1.2. PROPERTIES OF EIGEN VALUES AND EIGEN VECTORS

Property: 1(a) The sum of the Eigen values of a matrix is equal to the sum of the elements of the principal (main) diagonal.

(or)

The sum of the Eigen values of a matrix is equal to the trace of the matrix.

1. (b) product of the Eigen values is equal to the determinant of the matrix.

Proof:

Let A be a square matrix of order n.

The characteristic equation of A is $|A - \lambda I| = 0$

$(i.e.)\lambda^n - S_1\lambda^{n-1} + S_2\lambda^{n-2} - \cdots + (-1)S_n = 0$ ... (1)

where $\mathrm{S}_1 =$ Sum of the diagonal elements of A.

. . .

. . .

. . .

$\mathrm{S}_\mathrm{n} =$ determinant of A.

We know the roots of the characteristic equation are called Eigen values of the given matrix.

Solving (1) we get n roots.

Let the n be $\lambda_1, \lambda_2, \ldots\ \lambda_n$.

i.e., $\lambda_1, \lambda_2, \ldots\ \lambda_n$. are the Eignvalues of A.

We know already,

$\lambda^n -$ (Sum of the roots $\lambda^{n-1} +$ [sum of the product of the roots taken two at a time] $\lambda^{n-2} - \ \ldots + (-1)^n$(Product of the roots) $= 0$ ... (2)

Sum of the roots $= S_1 by$ (1)&(2)

$(i.e.)\ \lambda_1 + \lambda_2 + \cdots + \lambda_n = S_1$

$(i.e.)\ \lambda_1 + \lambda_2 + \cdots + \lambda_n = a_{11} + a_{22} + \cdots + a_{nn}$

Sum of the Eigen values = Sum of the main diagonal elements

Product of the roots $= \mathrm{S}_\mathrm{n}$by (1)&(2)

$(i.e.)\lambda_1\lambda_2 \ldots \lambda_n = \det \text{of } A$

$$\boxed{\text{Product of the Eigen values} = |A|}$$

Property: 2 A square matrix A and its transpose A^T have the same Eigenvalues.

(or)

A square matrix A and its transpose A^T have the same characteristic values.

Proof:

Let A be a square matrix of order n.

The characteristic equation of A and A^T are

$|A - \lambda I| = 0 \quad \ldots\ldots(1)$

and $|A^T - \lambda I| = 0 \quad \ldots\ldots(2)$

Since, the determinant value is unaltered by the interchange of rows and columns.

We know $|A| = |A^T|$

Hence, (1) and (2) are identical.

∴ The Eigenvalues of A and A^T are the same.

Property: 3 The characteristic roots of a triangular matrix are just the diagonal elements of the matrix.

(or)

The Eigen values of a triangular matrix are just the diagonal elements of the matrix.

Proof: Let us consider the triangular matrix.

$$A = \begin{bmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{22} & 0 \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

Characteristic equation of is

$$|A - \lambda I| = 0$$

i.e., $$\begin{vmatrix} a_{11} - \lambda & 0 & 0 \\ a_{21} & a_{22} - \lambda & 0 \\ a_{31} & a_{32} & a_{33} - \lambda \end{vmatrix} = 0$$

On expansion it gives $(a_{11} - \lambda)(a_{22} - \lambda)(a_{33} - \lambda) = 0$

i.e., $\lambda = a_{11}, a_{22}, a_{33}$

which are diagonal elements of the matrix A.

Property: 4 If λ is an Eigenvalue of a matrix A, then $\frac{1}{\lambda}, (\lambda \neq 0)$ is the Eignvalue of A^{-1}.

(or)

If λ is an Eigenvalue of a matrix A, what can you say about the Eigenvalue of matrix A^{-1}. Prove your statement.

Proof:

If X be the Eigenvector corresponding to λ,

then $AX = \lambda X$ ... (i)

Pre multiplying both sides by A^{-1}, we get

$$A^{-1}AX = A^{-1}\lambda X$$

$$(1) \Rightarrow X = \lambda A^{-1}X$$

$$X = \lambda A^{-1}X$$

$$\div \lambda \Rightarrow \quad \frac{1}{\lambda}X = A^{-1}X$$

$$(i.e.) \quad A^{-1}X = \frac{1}{\lambda}X$$

This being of the same form as (i), shows that $\frac{1}{\lambda}$ is an Eigenvalue of the inverse matrix A^{-1}.

