Joint probability distribution function for continuous random variables X &Y

The joint probability distribution function of a two dimension as random variables (X, Y) is denoted by $F_{XY}(x, y)$ and is given by

$f_{XY}(x, y) \geq 0$ and $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{XY}(x, y)\, dxdy = 1$

Properties of joint Distribution Functions

1. $F(-\infty, y) = 0 = F(x, \infty)$ and $F(-\infty, \infty) = 1$

2. $P(a_1 < X < b_1, a_2 < Y < b_2) = F(b_1, b_2) + F(a_1, a_2) - F(a_1, b_2) - F(b_1, a_2)$

Marginal probability density function

The marginal probability density function of the two random variables X and Y are defined as follows

$f(x) = f_X(x) = \int_{-\infty}^{\infty} f_{XY}(x, y)dy$ (Marginal pdf of X)

$f(y) = f_Y(y) = \int_{-\infty}^{\infty} f_{XY}(x, y)dx$ (Marginal pdf of Y)

Problems under Marginal Density function

1. The bivariate random variable X and Y has the pdf $f(x, y) = K\, x^2(8 - y), x < y < 2x, 0 \leq x \leq 2$. Find the value of K.

Solution:

We know that if $f(x, y)$ is a pdf then $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{XY}(x, y)\, dxdy = 1$

$$\Rightarrow \int_0^2 \int_x^2 K\, x^2(8 - y)\, dxdy = 1$$

$$\Rightarrow K \int_0^2 x^2 \left(8y - \frac{y^2}{2}\right)_x^{2x} dx = 1$$

$$\Rightarrow K \int_0^2 x^2 \left(16x - \frac{4x^2}{2} - 8x + \frac{x^2}{2}\right)_x^{2x} dx = 1$$

$$\Rightarrow K \int_0^2 \left(16x^3 - 2x^4 - 8x^3 + \frac{x^4}{2}\right)_x^{2x} dx = 1$$

$$\Rightarrow K \int_0^2 \left(8x^3 - \frac{3x^4}{2}\right)_x^{2x} dx = 1$$

$$\Rightarrow K \left[\frac{8x^4}{4} - \frac{3}{2}\frac{x^5}{5}\right]_0^2 = 1$$

$$\Rightarrow K \left[32 - \frac{48}{5}\right] = 1$$

$$\Rightarrow K \left(\frac{112}{5}\right) = 1 \Rightarrow K = \frac{5}{112}$$

2. The joint pdf of R.V X and Y is given by $f(x, y) = Kxye^{-(x^2+y^2)}, x > 0, y > 0$. Find the value of K and prove also that X and Y are independent.

Solution:

We know that if $f(x, y)$ is a p.d.f, then $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{XY}(x, y)\, dxdy = 1$

Here $f(x, y) = Kxye^{-(x^2+y^2)}, x > 0, y > 0$

We know that $\int_0^{\infty}\int_0^{\infty} Kxye^{-(x^2+y^2)}\, dydx = 1$

$$\Rightarrow \int_0^\infty \int_0^\infty Kxye^{-x^2} e^{-y^2} dydx = 1$$

$$\Rightarrow K \int_0^\infty ye^{-y^2} dy \int_0^\infty xe^{-x^2} dx = 1$$

$$\Rightarrow K \frac{1}{2}\frac{1}{2} = 1 \Rightarrow K = 4$$

To prove X and Y are independent we have to prove that $f(x).f(y) = f(x,y)$

Now, $f(x) = f_X(x) = \int_{-\infty}^{\infty} f(x,y)dy$

$$= \int_0^\infty Kxye^{-(x^2+y^2)} dy$$

$$= 4xe^{-x^2} \int_0^\infty ye^{-y^2} dy$$

$$= 4xe^{-x^2} \left(\frac{1}{2}\right)$$

$$= 2xe^{-x^2}, x > 0$$

Similarly $f(y) = f_Y(y) = 2ye^{-y^2}, y > 0$

Now, $f(x).f(y) = 2xe^{-x^2}\, 2ye^{-y^2}$

$$= 4xye^{-x^2} e^{-y^2}$$

$$= 4xye^{-(x^2+y^2)}$$

$$= f(x,y)$$

∴ X and Y are independent.

