UNIT – I – RANDOM VARAIBLES

Random Experiment

An experiment whose output is uncertain even though all the outcomes are known.

Example: Tossing a coin, Throwing a fair die, Birth of a baby.

Sample Space:

The set of all possible outcomes in a random experiment. It is denoted by S.

Example:

For tossing a fair coin, $S = \{H, T\}$

For throwing a fair die, $S = \{1, 2, 3, 4, 5, 6, \}$

For birth of a baby, $S = \{M, F\}$

Event:

A subset of sample space is event. It is denoted by A.

Mutually Exclusive Events:

Two events A and B are said to be mutually exclusive events if they do not occur simultaneously. If A and B are mutually exclusive, then $A \cap B = \Phi$

Example:

Tossing two unbiased coins $S = \{HH, HT, TH, TT\}$

(i) Let $A = \{HH\}, B = \{HT\}$

$$A \cap B = \{H\} \neq \Phi$$

Then A and B are not mutually exclusive.

(i) Let $A = \{HH\}, B = \{TT\}$

$$A \cap B = \Phi$$

Then A and B are mutually exclusive.

Probability:

Probability of an event A is $P(A) = \frac{n(A)}{n(S)}$

i.e., $P(A) = \frac{number\ of\ cases\ favourable\ to\ A}{Total\ number\ of\ cases}$

Axioms of Probability:

(i) $0 \leq P(A) \leq 1$

(ii) $P(S) = 1$

(iii) $P(A \cup B) = P(A) + P(B)$, if A and B are mutually exclusive.

Note:

(i) $P(\phi) = 0$

(ii) $P(\overline{A}) = 1 - P(A)$, for any event A

(iii) $P(A \cup B) = P(A) + P(B) - P(A \cap B)$, for any two events A and B.

Independent events:

Two events A and B are said to be independent if occurrence of A does not affect the occurrence of B.

Condition for two events and B are independent:

$$P(A \cap B) = P(A)\, P(B)$$

Conditional Probability:

If the probability of the event A provided the event B has already occurred is called the conditional probability and is defined as

$$P(A|B) = \frac{P(A \cap B)}{P(B)}, \text{ provided } P(B) \neq 0$$

The probability of an event B provided A has occurred already is given by

$$P(B|A) = \frac{P(A \cap B)}{P(A)}, \text{ provided } P(A) \neq 0$$

Random Variables:

A random variable is a function that assign a real number for all the outcomes in the sample space of a random experiment.

Example:

Toss two coins then the sample space $S = \{HH, HT, TH, TT\}$

Now we define a random variable X to denote the number of heads in 2 tosses.

X(HH) = 2

X(HT) = 1

X(TH) = 1

X(TT) = 0

Types of Random Variables:

(i) Discrete Random Variables

(ii) Continuous Random Variables

Probability mass function (PMF):

Let X be discrete random variable. Then $P(X = x_i) = p(x_i) = p_i$ is said to be a Probability mass function of X, if

(i) $0 \leq p(x_i) \leq 1$

(ii) $\sum_i p(x_i) = 1$

The collection of pairs $\{x_i, p_i\}, i = 1, 2, 3, \ldots$ is called the probability distribution of the random variable X, which is sometimes in the form of a table as given below:

$X = x_i$	x_1	x_2	. . .	x_r	. . .
P($X = x_i$)	p_1	p_2	. . .	p_r	. . .

