System of Simultaneous Linear Differential equations with constant coefficients

Simultaneous Linear equations

Linear differential equations in which there are two or more dependent variables and a single independent variable such equations are known as simultaneous linear equations.

Consider the simultaneous equation in two dependent variables x and y and one independent variable t.

$$f_1(D)x + g_1(D)y = h_1(t) \ldots (1)$$

$$f_2(D)x + g_2(D)y = h_2(t) \ldots (2)$$

Where f_1, f_2, g_1 and g_2 are polynomials in the operator D

The number of independent arbitrary constants appearing in the general solution of the system of differential equation (1) & (2) equal to the degree of D in the coefficient determinant

$$\Delta = \begin{vmatrix} f_1(D) & g_1(D) \\ f_2(D) & g_2(D) \end{vmatrix} \quad \text{Provided} \quad \Delta \neq 0$$

Example:5.67

Solve $\boldsymbol{\frac{dy}{dx} + x = t^2; \ \frac{dx}{dt} - y = t}$

Solution:

$$x + Dy = t^2 \ \ldots (1)$$

$$Dx - y = t \ \ldots (2)$$

Eliminate $'x'$

(1) x D $\Rightarrow$ $Dx + D^2 y = D\ (t^2)$

$$Dx + D^2 y = 2t \ldots (3)$$

(2) $\Rightarrow$ $Dx - y = t \ \ldots (4)$

(3) – (2) $D^2 y + y = t$

$$(D^2 + 1)y = t$$

Auxiliary Equation is $m^2 + 1 = 0$

$$m^2 = -1$$

$$m = \pm i$$

$$\propto = 0,\ \beta = 1$$

$$C.F = e^{at}[A cos\beta t + B sin\beta t]$$

$$= A cost + B sint$$

$$\text{P.I} = \frac{1}{D^2+1}(t)$$

$$= \frac{1}{1+D^2}(t)$$

$$= (1 + D^2)^{-1} t$$

$$= (1 - D^2)t$$

$$= t - D^2(t)$$

$$= t$$

$$y = Acost + Bsint + t$$

$$Dy = -Asint + Bcost + 1$$

$$(1) \Rightarrow \quad x = t^2 - Dy$$

$$= t^2 - [-Asint + Bcost + 1]$$

$$= t^2 + Asint - Bcost - 1$$

$$x = t^2 + Asint - Bcost - 1$$

$$y = Acost + Bsint + t$$

Example:5.68

Solve the simultaneous differential equations

$$\boldsymbol{\frac{dx}{dt} + 2y = sin2t \quad ; \quad \frac{dy}{dt} - 2x = cos2t}$$

Solution:

$$\frac{dx}{dt} + 2y = sin2t$$

$$Dx + 2y = sin2t \dots (1)$$

$$\frac{dy}{dt} - 2x = cos2t$$

$$-2x + Dy = cos2t \dots (2) \quad \text{Eliminate } 'x'$$

$$(1) \text{ x } 2 \quad \Rightarrow \quad 2Dx + 4y = 2sin2t \dots (3)$$

$$(2) \text{ x } D \quad \Rightarrow \quad -2Dx + D^2y = -2sin2t \dots (4)$$

$$(3) + (4) \quad \Rightarrow \quad D^2y + 4y = 0$$

$$(D^2 + 4)y = 0$$

Auxiliary Equation is $m^2 + 4 = 0$

$$m^2 = -4$$

$$m = \pm 2i$$

$$\propto = 0, \ \beta = 2$$

$$C.F = e^{\propto t}[A cos\beta t + B sin\beta t]$$

$$y = Acos2t + Bsin2t$$

To find x , (5) $\Rightarrow$ $Dy = -2Asin2t + 2Bcos2t$

(2) $\Rightarrow$ $2x = Dy - cos2t$

$$= -2Asin2t + 2Bcos2t - cos2t$$

$$x = -Asin2t + Bcos2t - \frac{cos2t}{2}$$

Example :5.69

Solve $Dx + y = sint,$ $x + Dy = cost$ given that $x = 2,\ y = 0\ at\ t = 0$

Solution:

