Exponential Distribution

The definition of exponential distribution is the probability distribution of the time between the events in a Poisson Process.

If you think about it, the amount of time until the event occurs means during the waiting period, not a single event has happened.

This is, in other words Poisson (X = 0).

$$\text{Poisson } P(X=k) = \frac{\lambda^k \cdot e^{-\lambda}}{k!}$$

$$\text{"} \quad P(X=0) = \frac{\lambda^0 \cdot e^{-\lambda}}{0!} = e^{-\lambda}$$

Why did we have to invent Exponential Distribution?

To predict the amount of waiting time until the next event (i.e., success, failure, arrival, etc.).

For example, we want to predict the following:

- The amount of time until the customer finishes browsing and actually purchases something in your store (success).
- The amount of time until the hardware on AWS EC2 fails (failure).

- The amount of time you need to wait until the bus arrives (arrival).

Relationship between a Poisson and an Exponential Distribution:

If the number of events per unit time follows a Poisson distribution, then the amount of time between events follows the exponential distribution.

Assuming that the time between events is not affected by the times between previous events (i.e., they are independent), then the number of events per unit time follows a Poisson distribution with the rate $\lambda = 1/\mu$.

Who else has Memoryless property?

The exponential distribution is the only continuous distribution that is memoryless (or with a constant failure rate). Geometric distribution, its discrete counterpart, is the only discrete distribution that is memory less.

Find the MGF of Exponential distribution and hence find Mean and variance.

Solution:

Let X follows the exponential distribution

By definition, $f(x) = \lambda e^{-\lambda x}; x \geq 0$

$$M_X(t) = E[e^{tX}] = \int_0^{\infty} e^{tx} f(x)dx$$

$$= \int_0^{\infty} e^{tx} \lambda e^{-\lambda x} dx$$

$$= \lambda \int_0^{\infty} e^{-(\lambda - t)x} dx$$

$$= \lambda \left[\frac{e^{-(\lambda-t)x}}{-(\lambda-t)}\right]_0^\infty$$

$$= \frac{-\lambda}{\lambda-t}\left[e^{-\infty} - e^0\right]$$

$$= \frac{-\lambda}{\lambda-t}[0-1]$$

$$= \frac{\lambda}{\lambda-t}$$

To find the mean and variance:

$$M_X(t) = \frac{\lambda}{\lambda\left(1-\frac{t}{\lambda}\right)} = \left(1-\frac{t}{\lambda}\right)^{-1}$$

$$= 1 + \frac{t}{\lambda} + \frac{t^2}{\lambda^2} + \cdots$$

Coefficient of $t = \frac{1}{\lambda}$; Coefficient of $t^2 = \frac{1}{\lambda^2}$

$E(X) = 1! \times$ coefficient of $t = \frac{1}{\lambda}$

$E(X^2) = 2! \times \frac{1}{\lambda^2} = \frac{2}{\lambda^2}$

Variance $= E(X^2) - [E(X)]^2 = \frac{2}{\lambda^2} - \left[\frac{1}{\lambda}\right]^2$

Variance $= \frac{1}{\lambda^2}$

Problems based on Exponential Distribution:

1. The time in hours required to repair a machine is exponentially distributed with parameter $\lambda = \frac{1}{2}$ (i) What is the probability that the required time exceeds 2 hours. (ii) What is the conditional probability that the repair takes at least 11 hours given that its duration exceeds 8 hours.

Solution:

Given X is exponentially distributed with parameter $\lambda = \frac{1}{2}$

The exponential distribution is given by $f(x) = \lambda e^{-\lambda x};\ x \geq 0$

$$f(x) = \frac{1}{2} e^{-\frac{x}{2}};\ x \geq 0$$

(i) P(repair time exceeds 2 hours) = $P(X > 2)$

$$= \int_2^\infty f(x)dx$$

$$= \frac{1}{2}\int_2^\infty e^{-\frac{x}{2}} dx$$

$$= \frac{1}{2}\left[\frac{e^{-\frac{x}{2}}}{\frac{-1}{2}}\right]_2^\infty$$

$$= -\left[e^{-\frac{x}{2}}\right]_2^\infty$$

$$= -[0 - e^{-1}]$$

$$= e^{-1}$$

(ii) P(time required atleast 11 hours / exceeds 8 hours)= $P(X \geq 11/X > 8)$

$$= P(X > 3)$$

$$= \int_3^\infty f(x)dx$$

$$= \frac{1}{2}\int_{3}^{\infty} e^{-\frac{x}{2}}dx$$

$$= \frac{1}{2}\left[\frac{e^{-\frac{x}{2}}}{\frac{-1}{2}}\right]_{3}^{\infty}$$

$$= -\left[e^{-\frac{x}{2}}\right]_{3}^{\infty}$$

$$= -\left[0 - e^{-\frac{3}{2}}\right]$$

$$= e^{-\frac{3}{2}}$$

2. The length of time a person speaks over phone follows exponential distribution with mean 6 minutes. What is the probability that the person will talk for (i) more than 8 minutes (ii) between 4 and 8 minutes.