Problems on Discrete Random Variables

1.A Discrete Random Variable X has the following probability distribution

X	0	1	2	3	4	5	6	7	8
P(x)	a	3a	5a	7a	9a	11a	13a	15a	17a

(i) **Find the value of "a".**

(ii) **Find $P[X < 3], P[0 < X < 3], P\ [X \geq 3]$**

(iii) **Find the distribution of X.**

Solution:

(i) We know that $\sum P(x) = 1$

$$\Rightarrow a + 3a + 5a + 7a + 9a + 11a + 13a + 15a + 17a = 1$$

$$\Rightarrow 81a = 1$$

$$\Rightarrow a = \frac{1}{81}$$

(ii) $P[X < 3] = P[X = 0] + P[X = 1] + P[X = 2]$

$$= a + 3a + 5a$$

$$= 9a$$

$$= \frac{9}{81}$$

$P[0 < X < 3] = P[X = 1] + P[X = 2]$

$$= 3a + 5a$$

$$= 8a$$

$$= \frac{8}{81}$$

$P\ [X \geq 3] = 1 - P[X < 3]$

$$= 1 - \frac{9}{81}$$

$$= \frac{72}{81}$$

(iii) Distribution of X:

X	P(x)	F(X) = P[X≤ x]
0	a	F(0) = P[X≤ 0] = $\frac{1}{81}$
1	3a	F(1) = P[X≤ 1] = $F(0) + P(1) = \frac{1}{81} + \frac{3}{81} = \frac{4}{81}$
2	5a	F(2) = P[X≤ 2] = $F(1) + P(2) = \frac{4}{81} + \frac{5}{81} = \frac{9}{81}$
3	7a	F(3) = P[X≤ 3] = $F(2) + P(3) = \frac{9}{81} + \frac{7}{81} = \frac{16}{81}$
4	9a	F(4) = P[X≤ 4] = $F(3) + P(4) = \frac{16}{81} + \frac{9}{81} = \frac{25}{81}$
5	11a	F(5) = P[X≤ 5] = $F(4) + P(5) = \frac{25}{81} + \frac{11}{81} = \frac{36}{81}$
6	13a	F(6) = P[X≤ 6] = $F(5) + P(6) = \frac{36}{81} + \frac{13}{81} = \frac{49}{81}$
7	15a	F(7) = P[X≤ 7] = $F(6) + P(7) = \frac{49}{81} + \frac{15}{81} = \frac{64}{81}$
8	17a	F(8) = P[X≤ 8] = $F(7) + P(8) = \frac{64}{81} + \frac{17}{81} = \frac{81}{81}$

2. A Discrete Random Variable X has the following probability distribution

X	0	1	2	3	4	5	6	7
P(x)	0	k	2k	2k	3k	k^2	$2k^2$	$7k^2 + k$

(i) **Find the value of "k".**

(ii) **Find $P[X < 6], P[1 < X < 5], P\ [X \geq 6], P[X > 2]$**

(iii) **Find $P[1.5 < X < 4.5\ /\ X > 2]$**

(iv) **Find the distribution of X and find the value of k if $P[X < k] > \frac{1}{2}$**

Solution:

(i) We know that $\sum P(x) = 1$

$$\Rightarrow 0 + k + 2k + 2k + 3k + k^2 + 2k^2 + 7k^2 + k = 1$$

$$\Rightarrow 10k^2 + 9k = 1$$

$$\Rightarrow 10k^2 + 9k - 1 = 0$$

$$\Rightarrow (k + 1)(10k - 1) = 0$$

$$\Rightarrow k = -1\ (or) k = \frac{1}{10}$$

(ii) $P\ [X \geq 6] = P[X = 6] + P[X = 7]$

$$= 2k^2 + 7k^2 + k$$

$$= 9k^2 + k$$

$$= \frac{9}{100} + \frac{1}{10}$$

$$= \frac{19}{100}$$

(iii) $P[X < 6] = 1 - P[X \geq 6]$

$$= 1 - \frac{19}{100}$$

$$= \frac{81}{100}$$

(iv) $P[1 < X < 5] = P[X = 2] + P[X = 3] + P[X = 4]$

$$= 2k + 2k + 3k$$

$$= 7k$$

$$= \frac{7}{10}$$

(v) $P[1.5 < X < 4.5 \,/\, X > 2] = \frac{P[1\cdot5 < X < 4\cdot5 \cap X > 2]}{P[X>2]}$

$$= \frac{P[2 < X < 4\cdot5\,]}{P[X>2]}$$

$$= \frac{P[X=3]+P[X=4]}{P[X>2]}$$

$$= \frac{\frac{5}{10}}{\frac{7}{10}}$$

$$= \frac{5}{7}$$

Distribution of X:

X	P(x)	F(X) = P[X≤ x]
0	0	F(0) = P[X≤ 0] = 0
1	k	F(1) = P[X≤ 1] $= F(0) + P(1) = 0 + \frac{1}{10} = \frac{1}{10}$

2	2k	F(2) = P[X≤ 2] = $F(1)+P(2)=\frac{1}{10}+\frac{2}{10}=\frac{3}{10}$
3	2k	F(3) = P[X≤ 3] = $F(2)+P(3)=\frac{3}{10}+\frac{2}{10}=\frac{5}{10}$
4	3k	F(4) = P[X≤ 4] = $F(3)+P(4)=\frac{5}{10}+\frac{3}{10}=\frac{8}{10}$
5	k^2	F(5) = P[X≤ 5] = $F(4)+P(5)=\frac{8}{10}+\frac{1}{100}=\frac{81}{100}$
6	$2k^2$	F(6) = P[X≤ 6] = $F(5)+P(6)=\frac{81}{100}+\frac{2}{100}=\frac{83}{100}$
7	$7k^2+k$	F(7) = P[X≤ 7] = $F(6)+P(7)=\frac{83}{100}+\frac{7}{100}+\frac{1}{10}=\frac{100}{100}$

The value of k = 4 when $P[X<k]>\frac{1}{2}$

3. If the random variable X takes the values 1, 2, 3 and 4 such that $2P(X=1)=3P(X=2)=P(X=3)=5P(X=4)$. Find the probability distribution.

Solution:

Let $2P(X=1)=3P(X=2)=P(X=3)=5P(X=4)=k$

$\Rightarrow 2P(X=1)=k$

$\Rightarrow P(X=1)=\frac{k}{2}$

$\Rightarrow 3P(X=2)=k$

$\Rightarrow P(X=2)=\frac{k}{3}$

$\Rightarrow P(X = 3) = k$

$\Rightarrow 5P(X = 3) = k$

$\Rightarrow P(X = 3) = \frac{k}{5}$

We know that $\sum P(x) = 1$

$\Rightarrow P(1) + P(2) + P(3) + P(4) = 1$

$\Rightarrow \frac{k}{2} + \frac{k}{3} + k + \frac{k}{5} = 1$

$\Rightarrow \frac{15k+10k+30k+6k}{30} = 1$

$\Rightarrow \frac{61k}{30} = 1$

$\Rightarrow k = \frac{30}{61}$

The Probability Distribution is

X	1	2	3	4
P(x)	$\frac{k}{2} = \frac{1}{2} \times \frac{30}{61} = \frac{15}{61}$	$\frac{k}{3} = \frac{1}{3} \times \frac{30}{61} = \frac{10}{61}$	$k = \frac{30}{61}$	$\frac{k}{5} = \frac{1}{5} \times \frac{30}{61} = \frac{6}{61}$

4. Suppose that the random variable X assumes three values 0, 1 and 2 with probabilities 1/3, 1/6 and ½ respectively. Obtain the distribution function of X.

Solution:

Values of X = x	0	1	2
P(x)	1/3	1/6	1/2
	P(0)	P(1)	P(2)

The distribution of X

X	**P(x)**	**F(X) = P[X≤ x]**
0	0	F(0) = P[X≤ 0] $= \frac{1}{3}$
1	k	F(1) = P[X≤ 1] $= F(0) + P(1) = \frac{1}{3} + \frac{1}{6} = \frac{1}{2}$
2	2k	F(2) = P[X≤ 2] $= F(1) + P(2) = \frac{1}{2} + \frac{1}{2} = 1$

Mathematical expectation for discrete random variable

Note:

(i) $E(c) = c$

(ii) $Var(c) = 0$

(iii) $E(aX) = aE(X)$

(iv) $E(aX + b) = aE(X) + b$

(v) $Var(aX) = a^2 Var(X)$

(vi) $Var(aX \pm b) = a^2 Var(X)$

Problems:

If $Var(X) = 4$, find $Var(4X + 5)$, where X is a random variable.

