b_1\ddot{u} + b_2\dot{u} + b_3 u$$

On taking Laplace transform with zero initial conditions, we get,

$$s^3Y(s) + a_1s^2Y(s) + a_2sY(s) + a_3Y(s)$$
$$= b_0s^3U(s) + b_1s^2U(s) + b_2sU(s) + b_3U(s)$$

$$[s^3 + a_1s^2 + a_2s + a_3]Y(s) = [b_0s^3 + b_1s^2 + b_2s + b_3]U(s)$$

$$\frac{Y(s)}{U(s)} = \frac{[b_0s^3 + b_1s^2 + b_2s + b_3]}{[s^3 + a_1s^2 + a_2s + a_3]} = \frac{s^3[b_0 + \frac{b_1}{s} + \frac{b_2}{s^2} + \frac{b_3}{s^3}]}{s^3[1 + \frac{a_1}{s} + \frac{a_2}{s^2} + \frac{a_3}{s^3}]} = \frac{[b_0 + \frac{b_1}{s} + \frac{b_2}{s^2} + \frac{b_3}{s^3}]}{1 - [-\frac{a_1}{s} - \frac{a_2}{s^2} - \frac{a_3}{s^3}]}$$

From Mason's gain formula, the transfer function of the system is given by,

$$T(s) = \frac{1}{\Delta}\sum_K P_K\,\Delta_K$$

where, P_K – path gain of Kth forward path

Δ = 1 – (sum of loop gain of all individual loops) + (sum of gain products of all possible combinations of two non-touching loops) – ……..

Δ_K = Δ for that part of the graph which is not touching Kth forward path

The transfer function of the system with four forward paths and three feedback loops (touching each other) is given by,

$$T(s) = \frac{P_1 + P_2 + P_3 + P_4}{1 - (P_{11} + P_{12} + P_{13})}$$

By comparing the above equations,

$$P_1 = b_0;\ P_2 = \frac{b_1}{s};\ P_3 = \frac{b_2}{s^2};\ P_4 = \frac{b_3}{s^3};\ P_{11} = -\frac{a_1}{s};\ P_{12} = -\frac{a_2}{s^2};\ P_{13} = -\frac{a_3}{s^3}$$

On arranging the above equations in the matrix form, we get,

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \vdots \\ \dot{x}_{n-1} \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} -a_1 & 1 & 0 & 0 & \vdots & 0 \\ -a_2 & 0 & 1 & 0 & \vdots & 0 \\ -a_3 & 0 & 0 & 1 & \vdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ -a_{n-1} & 0 & 0 & 0 & \vdots & 1 \\ -a_n & 0 & 0 & 0 & \vdots & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix} + \begin{bmatrix} b_1 - a_1 b_0 \\ b_2 - a_2 b_0 \\ b_3 - a_3 b_0 \\ \vdots \\ b_{n-1} - a_{n-1} b_0 \\ b_n - a_n b_0 \end{bmatrix} u$$

$$\dot{X} = AX + BU$$

$$y = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix} + b_0 u$$

$$Y = CX + DU$$

METHOD 3

Consider the following n[th] order linear differential equation relating the output y(t) to the input u(t) of a system,

$$\dot{y}^n + a_1 \dot{y}^{n-1} + a_2 \dot{y}^{n-2} + \cdots + a_{n-2} \ddot{y} + a_{n-1} \dot{y} + a_n y = bu$$

Let n = m = 3

$$\dddot{y} + a_1 \ddot{y} + a_2 \dot{y} + a_3 y = b_0 \dddot{u} + b_1 \ddot{u} + b_2 \dot{u} + b_3 u$$

On taking Laplace transform with zero initial conditions, we get,

$$s^3 Y(s) + a_1 s^2 Y(s) + a_2 s Y(s) + a_3 Y(s)$$
$$= b_0 s^3 U(s) + b_1 s^2 U(s) + b_2 s U(s) + b_3 U(s)$$

$$[s^3 + a_1 s^2 + a_2 s + a_3] Y(s) = [b_0 s^3 + b_1 s^2 + b_2 s + b_3] U(s)$$

$$\frac{Y(s)}{U(s)} = \frac{[b_0 s^3 + b_1 s^2 + b_2 s + b_3]}{[s^3 + a_1 s^2 + a_2 s + a_3]}$$

Let,

$$\frac{Y(s)}{U(s)} = \frac{Y(s)}{X_1(s)} . \frac{X_1(s)}{U(s)}$$

$$\frac{X_1(s)}{U(s)} = \frac{1}{[s^3 + a_1 s^2 + a_2 s + a_3]}$$

$$\frac{Y(s)}{X_1(s)} = [b_0 s^3 + b_1 s^2 + b_2 s + b_3]$$

State Equation

On cross multiplying the equation, we get,

$$X_1(s)[s^3 + a_1 s^2 + a_2 s + a_3] = U(s)$$

$$s^3 X_1(s) + a_1 s^2 X_1(s) + a_2 s X_1(s) + a_3 X_1(s) = U(s)$$

$$\dddot{x}_1 + a_1 \ddot{x}_1 + a_2 \dot{x}_1 + a_3 x_1 = u$$

Let the state variable be x_1, x_2, x_3.