Solution:

Given X is exponentially distributed with parameter $\lambda = \frac{1}{6}$

The exponential distribution is given by $f(x) = \lambda e^{-\lambda x};\ \ x \geq 0$

$$f(x) = \frac{1}{6}e^{-\frac{x}{6}};\ \ x \geq 0$$

(i))$P(X > 8)\ = \int_{8}^{\infty} f(x)dx$

$$= \frac{1}{6}\int_{2}^{\infty} e^{-\frac{x}{6}}dx$$

$$= \frac{1}{6}\left[\frac{e^{-\frac{x}{6}}}{\frac{-1}{6}}\right]_{8}^{\infty}$$

$$= -\left[e^{-\frac{x}{6}}\right]_{8}^{\infty}$$

$$= -\left[0 - e^{-\frac{8}{6}}\right]$$

$$= e^{-\frac{8}{6}}$$

(ii) P(between 4 and 8 minutes) $= P(4 < X < 8)$

$$= \int_4^8 f(x)dx$$

$$= \frac{1}{6}\int_4^8 e^{-\frac{x}{6}}dx$$

$$= \frac{1}{6}\left[\frac{e^{-\frac{x}{6}}}{\frac{-1}{6}}\right]_4^8$$

$$= -\left[e^{-\frac{8}{6}} - e^{-\frac{4}{6}}\right] = 0.25$$

Uniform Distribution (Rectangular Distribution)

A random variable X is said to have a continuous uniform distribution if its probability density function is given by

$$f(x) = \begin{cases} \dfrac{1}{b-a}, a < x < b \\ 0, otherwise \end{cases}$$

Find the MGF of uniform distribution and hence find its mean and variance.

Solution:

Let X follows uniform distribution in (a, b).

By Definition, $f(x) = \frac{1}{b-a}; a < x < b$.

$$M_X(t) = E[e^{tx}]$$

$$= \int_a^b f(x)e^{tx}dx$$

$$= \int_a^b \frac{1}{b-a} e^{tx} dx$$

$$= \frac{1}{b-a} \left[\frac{e^{tx}}{t}\right]_a^b$$

$$M_X(t) = \frac{1}{t(b-a)} [e^{bt} - e^{at}]$$

To find mean and variance:

$$M_X(t) = \frac{1}{t(b-a)} \left\{ \left[1 + \frac{bt}{1!} + \frac{(bt)^2}{2!} + \frac{(bt)^3}{3!} + \cdots \right] - \left[1 + \frac{at}{1!} + \frac{(at)^2}{2!} + \frac{(at)^3}{3!} + \cdots \right] \right\}$$

$$= \frac{1}{b-a} \left\{ \left[\frac{1}{t} + \frac{b}{1!} + \frac{b^2 t}{2!} + \frac{b^3 t^2}{3!} + \cdots \right] - \left[\frac{1}{t} + \frac{a}{1!} + \frac{a^2 t}{2!} + \frac{a^3 t^2}{3!} + \cdots \right] \right\}$$

Coefficient of $t = \frac{1}{b-a} \left[\frac{b^2}{2!} - \frac{a^2}{2!}\right] = \frac{1}{b-a} \frac{b^2 - a^2}{2!}$

$$= \frac{1}{2(b-a)} [(b+a)(b-a)]$$

$$= \frac{a+b}{2}$$

Coefficient of $t^2 = \frac{1}{b-a} \left[\frac{b^3}{3!} - \frac{a^3}{3!}\right]$

$$= \frac{1}{6(b-a)} [b^3 - a^3]$$

$$= \frac{1}{6} \frac{1}{(b-a)} (b-a)(b^2 + ba + a^2)$$

$$= \frac{1}{6}(b^2 + ab + a^2)$$

$$= \frac{b^2+ab+a^2}{6}$$

Mean $= E(X) = 1! \times$ coefficient of t

$$= 1! \times \left(\frac{a+b}{2}\right) = \frac{a+b}{2}$$

$E(X^2) = 2! \times$ coefficient of t^2

$$= 2 \times \frac{b^2+ab+a^2}{6}$$

$$= \frac{b^2+ab+a^2}{3}$$

Variance $= E(X^2) - [E(X)]^2$

$$= \frac{b^2+a^2+ab}{3} - \frac{(a+b)^2}{4}$$

$$= \frac{b^2+a^2+ab}{3} - \frac{(a^2+2ab+b^2)}{4}$$

$$= \frac{4b^2+4a^2+4ab-3a^2-6ab-3b^2}{12}$$

$$= \frac{a^2+b^2-2ab}{12}$$

$$= \frac{(a-b)^2}{12}$$

$$= \frac{(b-a)^2}{12} \quad \because a < b$$

Problem based on Uniform Distribution

1. A random variable X has a uniform distribution over (0, 10) compute

(i) $P(X < 2)$ (ii) $P(X > 8)$ (iii) $P(3 < X < 9)$

Solution:

