Binomial Distribution

Let us consider "n" independent trails. If the successes (S) and failures (F) are recorded successively as the trials are repeated we get a result of the type

S S F F S . . . F S

Let "x" be the number of success and hence we have $(n - x)$ number of failures.

$$P(S\,S\,F\,F\,S\ldots F\,S) = P(S)\,P(S)\,P(F)\,P(F)\,P(S)\ldots P(F)\,P(S)$$

$$= p\,p\,q\,q\,p\ldots q\,p$$

$$= p\,p\ldots p \times q\,q\,q\ldots q$$

$$= x \text{ factor} \times (n-x) \text{ factors}$$

$$= p^x \cdot q^{n-x}$$

But "x" success in "n" trials can occur in nC_x ways.

Therefore the probability of "x" successes in "n" trials is given by

$$P(X = x\,successes) = nC_x p^x q^{n-x}, x = 0, 1, 2, \ldots, n$$

Where $p + q = 1$

Assumptions in Binomial Distribution:

(i) There are only two possible outcomes for each trial (success or failure)

(ii) The probability of a success is the same for each trail.

(iii) There are "n" trials where "n" is constant.

(iv) The "n" trails are independent.

Mean and variance of a Binomial Distribution:

(i) Mean(μ) = np

(ii) Variance(σ^2) = npq

The variance of a Binomial Variable is always less than its mean.

$\therefore npq < np.$

Find the moment generating function of binomial distribution and hence find the mean and variance.

Solution:

Binomial distribution is $p(x) = nC_x p^x q^{n-x},\ x = 0,1,2, \dots \dots, n$

To find Mean and Variance:

$M_X(t) \ = E(e^{tX}) = \sum_{x=0}^{n} e^{tx} P(x)$

$= \sum_{x=0}^{n} e^{tx} nC_x p^x q^{n-x}$

$= \sum_{x=0}^{n} nC_x (pe^t)^x q^{n-x}$ $\quad \because \sum_{x=0}^{n} nC_x a^x b^{n-x} = (a+b)^n$

$M_X(t) \ = (pe^t + q)^n$

Mean $E(X) = \left[\frac{d}{dt}[M_*(t)]\right]_{t=0}$

$$= \left[\frac{d}{dt}[(pe^t + q)^n]\right]_{t=0}$$

$$= [n(pe^t + q)^{n-1}(pe^t + 0)]_{t=0}$$

$$= np[p + q]^{n-1}$$

$$E(X) \ = np$$

$$E(X^2) = \left[\frac{d^2}{dt^2}[M_X(t)]\right]_{t=0}$$

$$= \left[\frac{d}{dt}[n(pe^t + q)^{n-1}(pe^t)]\right]_{t=0}$$

$$= np\left[\frac{d}{dt}\left[(pe^t + q)^{(n-1)}e^t\right]\right]_{t=0}$$

$$= np[(pe^t + q)^{n-1}e^t + e^t(n-1)(pe^t + q)^{n-2}pe^t]_{t=0}$$

$$= np[(p + q)^{n-1} + (n-1)(p + q)^{n-2}p]$$

$$= np[1 + (n-1)p]$$

$$= np[1 + np - p]$$

$$= np[1 - p + np]$$

$$= np[q + np]$$

$$= npq + n^2p^2$$

$E(X^2) = (np)^2 + npq$

Variance $= E(X^2) - [E(X)]^2$

$= (np)^2 + npq - (np)^2$

$= n^2p^2 - n^2p^2 + npq$

Variance$= n$pq

Problems based on Binomial Distribution:

$$\textbf{Mean} = \boldsymbol{np}$$

$$\textbf{Variance} = \boldsymbol{npq}$$

1. Criticize the following statements " The mean of a binomial distribution is 5 and the standard deviation is 3"

Solution:

Given mean = np = 5 . . . (1)

Standard deviation = $\sqrt{npq} = 3$

⇒ Variance = npq = 9 . . . (2)

$$\frac{(2)}{(1)} \Rightarrow \frac{npq}{np} = \frac{9}{5} = 1.8 > 1$$

Which is impossible. Hence, the given statement is wrong.