Property: 5 If λ is an Eigenvalue of an orthogonal matrix, then $\frac{1}{\lambda}$ is an Eigenvalue.

Proof:

Definition: Orthogonal matrix. A square matrix A is said to be orthogonal if $AA^T = A^TA = I$ i.e., $A^T = A^{-1}$

Let A be an orthogonal matrix.

Given λ is an Eignevalue of A.

$\Rightarrow \frac{1}{\lambda}$ is an Eigenvalue of A^{-1}

Since, $A^T = A^{-1}$

$\therefore \frac{1}{\lambda}$ is an Eigenvalue of A^T

But, the matrices A and A^T have the same Eigenvalues, since the determinants $|A - \lambda I|$ and $|A^T - \lambda I|$ are the same.

Hence, $\frac{1}{\lambda}$ is also an Eigenvalue of A.

Property: 6 If $\lambda_1, \lambda_2, \dots \lambda_n$. are the Eignvalues of a matrix A, then A^m has the Eigenvalues $\lambda_1^m, \lambda_2^m, \dots, \lambda_n^m$ (m being a positive integer)

Proof:

Let A_i be the Eigenvalue of A and X_i the corresponding Eigenvector.

Then $AX_i = \lambda_i X_i \quad \dots (1)$

We have $A^2X_i = A(AX_i)$

$$= A(\lambda_i X_i)$$

$$= \lambda_i A(X_i)$$

$$= \lambda_i(\lambda_i X_i)$$

$$= \lambda_i^2 X_i$$

||| 1y $A^3X_i = \lambda_i^3 X_i$

In general, $A^m X_i = \lambda_i^m X_i \qquad \dots . (2)$

Hence, λ_i^m is an Eigenvalue of A^m.

The corresponding Eigenvector is the same X_i.

Note: If λ is the Eigenvalue of the matrix A then λ^2 is the Eigenvalue of A^2

Property: 7 The Eigen values of a real symmetric matrix are real numbers.

Proof:

Let λ be an Eigenvalue (may be complex) of the real symmetric matrix A. Let the corresponding Eigenvector be X. Let A denote the transpose of A.

We have $AX = \lambda X$

Pre-multiplying this equation by $1 \times n$ matrix $\bar{X}'$, where the bar denoted that all elements of $\bar{X}'$ are the complex conjugate of those of X', we get

$$\overline{X'}AX = \lambda \overline{X'}X \qquad \dots . (1)$$

Taking the conjugate complex of this we get $\mathrm{X}'\,\mathrm{A}\overline{\mathrm{X}} = \overline{\lambda}\mathrm{X}'\overline{\mathrm{X}}$ or

$$\mathrm{X}'\mathrm{A}\,\overline{\mathrm{X}} = \overline{\lambda}\,\mathrm{X}'\,\overline{\mathrm{X}} \text{ since, } \overline{\mathrm{A}} = \mathrm{A} \text{ for A is real.}$$

Taking the transpose on both sides, we get

$$\left(\mathrm{X}'\mathrm{A}\overline{\mathrm{X}}\right)' = \left(\overline{\lambda}\,\mathrm{X}'\,\overline{\mathrm{X}}\right)'(i.e.,)\overline{\mathrm{X}'}\,\mathrm{A}'\,\mathrm{X} = \overline{\lambda}\,\overline{\mathrm{X}'}\,X$$

$(i.e.)\overline{\mathrm{X}'}\,\mathrm{A}'\,\mathrm{X} = \overline{\lambda}\,\overline{\mathrm{X}'}\,X$ since $\mathrm{A}' = \mathrm{A}$ for A is symmetric.

But, from (1), $\overline{\mathrm{X}'}\,\mathrm{A}\,\mathrm{X} = \lambda\,\overline{\mathrm{X}'}\,X$ Hence $\lambda\,\overline{\mathrm{X}'}\,X = \overline{\lambda}\,\overline{\mathrm{X}'}\,X$

Since, $\overline{\mathrm{X}'}\,\mathrm{X}$ is an 1×1 matrix whose only element is a positive value, $\lambda = \overline{\lambda}$ $(i.e.)$ λ is real).

Property: 8 The Eigen vectors corresponding to distinct Eigen values of a real symmetric matrix are orthogonal.