3. The joint probability density function of a two dimensional random variable (X,Y) is given by $f(x,y) = xy^2 + \frac{x^2}{8}, 0 \le x \le 2, 0 \le y \le 1$ Compute

(i) $P\left(X > 1/Y < \frac{1}{2}\right)$ (ii) $P\left(Y < \frac{1}{2}/X > 1\right)$ (iii) $P(X < Y)$ (iv) $P(X + Y \le 1)$

Solution:

Given $f(x,y) = xy^2 + \frac{x^2}{8}, 0 \le x \le 2, 0 \le y \le 1$

Now $P\left(X > 1/Y < \frac{1}{2}\right) = \frac{P\left(X>1, Y<\frac{1}{2}\right)}{P(Y<\frac{1}{2})}$

$$= \int_0^{\frac{1}{2}}\int_1^2 (xy^2 + \frac{x^2}{8})dxdy$$

$$= \int_0^{\frac{1}{2}} (\frac{x^2y^2}{2} + \frac{x^3}{24})_1^2 dy$$

$$= \int_0^{\frac{1}{2}} \left(2y^2 + \frac{1}{3}\right) - (\frac{y^2}{2} + \frac{1}{24}) \quad dy$$

$$= \int_0^{\frac{1}{2}} (2y^2 + \frac{1}{3} - \frac{y^2}{2} - \frac{1}{24}) \, dy$$

$$= \int_0^{\frac{1}{2}} (\frac{3y^2}{2} + \frac{7}{24}) \, dy = \frac{5}{24}$$

$\Rightarrow P\left(X > 1/Y < \frac{1}{2}\right) = \frac{5}{24}$

P(Y< $\frac{1}{2}$) = $\int_0^{\frac{1}{2}}\int_1^2 (xy^2 + \frac{x^2}{8})dxdy$

$$= \int_0^{\frac{1}{2}} (\frac{x^2y^2}{2} + \frac{x^3}{24})_0^2 dy$$

$$= \int_0^{\frac{1}{2}} \left(2y^2 + \frac{1}{3}\right) dy$$

$$= (\frac{2y^3}{3} + \frac{y}{3})_0^{\frac{1}{2}} = \frac{1}{4}$$

$$\therefore P\left(Y < \frac{1}{2}\right) = \frac{1}{4}$$

$$P(X > 1) = \int_0^1 \int_1^2 (xy^2 + \frac{x^2}{8}) dx dy$$

$$= \int_0^1 (\frac{x^2y^2}{2} + \frac{x^3}{24})_1^2 dy$$

$$= \int_0^1 \left(\frac{3y^2}{2} + \frac{7}{24}\right) dy$$

$$= \left[\frac{3}{2}\frac{y^3}{3} + \frac{7y}{24}\right]_0^1 = \frac{19}{24}$$

(i) $P\left(X > 1 \,/\, Y < \frac{1}{2}\right) = \frac{P(X>1, Y<\frac{1}{2})}{P(Y<\frac{1}{2})}$

$$= \frac{5}{24} \times 4 = \frac{5}{6}$$

(ii) $P\left(Y < \frac{1}{2} / X > 1\right) = \frac{P(Y<\frac{1}{2}, X>1)}{P(X>1)} = \frac{5}{24} \times \frac{24}{19} = \frac{5}{19}$

(iii) $P(X < Y) = \int_0^1 \int_0^y (xy^2 + \frac{x^2}{8}) dx dy$

$$= \int_0^1 \left(\frac{x^2y^2}{2} + \frac{x^3}{24}\right)_0^y dy$$

$$= \int_0^1 \left(\frac{y^4}{2} + \frac{y^3}{24}\right) dy$$

$$= \left[\frac{y^5}{10} + \frac{y^4}{96}\right]_0^1 = \frac{53}{480}$$

(iv) $P(X + Y \leq 1) = \int_0^1 \int_0^{1-y} (xy^2 + \frac{x^2}{8}) dxdy$

$$= \int_0^1 \left(\frac{x^2y^2}{2} + \frac{x^3}{24}\right)_0^{1-y} dy$$

$$= \int_0^1 \left(\frac{(1-y)^2y^2}{2} + \frac{(1-y)^3}{24}\right) dy$$

$$= \int_0^1 \left(\frac{(1-y)^2y^2}{2} + \frac{(1-y)^3}{24}\right) dy$$

$$= \int_0^1 \left(\frac{1}{2}(y^2 + y^4 - 2y^3) + \frac{1}{24}(1-y)^3\right) dy$$

$$= \frac{1}{2}\left(\frac{y^3}{3} + \frac{y^5}{5} - \frac{2y^4}{4}\right)_0^1 + \frac{1}{24}\left(\frac{(1-y)^4}{-4}\right)_0^1 = \frac{13}{480}$$

4. The joint density function of X and Y is $f(x, y) = \begin{cases} e^{-(x+y)} & 0 \leq x, y \leq \infty \\ 0 & otherwise \end{cases}$.