Solution:

We know that $Var(aX + b) = a^2 Var(X)$

Here $a = 4, Var(X) = 4$

$Var(4X + 5) = 4^2 Var(X) = 16 \times 4 = 64$

Continuous Random Variable:

If X is a random variable4 which can take all the values in an interval then X is called continuous random variable.

Properties of Probability Density Function:

The Probability density function of the random variable X denoted by f(x) has the following properties.

(i) $f(x) \geq 0$

(ii) $\int_{-\infty}^{\infty} f(x)dx = 1$

Cumulative Distribution Function (CDF):

$$F_X(x) = P(X \leq x) = \int_{-\infty}^{x} f(x)dx$$

Properties of CDF:

(i) $F(-\infty) = 0$

(ii) $F(\infty) = 1$

(iii) $\frac{d}{dx}[F(x)] = f(x)$

(iv) $P(X \leq a) = F(a)$

(v) $P(X > a) = 1 - F(a)$

(vi) $P(a \leq X \leq b) = F(b) - F(a)$

Problems on Continuous Random Variables:

1. A continuous random variable X has a density function $f(x) = \frac{K}{1+x^2}$, $-\infty \leq X \leq \infty$. Find the values of K.

Solution:

We know that $\int_{-\infty}^{\infty} f(x)dx = 1$

$$\Rightarrow \int_{-\infty}^{\infty} \frac{K}{1+x^2} dx = 1$$

$$\Rightarrow K \int_{-\infty}^{\infty} \frac{dx}{1+x^2} = 1$$

$$\Rightarrow K \int_{-\infty}^{\infty} \frac{dx}{1+x^2} = 1$$

$$\Rightarrow K[tan^{-1}x]_{-\infty}^{\infty} = 1$$

$$\Rightarrow K[tan^{-1}\infty - tan^{-1}(-\infty)] = 1$$

$$\Rightarrow K\left[\frac{\pi}{2}+\frac{\pi}{2}\right]=1$$

$$\Rightarrow K\left[\frac{2\pi}{2}\right]=1$$

$$\Rightarrow K=\frac{1}{\pi}$$

2. If a random variable X has PDF $f(x)=\begin{cases}\frac{1}{4}, |x|<2\\ 0, |x|>2\end{cases}$ Find (i) $P[X<1]$

(ii) $P[|X|>1]$ $(iii) P[2X+3>5]$

Solution:

(i) $P[X<1]=\int_{-2}^{1} f(x)dx$

$$=\int_{-2}^{1}\frac{1}{4}dx$$

$$=\frac{1}{4}[x]_{-2}^{1}$$

$$=\frac{1}{4}[1-(-2)]$$

$$=\frac{3}{4}$$

(ii) $P[|X|>1]=1-P[-1<X<1]$

$$=1-\int_{-1}^{1} f(x)dx$$

$$=1-\int_{-1}^{1}\frac{1}{4}dx$$

$$=1-\frac{1}{4}[x]_{-1}^{1}$$

$$=1-\frac{1}{4}[1-(-1)]$$

$$= 1 - \frac{2}{4}$$

$$= \frac{2}{4}$$

(iii) $P[2X + 3 > 5] = P[2X > 5 - 3]$

$$= P\left[X > \frac{5-3}{2}\right]$$

$$= P\left[X > \frac{2}{2}\right]$$

$$= P[X > 1]$$

$$= \int_1^2 f(x)dx$$

$$= \int_1^2 \frac{1}{4} dx$$

$$= \frac{1}{4}\ [x]_1^2$$

$$= \frac{1}{4}[2 - (1)] = \frac{1}{4}$$

Mathematical expectation of continuous random variables

(i) $E(X) = \int_{-\infty}^{\infty} x\, f(x)dx$

(ii) $E(X^2) = \int_{-\infty}^{\infty} x^2\, f(x)dx$

(iii) $Var(X) = E(X^2) - E[(X)]^2$

Problems:

1. Let X be a continuous random variable with probability density function $f(x) = kx(2 - x), 0 < x < 2$. Find (i) k (ii) mean (iii) variance (iv) cumulative distribution function of X (v) rth moment.