Proof:

For a real symmetric matrix A, the Eigen values are real.

Let X_1, X_2 be Eigenvectors corresponding to two distinct eigen values λ_1, λ_2

$[\lambda_1, \lambda_2$ are real]

$$\mathrm{AX}_1 = \lambda_1\mathrm{X}_1 \qquad \ldots.(1)$$

$$\mathrm{AX}_2 = \lambda_2\mathrm{X}_2 \qquad \ldots.(2)$$

Pre multiplying (1) by $\mathrm{X}_2{}'$, we get

$$\mathrm{X}_2{}'\mathrm{AX}_1 = \mathrm{X}_2{}'\lambda_1\mathrm{X}_1$$

$$= \lambda_1\mathrm{X}_2{}'\mathrm{X}_1$$

Pre-multiplying (2) by $\mathrm{X}_1{}'$, we get

$$\mathrm{X}_1{}'\mathrm{AX}_2 = \lambda_2\mathrm{X}_1{}'\mathrm{X}_2 \qquad \ldots.(3)$$

$$\text{But}(\mathrm{X}_2{}'\mathrm{AX}_1)' = (\lambda_1\mathrm{X}_2{}'\mathrm{X}_1)'$$

$$\mathrm{X}_1{}'\mathrm{A}\,\mathrm{X}_2 = \lambda_1\mathrm{X}_1{}'\mathrm{X}_2$$

$$(i.e) \quad \mathrm{X}_1{}'\mathrm{A}\,\mathrm{X}_2 = \lambda_1\mathrm{X}_1{}'\mathrm{X}_2 \qquad \ldots..(4) \quad [\because \mathrm{A}' = \mathrm{A}]$$

From (3) and (4)

$$\lambda_1\mathrm{X}_1'\,\mathrm{X}_2 = \lambda_2\mathrm{X}_1{}'\,\mathrm{X}_2$$

$$(\text{i.e.,})(\lambda_1 - \lambda_2)\mathrm{X}_1'\,\mathrm{X}_2 = 0$$

$\lambda_1 \neq \lambda_2, X_1' X_2 = 0$

$\therefore X_1, X_2$ are orthogonal.

Property: 9 The similar matrices have same Eigen values.

Proof:

Let A, B be two similar matrices.

Then, there exists an non-singular matrix P such that $B = P^{-1} AP$

$B - \lambda I = P^{-1}AP - \lambda I$

$= P^{-1}AP - P^{-1} \lambda IP$

$= P^{-1}(A - \lambda I)P$

$|B - \lambda I| = |P^{-1}|\, |A - \lambda I|\, |P|$

$= |A - \lambda I|\, |P^{-1}P|$

$= |A - \lambda I|\, |I|$

$= |A - \lambda I|$

Therefore, A, B have the same characteristic polynomial and hence characteristic roots.

$\therefore$ They have same Eigen values.

Property: 10 If a real symmetric matrix of order 2 has equal Eigen values, then the matrix is a scalar matrix.

Proof :

Rule 1 : A real symmetric matrix of order n can always be diagonalised.

Rule 2 : If any diagonalized matrix with their diagonal elements are equal, then the matrix is a scalar matrix.

Given A real symmetric matrix 'A' of order 2 has equal Eigen values.

By Rule: 1 A can always be diagonalized, let λ_1 and λ_2 be their Eigenvalues then

we get the diagonlized matrix $= \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$

Given $\lambda_1 = \lambda_2$

Therefore, we get $= \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$

By Rule: 2 The given matrix is a scalar matrix.

Property: 11 The Eigen vector X of a matrix A is not unique.

Proof :

Let λ be the Eigenvalue of A, then the corresponding Eigenvector X such that $A X = \lambda X$.

Multiply both sides by non-zero K,

$$K (AX) = K (\lambda X)$$

$$\Rightarrow A (KX) = \lambda (KX)$$

$(i.e.)$an Eigenvector is determined by a multiplicative scalar.

$(i.e.)$ Eigenvector is not unique.

Property: 12 $\lambda_1, \lambda_2, \ldots \lambda_n$ be distinct Eigenvalues of an $n \times n$ matrix, then the corresponding Eigenvectors $X_1, X_2, \ldots X_n$ form a linearly independent set.

Proof:

Let $\lambda_1, \lambda_2, \ldots \lambda_m (m \leq n)$ be the distinct Eigen values of a square matrix A of order n.