Are X and Y independent. Find (i) $P(X < 1)$ (ii) $P(X + Y < 1)$

Solution:

Given $f(x, y) = \begin{cases} e^{-(x+y)} & 0 \leq x, y \leq \infty \\ 0 & otherwise \end{cases}$

The Marginal pdf of X is $f_X(x) = f(x) = \int_{-\infty}^{\infty} f(x, y)dy$

$$= \int_0^{\infty} e^{-(x+y)}\, dy$$

$$= e^{-x} \int_0^{\infty} e^{-y}\, dy$$

$$= e^{-x}[-e^{-y}]_0^{\infty}$$

$$= e^{-x}[0 + 1] = e^{-x}$$

Marginal pdf of ‘x’ is $f(x) = e^{-x}$

The Marginal pdf of Y is $f_Y(y) = f(y) = \int_{-\infty}^{\infty} f(x, y)dx$

$$= \int_0^{\infty} e^{-(x+y)}\, dx$$

$$= e^{-y} \int_0^{\infty} e^{-x}\, dx$$

$$= e^{-y}[-e^{-x}]_0^{\infty}$$

$$= e^{-y}[0 + 1] = e^{-y}$$

Marginal pdf of ‘Y’ is $f(y) = e^{-y}$

Now $f(x).f(y) = e^{-x}.e^{-y} = e^{-(x+y)}$

But $f(x, y) = e^{-(x+y)}$

$$\therefore f(x, y) = f(x).f(y)$$

Hence X and Y ae independent.

(i) $P(\,X < 1\,) = \int_0^1 \int_0^\infty e^{-(x+y)} \; dy\, dx$

$= \int_0^1 e^{-x} \left[\frac{e^{-y}}{-1}\right]_0^\infty dx$

$= \int_0^1 e^{-x} \left[\frac{0-1}{-1}\right] dx$

$= \int_0^1 e^{-x}\, dx$

$= [-e^{-x}]_0^1$

$= -e^{-1} + e^0 = 1 - e^{-1}$

(ii) $P(X + Y < 1) = \int_0^1 \int_0^{1-x} e^{-(x+y)}\, dy\, dx$

$= \int_0^1 e^{-x} \left[\frac{e^{-y}}{-1}\right]_0^{1-x} dx$

$= -\int_0^1 e^{-x}\, [e^{x-1} - e^0] dx$

$= -\int_0^1 e^{-x} [e^{x-1} - e^0]\, dx$

$= -\int_0^1 [e^{-x+x-1} - e^{-x}\,] dx$

$= -[e^{-1}x + e^{-x}]_0^1$

$= -[e^{-1} + e^{-1} - 0 - e^0]$

$= -[2e^{-1} - 1] = 1 - 2e^{-1}$

5. If the joint pdf of X and Y is given by $f(x,y) =$

$\begin{cases} \frac{1}{8}(6-x-y) & 0<x<2, 2<y<4 \\ 0 & otherwise \end{cases}$ **Find the value of** $(i) P\ (X<1 \cap Y<3)\ (ii)\ P(\ X<1\ \ /Y<3)$

Solution:

We know that $(i)\ P\ (X<1\ \cap Y<3) = \int_0^1 \int_2^3 f(x,y)\ dy\, dx$

$$= \int_0^1 \int_2^3 \frac{1}{8}\ (6-x-y)dy\ dx$$

$$= \frac{1}{8}\int_0^1 [6y - xy - \frac{y^2}{2}]_2^3\ dx$$

$$= \frac{1}{8}\int_0^1 \{(\ 18 - 3x - \frac{9}{2}) - (\ 12 - 2x - \frac{4}{2})\}\ dx$$

$$= \frac{1}{8}\int_0^1 (\ 8 - x - \frac{9}{2})\ dx$$

$$= \frac{1}{8}\int_0^1 (\ \frac{7-2x}{2})\ dx$$

$$= \frac{1}{16}[7x - x^2]_0^1 = \frac{6}{16} = \frac{3}{8}$$

$\therefore P(X<1\ \cap Y<3\ ) = \frac{3}{8}$ ... (1)