Solution:

(i) To find k

$$\int_0^2 f(x)dx = 1 \Rightarrow k\int_0^2 (2x - x^2)dx = 1$$

$$\Rightarrow k\left[\frac{2x^2}{2} - \frac{x^3}{3}\right]_0^2 = 1$$

$$\Rightarrow k\left[4 - \frac{8}{3}\right] = 1$$

$$\Rightarrow k\left(\frac{4}{3}\right) = 1$$

$$\Rightarrow k = \frac{3}{4}$$

(ii) To calculate mean of X

$$E(X) = \int_0^2 x\, f(x)dx$$

$$= \int_0^2 x^2\ \frac{3}{4}(2 - x)dx$$

$$= \frac{3}{4}\int_0^2 (2x^2 - x^3)dx$$

$$= \frac{3}{4}\left[\frac{2x^3}{3} - \frac{x^4}{4}\right]_0^2$$

$$= \frac{3}{4}\left(\frac{16}{3} - 4\right)$$

$$= \frac{3}{4} \times \frac{4}{3} = 1$$

(iii) To calculate variance of X

$$E(X^2) = \int_0^2 x^2\, f(x)dx$$

$$= \int_0^2 x^3\ \frac{3}{4}(2-x)dx$$

$$= \frac{3}{4}\int_0^2 (2x^3 - x^4)dx$$

$$= \frac{3}{4}\left[\frac{2x^4}{4} - \frac{x^5}{5}\right]_0^2$$

$$= \frac{3}{4}\left(8 - \frac{32}{5}\right)$$

$$= \frac{3}{4} \times \frac{8}{5} = \frac{6}{5}$$

$$Var(X) = E(X^2) - E[(X)]^2$$

$$= \frac{6}{5} - 1 = \frac{1}{5}$$

(iv) To calculate CDF of X

$$F_X(x) = P(X \le x) = \int_{-\infty}^{x} f(x)dx$$

$$= \int_0^x f(x)dx$$

$$= \int_0^x \frac{3}{4}x(2-x)dx$$

$$= \frac{3}{4}\int_0^x (2x - x^2)dx$$

$$= \frac{3}{4}\left[\frac{2x^2}{2} - \frac{x^3}{3}\right]_0^x$$

$$= \frac{3}{4}\left(x^2 - \frac{x^3}{2}\right)$$

$$= \frac{1}{4}(3x^2 - x^3)$$

$$F(x) = \begin{cases} 0; & x < 0 \\ \frac{1}{4}(3x^2 - x^3); & 0 \le x < 2 \\ 1; & x \ge 2 \end{cases}$$

(v) To find the rth moment:

$$E(x^r) = \int_{-\infty}^{\infty} x^r\, f(x)dx$$

$$= \int_0^2 x^r\, \frac{3}{4}x(2 - x)dx$$

$$= \frac{3}{4}\int_0^2 (2x^{r+1} - x^{r+2})\, dx$$

$$= \frac{3}{4}\left[\frac{2x^{r+2}}{r+2} - \frac{x^{r+3}}{r+3}\right]_0^2$$

$$= \frac{3}{4}\left[\left(2\frac{2^{r+2}}{r+2} - \frac{2^{r+3}}{r+3}\right) - (0 - 0)\right]$$

$$= \frac{3}{4} \times 2^r 2^2 \left[\frac{1}{r+2} - \frac{1}{r+3}\right]$$

$$= 6 \cdot \frac{2^r}{(r+2)(r+3)}$$

2. The probability distribution function of a random variable X is

$$\boldsymbol{f(x) = \begin{cases} x; & 0 < x < 1 \\ 2 - x; & 1 < x < 2 \\ 0; & x > 2 \end{cases}}$$ **Find the cdf of X.**