$$Dx + y = sint \ldots (1)$$

$$x + Dy = cost \ \ldots (2)$$

Eliminate $'x'$

(2) x D => $Dx + D^2y = D(cost) \ldots (3)$

(1) => $Dx + y \quad = sint \ldots (4)$

(3) - (4) => $D^2y - y = -2sint$

$$(D^2-1)y=-2sint$$

Auxiliary Equation is $m^2-1=0$

$$m^2=+1$$

$$m=\pm1$$

$$C.F=Ae^t+Be^{-t}$$

$$\text{P.I}=\frac{1}{D^2-1}(-2sint)$$

$$=\frac{1}{-1-1}(-2sint) \quad \text{Replace } D^2 \text{ by } -1$$

$$=sint$$

$$y=Ae^t+Be^{-t}+sint \ \dots(5)$$

To find x

$(5) \Rightarrow \quad Dy=Ae^t-Be^{-t}+cost$

$(2) \Rightarrow \quad x=cost-Dy$

$$=cost-[Ae^t-Be^{-t}+cost]$$

$$=cost-Ae^t+Be^{-t}-cost$$

$$x=-Ae^t+Be^{-t} \ \dots(6)$$

When $t=0,\ x=2$

$(2) \quad \Rightarrow \quad 2=-Ae^0+Be^0$

$$2=-A+B \ \dots(7)$$

When $t=0,\ y=0$

(5) $\Rightarrow$ $0 = Ae^0 + Be^0 + 0$

$0 = A + B \ \dots(8)$

(7) + (8) $\Rightarrow$ $2B = 2$

$B = 1$

(8) $\Rightarrow$ $A + 1 = 0$

$A = -1$

(6) $\Rightarrow$ $x = e^t + e^{-t}$

(5) $\Rightarrow$ $y = -e^t + e^{-t} + sint$

$\therefore\ x = e^t + e^{-t}$

$y = -e^t + e^{-t} + sint$

Example:5.70

Solve $\frac{dx}{dt} + 5x - 2y = t\ ;\ \frac{dy}{dt} + 2x + y = 0$

Solution:

Put $D = \frac{d}{dt}$

Given $\frac{dx}{dt} + 5x - 2y = t$

$Dx + 5x - 2y = t$

$(D + 5)x - 2y = t \dots(1)$

$\frac{dy}{dt} + 2x + y = 0$

$2x + (D + 1)y = 0 \dots(2)$

Eliminate x between (1) & (2)

(2) x (D+5) ⇒ $2(D+5)x+(D+5)(D+1)y=0$

$$\underline{2(D+5)x-4y \qquad\qquad} = 2t$$

(3) – (1) ⇒ $(D+5)(D+1)y+4y = -2t$

$$(D^2+6D+9)y=-2t$$

A.E is $m^2+6m+9=0$

$$(m+3)(m+3)=0$$

$$m=-3,-3$$

$$C.F=(At+B)e^{-3t}$$

$$\text{P.I}=\frac{1}{D^2+6D+9}(-2t)$$

$$=\frac{1}{9\left[1+\frac{D^2+6D}{9}\right]}(-2t)$$

$$=\frac{-2}{9}\left[1+\frac{D^2+6D}{9}\right]^{-1}t$$

$$=\frac{-2}{9}\left[1-\frac{D^2+6D}{9}\right]t$$

$$=\frac{-2}{9}\left[1-\frac{D^2(t)}{9}-\frac{6D(t)}{9}\right]$$

$$=\frac{-2t}{9}+\frac{4}{27}$$

$$y=(At+B)e^{-3t}-\frac{2t}{9}+\frac{4}{27}$$

To find x ⇒ $Dy=(At+B)(-3)e^{-3t}+e^{-3t}A-\frac{2}{9}$

$(2) \Rightarrow \qquad 2x = -(D+1)y$

$$= -\left[(At+B)(-3)e^{-3t} + e^{-3t}A - \frac{2}{9}\right] - \left[(At+B)e^{-3t} - \frac{2t}{9} + \frac{4}{27}\right]$$