(ii) $P(\ X<1/<3\ ) = \frac{P(X<1\ \cap Y<3)}{P(Y<3)}$ ... (2)

$$= \int_0^1 \int_2^3 \frac{1}{8}\ (6-x-y)dy\ dx$$

$$= \frac{1}{8}\int_0^2 [6y - xy - \frac{y^2}{2}]_2^3 \ dx$$

$$= \frac{1}{8}\int_0^2 \{(18 - 3x - \frac{9}{2}) - (12 - 2x - 2)\} \, dx$$

$$= \frac{1}{8}\int_0^2 (8 - x - \frac{9}{2}) \, dx$$

$$= \frac{1}{8}\int_0^2 \{(\frac{16-9}{2}) - x \} \, dx$$

$$= \frac{1}{8}[\int_0^2 (\frac{7}{2} - x) dx] = \frac{1}{8}[\frac{7}{2}x - \frac{x^2}{2}]_0^2 = \frac{5}{8}$$

$\therefore P(Y < 3) = \frac{5}{8}$ . . . (3)

Substituting (1) and (3) in (2) we get $P(X < 1/Y < 3) = \frac{\frac{3}{8}}{\frac{5}{8}} = \frac{3}{5}$

6. If the joint distribution function of X and Y is given by $f(x, y) =$

$$\begin{cases} (1 - e^{-x})(1 - e^{-y}) & for\ x > 0, y > 0 \\ 0 & otherwise \end{cases}$$

(i) Find the marginal densities of X and Y.

(ii) Are X and Y independent? (iii)P(1 < X < 3 , 1 < Y < 2)

Solution:

Given $f(x, y) = (1 - e^{-x})(1 - e^{-y})$

$$= 1 - e^{-x} - e^{-y} + e^{-(x+y)}$$

The joint pdf is given by $f(x,y) = \frac{\partial^2 F(x,y)}{\partial x \partial y}$

$$= \frac{\partial^2}{\partial x \partial y}[1 - e^{-x} - e^{-y} + e^{-(x+y)}]$$

$$= \frac{\partial}{\partial x}\left[e^{-y} + e^{-(x+y)}\right] = 0 + e^{-(x+y)}$$

$$\therefore f(x,y) = \begin{cases} (1-e^{-x})(1-e^{-y}) & for\ x > 0, y > 0 \\ 0 & otherwise \end{cases}$$

$$f(x,y) = e^{-(x+y)}$$

The marginal pdf of X is $f_{X(x)} = f(x) = \int_{-\infty}^{\infty} f(x,y)dy$

$$= \int_0^{\infty} e^{-(x+y)}dy$$

$$= [-e^{-(x+y)}]_0^{\infty}$$

$$= -e^{-\infty} + e^{-x} = e^{-x} \qquad \ldots (1)$$

The marginal pdf of Y is $f_{Y(y)} = f(y) = \int_{-\infty}^{\infty} f(x,y)dy$

$$= \int_0^{\infty} e^{-(x+y)}dx$$

$$= [-e^{-(x+y)}]_0^{\infty}$$

$$= -e^{-\infty} + e^{-y} = e^{-y} \qquad \ldots (2)$$

(ii) From (1) and (2) we get

$$f(x).f(y) = e^{-x}e^{-y} = e^{-(x+y)} = f(x,y)$$

Hence X and Y are independent.

(iii) $P(1 < X < 3, 1 < Y < 2) = P(1 < X < 3).P(1 < Y < 2)$

(Since X and Y are independent)

$$= \int_1^3 e^{-x}\,dx.\int_1^2 e^{-y}\,dy$$

$$= (-e^{-x})_1^3\ (-e^{-y})_1^2$$

$$= (-e^{-3} + e^{-1})(-e^{-2} + e^{-1})$$

$$= e^{-5} - e^{-4} - e^{-3} + e^{-2}$$