Solution:

We know that c.d.f $F(x) = \int_{-\infty}^{x} f(x)dx$

(i) When $0 < x < 1$

$$F(x) = \int_{-\infty}^{0} f(x)dx + \int_{0}^{x} f(x)dx$$

$$= 0 + \int_{0}^{x} x\, dx$$

$$= \left[\frac{x^2}{2}\right]_0^x$$

$$= \frac{x^2}{2} - 0$$

$$= \frac{x^2}{2}$$

(ii) When $1 < x < 2$

$$F(x) = \int_{-\infty}^{0} f(x)dx + \int_{0}^{1} f(x)dx + \int_{1}^{x} f(x)dx$$

$$= 0 + \int_{0}^{1} f(x)dx + \int_{1}^{x} x\, dx$$

$$= \left[\frac{x^2}{2}\right]_0^1 + \left[2x - \frac{x^2}{2}\right]_1^x$$

$$= \left(\frac{1}{2} - 0\right) + \left[\left(2x - \frac{x^2}{2}\right) - \left(2 - \frac{1}{2}\right)\right]$$

$$= \frac{1}{2} + 2x - \frac{x^2}{2} - \frac{3}{2}$$

$$= 2x - \frac{x^2}{2} - 1$$

(iii) When $x > 2$

$$F(x) = \int_{-\infty}^{0} f(x)dx + \int_{0}^{1} f(x)dx + \int_{1}^{2} f(x)dx + \int_{2}^{x} f(x)dx$$

$$= 0 + \int_0^1 f(x)dx + \int_1^2 f(x)dx + \int_2^x x\,dx$$

$$= 0 + \int_0^1 x\,dx + \int_1^2 (2-x)dx + 0$$

$$= \left[\frac{x^2}{2}\right]_0^1 + \left[2x - \frac{x^2}{2}\right]_1^2$$

$$= \left(\frac{1}{2} - 0\right) + \left[(4-2) - \left(2 - \frac{1}{2}\right)\right]$$

$$= \frac{1}{2} + 2 - \frac{3}{2} \quad = 1$$

$$F(x) = \begin{cases} \frac{x^2}{2}; & 0 < x < 1 \\ 2x - \frac{x^2}{2} - 1; & 1 < x < 2 \\ 1; & x > 2 \end{cases}$$

3. The Cumulative distribution function of a random variable X is given by

$$\boldsymbol{F(x) = \begin{cases} 0; & x < 0 \\ x^2; & 0 \le x < \frac{1}{2} \\ 1 - \frac{3}{25}(3-x)^2; & \frac{1}{2} \le x < 3 \\ 1, & x \ge 3 \end{cases}}$$

Find the pdf of X and evaluate $P(|X| \le 1)$ using both pdf and cdf.

Solution:

Given

$$F(x) = \begin{cases} 0; & x < 0 \\ x^2; & 0 \le x < \frac{1}{2} \\ 1 - \frac{3}{25}(3-x)^2; & \frac{1}{2} \le x < 3 \\ 1, & x \ge 3 \end{cases}$$

Pdf id $f(x) = \frac{d}{dx}[F(x)]$

$$f(x) = \begin{cases} 0; & x < 0 \\ 2x; & 0 \leq x < \frac{1}{2} \\ \frac{6}{25}(3-x); & \frac{1}{2} \leq x < 3 \\ 0, & x \geq 3 \end{cases}$$