The pdf $f(x) = \begin{cases} \frac{1}{b-a}, a < x < b \\ 0, otherwise \end{cases}$

$$f(x) = \begin{cases} \frac{1}{10}, 0 < x < 10 \\ 0, otherwise \end{cases}$$

(i) $P(X < 2) = \int_0^{10} f(x)dx$

$$= \frac{1}{10}\int_0^{10} dx$$

$$= \frac{1}{10} [x]_0^{10}$$

$$= \frac{1}{10}[10 - 0]$$

$$= \frac{1}{5}$$

(ii) $P(X > 8) = \int_8^{10} f(x)dx$

$$= \frac{1}{10}\int_8^{10} dx$$

$$= \frac{1}{10} [x]_8^{10}$$

$$= \frac{1}{10}[10 - 8]$$

$$= \frac{1}{5}$$

(iii) $P(3 < X < 9) = \int_3^9 f(x)dx$

$$= \frac{1}{10}\int_3^9 dx$$

$$= \frac{1}{10}\,[x]_3^9$$

$$= \frac{1}{10}[9 - 3]$$

$$= \frac{3}{5}$$

2. A random variable X has a uniform distribution over (-3, 3) compute

(i) $P(X < 2)$ (ii) $P(|X| < 2)$ (iii) $P(|X - 2| < 2)$

Solution:

The pdf $f(x) = \begin{cases} \frac{1}{b-a}, a < x < b \\ 0, otherwise \end{cases}$

$$f(x) = \begin{cases} \frac{1}{6}, -3 < x < 3 \\ 0, otherwise \end{cases}$$

(i) $P(X < 2) = \int_{-3}^{2} f(x)dx$

$$= \frac{1}{6}\int_{-3}^{2} dx$$

$$= \frac{1}{6}\ [x]_{-3}^{2}$$

$$= \frac{1}{6}[2+3]$$

$$= \frac{5}{6}$$

(ii) $P(|X| < 2) = P(-2 < X < 2)$

$$= \frac{\mathbf{1}}{\mathbf{6}}\int_{-2}^{2} dx$$

$$= \frac{1}{6}\ [x]_{-2}^{2}$$

$$= \frac{1}{6}[2+2]$$

$$= \frac{4}{6}$$

(iii) $P(|X-2| < 2) = P(-2 < X-2 < 2)$

$$= P(-2+2 < X < 2+2)$$

$$= P(0 < X < 4)$$

$$= \frac{\mathbf{1}}{\mathbf{6}}\int_{0}^{3} dx$$

$$= \frac{1}{6}\ [x]_{0}^{3}$$

$$= \frac{1}{6}[3-0]$$

$$= \frac{3}{6}$$

3. 4 buses arrive at a specified stop at 15 minute intervals starting at 7 am. That is, they arrive at 7, 7.15, 7.30, 7.45 am and so on. If a passenger arrives at the stop at a time that is uniformly distributed between 7 and 7.30 am. Find the probability that he waits (i) less than 5 minutes for a bus (ii) more than 10 minutes for a bus.

Solution:

The pdf is $f(x) = \begin{cases} \frac{1}{30}, 0 < x < 30 \\ 0, otherwise \end{cases}$

(i) P(a person arrives between 7.10 and 7.15 or 7.25 and 7.30)

$$= P(10 < X < 15) + P(25 < X < 30)$$

$$= \int_{10}^{15} f(x)dx + \int_{25}^{30} f(x)dx$$

$$= \frac{\mathbf{1}}{\mathbf{30}} \int_{10}^{15} dx + \frac{\mathbf{1}}{\mathbf{30}} \int_{25}^{30} dx$$

$$= \frac{1}{30}\ [x]_{10}^{15} + \frac{1}{30}\ [x]_{25}^{30}$$

$$= \frac{1}{30}[15 - 10] + \frac{1}{30}[30 - 25]$$

$$= \frac{1}{3}$$

(ii) P(a person arrives between 7.00 and 7.05 or 7.15 and 7.20)

$$= P(0 < X < 5) + P(15 < X < 20)$$

$$= \int_{0}^{5} f(x)dx + \int_{15}^{0} f(x)dx$$

$$= \frac{\mathbf{1}}{\mathbf{30}}\int_0^5 dx + \frac{\mathbf{1}}{\mathbf{30}}\int_{15}^{20} dx$$

$$= \frac{1}{30}\ [x]_0^5 + \frac{1}{30}\ [x]_{15}^{20}$$

$$= \frac{1}{30}[5-0] + \frac{1}{30}[20-15] = \frac{1}{3}$$

4. Subway trains on a certain line run every half an hour between mid- night and six in the morning. What is the probability that a man entering the station at a random time during this period will have to wait atleast twenty minutes.

Solution:

The pdf is $f(x) = \begin{cases} \frac{1}{30-0}, 0 < x < 30 \\ 0, otherwise \end{cases}$

(i) P(a man waiting for atleast 20 minutes) $= P(X \geq 20)$

$$= \int_{20}^{30} f(x)dx$$

$$= \frac{\mathbf{1}}{\mathbf{30}}\int_{20}^{30} dx$$

$$= \frac{1}{30}\ [x]_{20}^{30}$$

$$= \frac{1}{30}[30-20]$$

$$= \frac{1}{3}$$