$$= 3(At+B)e^{-3t} - e^{-3t}A + \frac{2}{9} - (At+B)e^{-3t} + \frac{2t}{9} - \frac{4}{27}$$

$$2x = 2(At+B)e^{-3t} - Ae^{-3t} + \frac{2t}{9} + \frac{2}{27}$$

$$x = (At+B)e^{-3t} - \frac{A}{2}e^{-3t} + \frac{t}{9} + \frac{1}{27}$$

$$\therefore\ x = (At+B)e^{-3t} - \frac{A}{2}e^{-3t} + \frac{t}{9} + \frac{1}{27}$$

$$y = (At+B)e^{-3t} - \frac{2t}{9} + \frac{4}{27}$$

Example:5.71

Solve $\frac{dx}{dt} + 2x + 3y = 2e^{2t}$; $\frac{dy}{dt} + 3x + 2y = 0$

Solution:-

Given $Dx + 3x + 2y = 0$

$(D+2)x + 3y = 2e^{2t}$ ... (1)

$Dy + 3x + 2y = 0$

$3x + (D+2)y = 0$ ... (2)

Eliminate x

(2) x (D+2) $\Rightarrow \qquad 3(D+2)x + (D+2)^2 y = 0$ ... (3)

(1) x (3) $\Rightarrow \qquad 3(D+2)x + 9y \quad = 6e^{2t}$ ... (4)

(3) – (4) $\Rightarrow \qquad [(D+2)^2 - 9]y = -6e^{2t}$

$$(D^2 + 4D - 5)y = -6e^{2t}$$

Auxiliary Equation is $m^2 + 4m - 5 = 0$

$$m = -5, 1$$

$$C.F = Ae^{-5t} + Be^{t}$$

P.I $= \frac{1}{D^2+4D-5}(-6e^{2t})$ Replace D by 2

$$= \frac{-6}{7}e^{2t}$$

$$y = Ae^{-5t} + Be^{t} - \frac{6}{7}e^{2t} \text{ ... (5)}$$

To find x (5) $\Rightarrow$ $Dy = -5Ae^{-5t} + Be^{t} - \frac{12}{7}e^{2t}$

(2) $\Rightarrow$ $3x = -(D+2)y$

$$= -Dy - 2y$$

$$= 5Ae^{-5t} - Be^{t} + \frac{12}{7}e^{2t} - 2Ae^{-5t} - 2Be^{t} + \frac{12}{7}e^{2t}$$

$$3x = 3Ae^{-5t} - 3Be^{t} + \frac{24}{7}e^{2t}$$

$$x = Ae^{-5t} - Be^{t} + \frac{8}{7}e^{2t}$$

Example :5.72

Solve $\frac{dx}{dt} + y = e^{t}$, $x - \frac{dy}{dt} = t$

Solution:

Given $\frac{dx}{dt} + y = e^{t}$

$$Dx + y = e^{t} \text{ ... (1)}$$

Given $x - \frac{dy}{dt} = t$

$x - Dy = t \ \dots (2)$

2 x D $\Rightarrow$ $Dx - D^2y = 1 \ \dots (3)$

(1) − (3) $\Rightarrow$ $(D^2+1)y = e^t - 1$

Auxiliary Equation is $m^2 + 1 = 0$

$$m = \pm i$$

$$C.F = Acost + Bsint$$

$$\text{P.I}_1 = \frac{1}{D^2+1}e^t$$

$= \frac{1}{1+1}e^t$ Replace D by1

$$= \frac{1}{2}e^t$$

$$\text{P.I}_2 = \frac{1}{D^2+1}(-e^{0t})$$

$= -1$ Replace D by0

$$y = C.F + P.I_1 + P.I_2$$

$$y = Acost + Bsint + \frac{1}{2}e^t - 1$$

$$Dy = -Asint + Bcost + \frac{1}{2}e^t$$

(2) $\Rightarrow$ $x = Dy + t$

$$= -Asint + Bcost + \frac{1}{2}e^t + t$$

$$\therefore \ x = -Asint + Bcost + \frac{1}{2}e^t + t$$

$$y = Acost + Bsint + \frac{1}{2}e^{t} - 1$$

Example:5.73

Solve $\frac{dx}{dt} + 2x - 3y = t$; $\frac{dy}{dt} - 3x + 2y = e^{2t}$

Solution:

Given $\frac{dx}{dt} + 2x - 3y = t \ \ldots (1)$

$$Dx + 2x - 3y = t$$

$$(D+2)x - 3y = t \ \ldots (2)$$

Given $\frac{dy}{dt} - 3x + 2y = e^{2t} \ \ldots (3)$

$$Dx - 3x + 2y = e^{2t}$$

$$-3x + (D+2)y = e^{2t} \ \ldots (4)$$

(2) x (3) $\Rightarrow$ $3(D+2)x - 9y = 3t$

(4) x (D+2) $\Rightarrow$ $-3(D+2)x + (D+2)^2 y = (D+2)e^{2t}$

$$-9y + (D+2)^2 y = 3t + (D+2)e^{2t}$$

$$(-9 + D^2 + 4D + 4)y = 3t + 4e^{2t}$$

$$(D^2 + 4D - 5)y = 3t + 4e^{2t}$$

Auxiliary Equation is $m^2 + 4m - 5 = 0$

$$m = -5, 1$$

$$C.F = Ae^{t} + Be^{-5t}$$

$$\text{P.I}_1 = \frac{1}{D^2 + 4D - 5}(3t)$$

$$= \frac{3}{-5\left[1-\frac{D^2+4D}{5}\right]} t$$

$$= \frac{3}{-5\left[1-\frac{D^2+4D}{5}\right]} t$$

$$= \frac{-3}{5}\left[1-\left(\frac{D^2+4D}{5}\right)\right]^{-1} (t)$$

$$= \frac{-3}{5}\left[1+\left(\frac{D^2+4D}{5}\right)+\left(\frac{D^2+4D}{5}\right)^2+---\right](t)$$

$$= \frac{-3}{5}\left[t+\frac{4}{5}\right]$$

$$= \frac{-3}{5}t-\frac{12}{25}$$

$$P.I_2 = \frac{1}{D^2+4D-5}\ 4e^{2t}$$

$$= \frac{4}{4-8-5}e^{2t}$$ Replace D by 2

$$= \frac{4}{7}e^{2t}$$

$$y = C.F + P.I_1 + P.I_2$$

$$y = Ae^t + Be^{-5t} - \frac{3}{5}t - \frac{12}{25} + \frac{4}{7}e^{2t}$$

$$Dy = Ae^t - 5Be^{-5t} - \frac{3}{5} + \frac{8}{7}e^{2t}$$

(3) ⇒ $3x = \frac{dy}{dt} + 2y - e^{2t}$

$$= \left[Ae^t - 5Be^{-5t} - \frac{3}{5} + \frac{8}{7}e^{2t}\right] + 2\left[Ae^t + Be^{-5t} - \frac{3}{5}t - \frac{12}{25} + \frac{4}{7}e^{2t}\right] - e^{2t}$$

$$= Ae^t - 5Be^{-5t} - \frac{3}{5} + \frac{8}{7}e^{2t} + 2Ae^t + 2Be^{-5t} - \frac{6}{5}t + \frac{24}{25} + \frac{8}{7}e^{2t} - e^{2t}$$

$$3x = 3Ae^{t} - 3Be^{-5t} - \frac{6}{5}t - \frac{39}{25} + \frac{9}{7}e^{2t}$$

$$x = Ae^{t} - Be^{-5t} - \frac{2}{5}t - \frac{13}{25} + \frac{3}{7}e^{2t}$$

$$\therefore x = Ae^{t} - Be^{-5t} - \frac{2}{5}t - \frac{13}{25} + \frac{3}{7}e^{2t}$$

$$y = Ae^{t} + Be^{-5t} - \frac{3}{5}t + \frac{4}{7}e^{2t} - \frac{12}{25}$$