$$x_2 = \dot{x}_1$$

$$x_3 = \ddot{x}_1 = \dot{x}_2$$

$$\dot{x}_3 = \dddot{x}_1$$

On substituting the state variables, we get,

$$\dot{x}_3 + a_1 x_3 + a_2 x_2 + a_3 x_1 = u$$

The state equations are

$$\dot{x}_1 = x_2 \; ; \dot{x}_2 = x_3$$

$$\dot{x}_3 = -a_1 x_3 - a_2 x_2 - a_3 x_1 + u$$

Output Equation

On cross multiplying the equation, we get,

$$Y(s) = [b_0 s^3 X_1(s) + b_1 s^2 X_1(s) + b_2 s X_1(s) + b_3 X_1(s)]$$

Taking inverse Laplace transform, we get,

$$y = b_0 \dddot{x}_1 + b_1 \ddot{x}_1 + b_2 \dot{x}_1 + b_3 x_1$$

On substituting the state variables, we get,

$$y = b_0 \dot{x}_3 + b_1 x_3 + b_2 x_2 + b_3 x_1$$

Substituting $\dot{x}_3 = -a_1 x_3 - a_2 x_2 - a_3 x_1 + u$, we get,

$$y = b_0(-a_1 x_3 - a_2 x_2 - a_3 x_1 + u) + b_1 x_3 + b_2 x_2 + b_3 x_1$$

$$y = (b_3 - a_3 b_0) x_1 + (b_2 - a_2 b_0) x_2 + (b_1 - a_1 b_0) x_3 + b_0 u$$

Framing the state and output equation in matrix form, we get,

$$\begin{bmatrix} \dot{x_1} \\ \dot{x_2} \\ \dot{x_3} \\ \vdots \\ \dot{x}_{n-1} \\ \dot{x_n} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 & \vdots & 0 \\ 0 & 0 & 1 & 0 & \vdots & 0 \\ 0 & 0 & 0 & 1 & \vdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \vdots & 1 \\ -a_n & -a_{n-1} & -a_{n-2} & -a_{n-3} & \vdots & -a_1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} u$$

$$\dot{X} = AX + BU$$

This form of matrix A is known as ***Bush form (or) Companion form***.

$$y = [b_n - a_n b_0 \quad b_{n-1} - a_{n-1} b_0 \quad \cdots \quad \cdots \quad b_2 - a_2 b_0 \quad b_1 - a_1 b_0] \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix} + b_0 u$$

$$Y = CX$$

STATE SPACE REPRESENTATION USING CANONICAL VARIABLES

In canonical form (or normal form) of state model, the system matrix A will be a diagonal matrix. The elements on the diagonal are the poles of the transfer function of the system. By partial fraction expansion, the transfer function Y(s)/U(s) of the nth order system can be expressed as,

$$\frac{Y(s)}{U(s)} = b_0 + \frac{C_1}{s+\lambda_1} + \frac{C_2}{s+\lambda_2} + \cdots + \frac{C_n}{s+\lambda_n}$$

where, C_1, C_2, C_3, ….., C_n are residues and λ_1, λ_2, λ_3, ….., λ_n are roots of denominator polynomial (or poles of the system).

$$\frac{Y(s)}{U(s)} = b_0 + \frac{C_1/s}{1+\lambda_1/s} + \frac{C_2/s}{1+\lambda_2/s} + \cdots + \frac{C_n/s}{1+\lambda_n/s}$$

$$Y(s) = b_0 U(s) + \frac{C_1/s}{1+\lambda_1/s} U(s) + \frac{C_2/s}{1+\lambda_2/s} U(s) + \cdots + \frac{C_n/s}{1+\lambda_n/s} U(s)$$

The state equation can be framed as,

$$\dot{x}_1 = -\lambda_1 x_1 + u$$

$$\dot{x}_2 = -\lambda_2 x_2 + u$$

$$\vdots$$

$$\dot{x}_n = -\lambda_n x_n + u$$

The output equation can be framed as,

$$y = C_1 x_1 + C_2 x_2 + \cdots + C_n x_n + b_0 u$$

The canonical form of state model in the matrix form is given by,

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \vdots \\ \dot{x}_{n-1} \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} -\lambda_1 & 0 & 0 & 0 & \vdots & 0 \\ 0 & -\lambda_2 & 0 & 0 & \vdots & 0 \\ 0 & 0 & -\lambda_3 & 0 & \vdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \vdots & 0 \\ 0 & 0 & 0 & 0 & \vdots & -\lambda_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ 1 \\ \vdots \\ 1 \\ 1 \end{bmatrix} u$$

$$y = [C_1 \quad C_2 \quad C_3 \quad \cdots \quad C_{n-1} \quad C_n] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix} + b_0 u$$

JORDAN CANONICAL FORM

$$A = J = \begin{bmatrix} -\lambda_1 & 0 & 0 & 0 & \vdots & 0 \\ 0 & -\lambda_1 & 0 & 0 & \vdots & 0 \\ 0 & 0 & -\lambda_1 & 0 & \vdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \vdots & 0 \\ 0 & 0 & 0 & 0 & \vdots & -\lambda_n \end{bmatrix}$$

$$B = \begin{bmatrix} 0 \\ 0 \\ 1 \\ \vdots \\ 1 \\ 1 \end{bmatrix}$$

$$\dot{Z} = JZ + \tilde{B}U$$

$$Y = \tilde{C}Z + DU$$

where,

$$J = M^{-1}AM; \quad \tilde{B} = M^{-1}B; \quad \tilde{C} = CM$$