2. If $M_X(t) = \frac{(2e^t+1)^4}{81}$, then find Mean and Variance.

Solution:

Given $M_X(t) = \frac{(2e^t+1)^4}{81}$

$\Rightarrow M_X(t) = \left(\frac{2}{3}e^t + \frac{1}{3}\right)^4$

Comparing with MGF of Binomial Distribution, $M_X(t) = (pe^t + q)^n$, we get

$p = \frac{2}{3}$ and $= \frac{1}{3}$, n = 4

(i) Mean = $np \ = 4 \times \frac{2}{3} = \frac{8}{3}$

(ii) Variance = $npq \ = \frac{8}{3} \times \frac{1}{3} = \frac{8}{9}$

3. Six dice are thrown 729 times. How many times do you expect atleast 3 dice to show a five or six.

Solution:

Given $n = 6$ and $N = 729$

Probability of getting (5 or 6) $p = \frac{2}{6} = \frac{1}{3}$

and $q = 1 - \frac{1}{3} = \frac{2}{3}$

$P(X = x) = nC_x p^x q^{n-x}, x = 0, 1, 2, \dots, n$

$$= 6C_x \left(\frac{1}{3}\right)^x \left(\frac{2}{3}\right)^{6-x}, x = 0, 1, 2, \dots, 6$$

P(atleast 3 dice to show a five or six) $= P(X \geq 3) = 1 - P(X < 3)$

$$= 1 - [P(X = 0) + P(X = 1) + P(X = 2)]$$

$$= 1 - \left[6C_0\left(\frac{1}{3}\right)^0\left(\frac{2}{3}\right)^{6-0} + 6C_1\left(\frac{1}{3}\right)^1\left(\frac{2}{3}\right)^{6-1} + 6C_2\left(\frac{1}{3}\right)^2\left(\frac{2}{3}\right)^{6-2}\right]$$

$$= 1 - [0.0877 + 0.2634 + 0.3292]$$

$$= 1 - 0.6803$$

$= 0.3197$

Number of times expecting atleast 3 dice to show 5 or 6 $= 729 \times 0.3197$

$= 233$ times

4. A machine manufacturing screw is known to produce 5% defective. In a random sample of 15 screws, what is the probability that there are (i) exactly 3 defectives (ii) not more than 3 defectives.

Solution:

Given $n = 15$

$p = 5\% = 0.05$

$q = 1 - p = 1 - 0.05 = 0.95$

$P(X = x) = nC_x p^x q^{n-x}, x = 0, 1, 2, \ldots, n$

$= 15C_x (0.05)^x (0.95)^{15-x}, x = 0, 1, 2, \ldots, 15$

(i) P(exactly 3 defectives) $= P(X = 3)$

$= 15C_3 (0.05)^3 (0.95)^{15-3}$

$= 0.056(0.95)^{12}$

$= 0.0307$

(ii) P(no kore than 3 defectives) $= P(X \leq 3)$

$= P(X = 0) + P(X = 1) + P(X = 2) + P(X = 3)$

$= 15C_0 (0.05)^0 (0.95)^{15-0} + 15C_1 (0.05)^1 (0.95)^{15-1}$

$+ 15C_3 (0.05)^2 (0.95)^{15-2}$

$+ 15C_3 (0.05)^3 (0.95)^{15-3}$

$$= 15C_0(0.05)^0(0.95)^{15} + 15C_1(0.05)^1(0.95)^{14} + 15C_3(0.05)^2(0.95)^{13}$$

$$+15C_3(0.05)^3(0.95)^{12}$$

$$= 0.4633 + 0.3658 + 0.1348 + 0.0307$$

$$= 0.9946$$

Poisson Distribution

Poisson Distribution is a limiting case of Binomial Distribution under the following assumptions.

(i) The number of trails "n" should be independently large. i.e., $n \to \infty$

(ii) The probability of successes "p" for each trail is indefinitely small.

(iii) $np = \lambda$ should be finite where λ is a constant.

Application of Poisson Distribution:

Determining the number of calls received per minute at a call Centre or the number of unbaked cookies in a batch at a bakery, and much more.

Find the MGF for Poisson distribution and hence find the mean and variance.