Let $X_1, X_2, \ldots X_m$ be their corresponding Eigenvectors we have to prove $\sum_{i=1}^{m} \alpha_i X_i = 0$ implies each $\alpha_i = 0, i = 1,2, \ldots, m$

Multiplying $\sum_{i=1}^{m} \alpha_i X_i = 0$ by $(A - \lambda_1 I)$, we get

$$(A - \lambda_1 I)\alpha_1 X_1 = \alpha_1 (AX_1 - \lambda_1 X_1) = \alpha_1(0) = 0$$

When $\sum_{i=1}^{m} \alpha_i X_i = 0$ Multiplied by

$$(A - \lambda_2 I)(A - \lambda_2 I) \ldots (A - \lambda_{i-1} I)(A - \lambda_i I)\ (A - \lambda_{i+1} I) \ldots\ (A - \lambda_m I)$$

We get, $\alpha_i (\lambda_i - \lambda_1)(\lambda_i - \lambda_2) \ldots\ (\lambda_i - \lambda_{i-1})(\lambda_i - \lambda_{i+1}) \ldots (\lambda_i - \lambda_m) = 0$

Since, λ's are distinct, $\alpha_i = 0$

Since, i is arbitrary, each $\alpha_i = 0, i = 1, 2, \ldots, m$

$\sum_{i=1}^{m} \alpha_i X_i = 0$ implies each $\alpha_i = 0, i = 1,2, \ldots, m$

Hence, $X_1, X_2, \ldots X_m$ are linearly independent.

Property: 13 If two or more Eigen values are equal it may or may not be possible to get linearly independent Eigenvectors corresponding to the equal roots.

Property: 14 Two Eigenvectors X_1 and X_2 are called orthogonal vectors if $X_1^T X_2 = 0$

Property: 15 If A and B are $n \times n$ matrices and B is a non singular matrix, then A and B^{-1} AB have same eigenvalues.

Proof:

Characteristic polynomial of B^{-1} AB

$$= |B^{-1} AB - \lambda I| = |B^{-1} AB - B^{-1}(\lambda I)B|$$

$$= |B^{-1} (A - \lambda I)B| = |B^{-1}||A - \lambda I||B|$$

$$= |B^{-1}|\,|B|\,|A - \lambda I| = |B^{-1}B||A - \lambda I|$$

$= \text{Characterisstisc polynomial of A}$

Hence, A and B^{-1} AB have same Eigenvalues.

Example: Find the sum and product of the Eigen values of the matrix $\begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$

Solution:

Sum of the Eigen values =Sum of the main diagonal elements

$$= (-2) + (1) + (0)$$

$$= -1$$

Product of the Eigen values = $\begin{vmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{vmatrix}$

$$= -2(0 - 12) - 2(0 - 6) - 3(-4 + 1)$$

$$= 24 + 12 + 9 = 45$$

Example: Find the sum and product of the Eigen values of the matrix A= $\begin{bmatrix} 1 & 2 & 3 \\ -1 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix}$

Solution:

Sum of the Eigen values = Sum of its diagonal elements $= 1 + 2 + 1 = 4$

$$\text{Product of Eigen values } = |C| = \begin{vmatrix} 1 & 2 & 3 \\ -1 & 2 & 1 \\ 1 & 1 & 1 \end{vmatrix}$$

$$= 1(2-1) - 2(-1-1) + 3(-1-2)$$

$$= 1(1) - 2(-2) + 3(-3)$$

$$= 1 + 4 - 9 = -4$$

Example: The product of two Eigen values of the matrix $A = \begin{bmatrix} \mathbf{6} & \mathbf{-2} & \mathbf{2} \\ \mathbf{-2} & \mathbf{3} & \mathbf{-1} \\ \mathbf{2} & \mathbf{-1} & \mathbf{3} \end{bmatrix}$ is 16. Find the third Eigenvalue.

Solution:

Let Eigen values of the matrix A be $\lambda_1, \lambda_2, \lambda_3$.

Given $\lambda_1\lambda_2 = 16$

We know that, $\lambda_1\lambda_2\lambda_3 = |A|$

[Product of the Eigen values is equal to the determinant of the matrix]

$$\therefore \lambda_1\lambda_2\lambda_3 = \begin{vmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{vmatrix}$$

$$= 6(9-1) + (-6+2) + 2(2-6)$$

$$= 6(8) + 2(-4) + 2(-4)$$

$$= 48 - 8 - 8$$

$$\Rightarrow \lambda_1\lambda_2\lambda_3 = 32$$

$$\Rightarrow 16\,\lambda_3 = 32$$

$$\Rightarrow \lambda_3 = \frac{32}{16} = 2$$

Example: Two of the Eigen values of $\begin{bmatrix} \mathbf{6} & \mathbf{-2} & \mathbf{2} \\ \mathbf{-2} & \mathbf{3} & \mathbf{-1} \\ \mathbf{2} & \mathbf{-1} & \mathbf{3} \end{bmatrix}$ are 2 and 8. Find the third Eigen value.

Solution:

We know that, Sum of the Eigen values = Sum of its diagonal elements

$$= 6 + 3 + 3 = 12$$

Given $\lambda_1 = 2, \lambda_2 = 8, \lambda_3 = ?$

We get, $\lambda_1 + \lambda_2 + \lambda_3 = 12$

$$2 + 8 + \lambda_3 = 12$$

$$\lambda_3 = 12 - 10$$

$$\lambda_3 = 2$$

$\therefore$ The third Eigenvalue $= 2$

Example: If 3 and 15 are the two Eigen values of $A = \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$ **find** $|A|$**, without expanding the determinant.**

Solution:

Given $\lambda_1 = 3, \lambda_2 = 15, \lambda_3 = ?$

We know that, Sum of the Eigen values $=$ Sum of the main diagonal elements

$$\Rightarrow \lambda_1 + \lambda_2 + \lambda_3 = 8 + 7 + 3$$

$$3 + 15 + \lambda_3 = 18$$

$$\Rightarrow \lambda_3 = 0$$

We know that, Product of the Eigen values $= |A|$

$$\Rightarrow (3)(15)(0) = |A|$$

$$\Rightarrow |A| = (3)(15)(0)$$

$$\Rightarrow |A| = 0$$

Example: If 2, 2, 3 are the Eigen values of $A = \begin{bmatrix} 3 & 10 & 5 \\ -2 & -3 & -4 \\ 3 & 5 & 7 \end{bmatrix}$ **find the Eigen values of** A^T**.**

Solution:

By Property "A square matrix A and its transpose A^T have the same Eigen values".

Hence, Eigen values of A^T are 2, 2, 3

Example: If the Eigen values of the matrix $A = \begin{bmatrix} 1 & 1 \\ 3 & -1 \end{bmatrix}$ **are** $2, -2$ **then find the Eigen values of** A^T**.**

Solution:

Eigen values of $A =$ Eigen values of A^{T}

$\therefore$ Eigen values of A^{T} are $2, -2$.

Example: Two of the Eigen values of $A = \begin{bmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{bmatrix}$ are 3 and 6. Find the Eigen values of A^{-1}.

Solution:

Sum of the Eigen values = Sum of the main diagonal elements

$$= 3 + 5 + 3 = 11$$

Let K be the third Eigen value

$$\therefore 3 + 6 + k = 11$$

$$\Rightarrow 9 + k = 11$$

$$\Rightarrow k = 2$$

$\therefore$ The Eigenvalues of A^{-1} are $\frac{1}{2}, \frac{1}{3}, \frac{1}{6}$

Example: Two Eigen values of the matrix $\mathrm{A} = \begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}$ are equal to 1 each. Find the Eigenvalues of A^{-1}.

Solution:

Given $\mathrm{A} = \begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}$

Let the Eigen values of the matrix A be $\lambda_1, \lambda_2, \lambda_3$

Given condition is $\lambda_2 = \lambda_3 = 1$

We have, Sum of the Eigen values = Sum of the main diagonal elements

$$\Rightarrow \lambda_1 + \lambda_2 + \lambda_3 = 2 + 3 + 2$$

$$\Rightarrow \lambda_1 + 1 + 1 = 7$$

$$\Rightarrow \lambda_1 + 2 = 7$$

$$\Rightarrow \lambda_1 = 5$$

Hence, the Eigen values of A are 1, 1, 5

Eigen values of A^{-1} are $\frac{1}{1}, \frac{1}{1}, \frac{1}{5}$, i.e., $1, 1, \frac{1}{5}$