Solution:

Poisson distribution is $p(x) = \frac{e^{-\lambda}\lambda^x}{x!}$ $x = 0,1,2, \dots \dots,$

$M_X(t) = E[e^{tx}]$

$= \sum_{x=0}^{\infty} e^{tx} p(x)$

$$= \sum_{x=0}^{\infty} e^{-x} e^{-\lambda} \frac{\lambda^x}{x!} = \sum_{x=0}^{\infty} \frac{(e^t)}{0}$$

$$= e^{-\lambda}\left[1 + \frac{\lambda e^t}{1!} + \frac{(\lambda e^t)^2}{2!} + \frac{(\lambda e^t)^3}{3!} + \cdots\right]$$

$$= e^{-\lambda} e^{\lambda e^t} = e^{-\lambda+\lambda\epsilon^t} \quad \because 1 + \frac{x}{1!} + \frac{x^2}{2!} + \cdots = e^x$$

$$M_X(t) = e^{\lambda(e^t-1)}$$

To find the mean and variance of :

$$\text{Mean } E(X) = \left[\frac{d}{dt}[M_X(t)]\right]_{t=0}$$

$$= \left[\frac{d}{dt}\left[e^{\lambda(e^t-1)}\right]\right]_{t=0}$$

$$= \left[e^{\lambda(e^t-1)}\lambda(e^t)\right]_{t=0}$$

$$= e^{\lambda(e^0-1)}\lambda e^0 = e^0\lambda$$

$$\text{Mean } = \lambda$$

$$E(X^2) = \left[\frac{d^2}{dt^2}[M_X(t)]\right]_{t=0}$$

$$= \left[\frac{d^2}{dt^2}\left[e^{\lambda(e^t-1)}\right]\right]_{t=0}$$

$$= \left[\frac{d}{dt}\left[e^{\lambda(e^t-1)}\lambda e^t\right]\right]_{t=0}$$

$$= \lambda\left[\frac{d}{dt}\left[e^{\lambda(e^t-1)+t}\right]\right]_{t=0}$$

$$= \lambda\left[e^{\lambda(e^t-1)+t}(\lambda e^t + 1)\right]_{t=0}$$

$$= \lambda[e^0(\lambda + 1)]$$

$$= \lambda(\lambda + 1)$$

$$E(X^2) = \lambda^2 + \lambda$$

$$\text{Variance } = E(X^2) - [E(X)]^2$$

$$= \lambda^2 + \lambda - \lambda^2$$

$$\text{Variance } = \lambda$$

Problems based on Poisson Distribution:

1. Write down the probability mass function of the Poisson distribution which is approximately equivalent to B(100, 0.02)

Solution:

Given $n = 1000, p = 0.02$

$$\lambda = np = 100 \times 0.02 = 2$$

The probability mass function of the Poisson distribution

$$P(x) = \frac{e^{-\lambda}\lambda^x}{x!}; x = 0, 1, 2, \ldots, \infty$$

$$= \frac{e^{-2}2^x}{x!}; x = 0, 1, 2, \ldots, \infty$$

2. One percent of jobs arriving at a computer system need to wait until weekends for scheduling, owing to core – size limitations. Find the probability that among a sample of 200 jobs there are no jobs hat have to wait until weekends.

Solution:

Given $n = 200$, p = 0.01

$$\lambda = np = 200 \times 0.01 = 2$$

The Poisson distribution is

$$P(x) = \frac{e^{-\lambda}\lambda^x}{x!}; x = 0, 1, 2, \ldots, \infty$$

P(no jobs to wait until weekends) = $P(X = 0)$

$$P(X = 0) = \frac{e^{-2}2^0}{0!} = e^{-2} = 0.1353$$

3. The proofs of a 500 pages book containing 500 misprints. Find the probability that there are atleast 4 misprints in a randomly chosen page.

Solution:

Given $n = 500$

p = P(getting a misprint in a given page) = $\frac{1}{500}$

$\lambda = np = 500 \times \frac{1}{500} = 1$

The Poisson distribution is

$P(x) = \frac{e^{-\lambda}\lambda^{x}}{x!}; x = 0, 1, 2, \ldots, \infty$

$P(X \geq 4) = 1 - P(X < 4)$

$= 1 - [P(X = 0) + P(X = 1)] + P(X = 2) + P(X = 3)$

$= 1 - \left[\frac{e^{-1}1^{0}}{0!} + \frac{e^{-1}1^{1}}{1!} + \frac{e^{-1}1^{2}}{2!} + \frac{e^{-1}1^{3}}{3!}\right]$

$= 1 - e^{-1}\left[1 + 1 + \frac{1}{2} + \frac{1}{6}\right]$

$= 1 - 0.3679[2.666]$

$= 1 - 0.9809$

$= 0.0192$