To find $P(|X| \leq 1)$ using cdf:

$$P(|X| \leq 1) = P(-1 \leq X \leq 1)$$

$$= F(1) - F(-1)$$

$$= \left[1 - \frac{3}{25}(3-1)^2\right] - 0$$

$$= 1 - \frac{12}{25}$$

$$= \frac{25-12}{25} = \frac{13}{25}$$

To find $P(|X| \leq 1)$ using pdf:

$$P(|X| \leq 1) = P(-1 \leq X \leq 1)$$

$$= \int_{-1}^{1} f(x)dx$$

$$= \int_{-1}^{0} f(x)dx + \int_{0}^{\frac{1}{2}} f(x)dx + \int_{\frac{1}{2}}^{1} f(x)dx$$

$$= 0 + \int_{0}^{\frac{1}{2}} 2x\,dx + \int_{\frac{1}{2}}^{1} \frac{6}{25}(3-x)dx$$

$$= 2\left(\frac{x^2}{2}\right)_0^{\frac{1}{2}} + \frac{6}{25}\left[3x - \frac{x^2}{2}\right]_{\frac{1}{2}}^{1}$$

$$= \frac{1}{4} + \frac{6}{25}\left[3 - \frac{1}{2} - \frac{3}{2} + \frac{1}{8}\right]$$

$$= \frac{13}{25}$$

Moment Generating Function:

The moment generating function (MGF) of a random variable "X" (about origin) whose probability function f(x) is given by $M_X(t) = E(e^{tx})$

$$= \begin{cases} \displaystyle\int_{-\infty}^{\infty} e^{tx} f(x)dx, for\ a\ continuous\ random\ variable \\ \displaystyle\sum_{x=-\infty}^{\infty} e^{tx} p(x), for\ a\ discrete\ probability\ distribution \end{cases}$$

Problems:

1. If a random variable "X" has the MGF, $M_X(t) = \frac{2}{2-t}$, find the variance of X.

Solution:

Given $M_X(t) = \frac{2}{2-t} = 2(2-t)^{-1}$

$$M_X{}'(t) = -2(2-t)^{-2}(-1)$$

$$= 2(2-t)^{-2}$$

$M_X{}'(t=0) = 2(2-0)^{-2} = \frac{2}{4} = \frac{1}{2}$

$M_X{}''(t) = -4(2-t)^{-3}(-1)$

$= 4(2-t)^{-3}$

$M_X{}''(t=0) = 4(2-0)^{-3} = \frac{4}{8} = \frac{1}{2}$

$Var(X) = E(X^2) - E[(X)]^2$

$= \frac{1}{2} - \frac{1}{4} = \frac{1}{4}$

Moments

The rth moment about origin is $\mu'_r = E[\,x'_r]$

First moment about origin $\mu'_1 = E[X]$= $E(X^2) - [E(X)]^2$

Variance $\sigma^2 = \mu'_2 - (\mu'_1)^2$

The rth moment about mean is $\mu_r = E[(X-\mu)^r], where\ \mu\ is\ mean\ of\ X.$

$\Rightarrow \mu_1 = E[(X-\mu)^1]$

$= E[X] - E[\mu] = \mu - \mu = 0$

$\Rightarrow \mu_1 = 0$

$\Rightarrow \mu_2 = E[(X-\mu)^2]$

$= E[X^2 + \mu^2 - 2X\mu]$

$= E[X^2] + \mu^2 - 2E[X]\mu$

$= E(X^2) + [E(X)]^2 - 2E(X)E(X)$

$= E(X^2) + [E(X)]^2 - 2[E(X)]^2$

$= E(X^2) - [E(X)]^2 = \sigma^2$

$\Rightarrow \mu_2 = \sigma^2$

1. If the probability density of X is given $f(x) = \begin{cases} 2(1-x) & 0 < x < 1 \\ 0 & otherwise \end{cases}$ **Find its r^th moment about origin. Hence find evaluate** $E[(2X+1)^2]$

Solution:

The r^th moment about origin is given by

$\mu_r' = E[\, x_r'] = \int_0^1 x^r\, f(x)dx$

$= \int_0^1 x^r\, 2(1-x)dx$

$= 2\int_0^1 (x^r - x^{r+1})dx$

$= 2\left[\frac{x^{r+1}}{r+1} - \frac{x^{r+1+1}}{r+2}\right]_0^1$

$= 2\left[\frac{1}{r+1} - \frac{1}{r+2}\right]$

$$= 2\left[\frac{(r+2)-(r+1)}{(r+2)(r+1)}\right] = \frac{2}{r^2+3r+2}$$

$$E[(2X+1)^2] = E[4X^2 + 4X + 1]$$

$$= 4E[X^2] + 4E[X] + 1$$

$$= 4\mu_2' + 4\mu_1' + 1$$

$$= 4\frac{2}{2^2+3(2)+2} + 4\frac{2}{2^2+3(2)+2} + 1$$

$$= \frac{8}{12} + \frac{8}{6} + 1 = 3$$

$$\therefore E[(2X+1)^2] = 3$$

Moment Generating Function (MGF)

Let X be a random variable. Then the MGF of X is $M_X(t) = E[e^{tx}]$

If X is a discrete random variable, then the MGF is given by

$$M_X(t) = \sum_x e^{tx} p(x)$$

If X is a continuous random variable, then the MGF is given by

$$M_X(t) = \int_{-\infty}^{\infty} e^{tx} f(x)dx$$

Define MGF and why it is called so?

Solution:

Let X be a random variable. Then the MGF of X is $M_x(t) = E[e^{tX}]$.

Let X be a continuous random variable. Then

$M_X(t)$= $\int_{-\infty}^{\infty} e^{tx}f(x)dx = \int_{-\infty}^{\infty}\left[1+\frac{tx}{1!}+\frac{t^2x^2}{2!}+\cdots\frac{t^rx^r}{r!}+\cdots\right]f(x)dx$

=$\int_{-\infty}^{\infty}\left[f(x)+\frac{tx}{1!}f(x)+\frac{t^2x^2}{2!}f(x)+\cdots\frac{t^rx^r}{r!}f(x)+\cdots\right]dx$

=$\int_{-\infty}^{\infty} f(x)dx+\frac{t}{1!}\int_{-\infty}^{\infty} xf(x)dx+\frac{t^2}{2!}\int_{-\infty}^{\infty} x^2f(x)dx\ldots+\frac{t^r}{r!}\int_{-\infty}^{\infty} x^rf(x)dx+\cdots$

$M_X(t) = 1+\frac{t}{1!}\mu_1'+\frac{t^2}{2!}\mu_2'+\cdots\frac{t^r}{r!}\mu_r'+\cdots\ldots\ldots$

$\because M_X(t)$ generates moments therefore it is moment generation function

Note:

If X is a discrete RV and if $M_X(t)$ is known, then $\mu_r' = \left[\frac{d^r}{dt^r}[M_X(t)]\right]_{t=0}$

If X is a continuous RV and if $M_X(t)$ is known, then μ_r'

$$= r! \times \text{ coeff of } t^r \text{ in } M_X(t)$$

Problems under MGF of discrete random variable

$$M_X(t) = \sum_x e^{tx}p(x)$$

If X is a discrete RV and if $M_X(t)$ is known, then $\mu_r' = \left[\frac{d^r}{dt^r}[M_X(t)]\right]_{t=0}$

1. Let X be the number occur when a die is thrown. Find the MGF mean and variance of X.

Solution:

x	1	2	3	4	5	6
$p(x)$	1/6	1/6	1/6	1/6	1/6	1/6

(i) $M_X(t) = \sum_{x=1}^{6} e^{tx} p(x)$

$$= e^t P(1) + e^{2t} P(2) + e^{3t} P(3) + e^{4t} P(4) + e^{5t} P(5) + e^{6t} P(6)$$

$$= e^t \frac{1}{6} + e^{2t} \frac{1}{6} + e^{3t} \frac{1}{6} + e^{4t} \frac{1}{6} + e^{5t} \frac{1}{6} + e^{6t} \frac{1}{6}$$

$$M_X(t) = \frac{1}{6}[e^t + e^{2t} + e^{3t} + e^{4t} + e^{5t} + e^{6t}]$$

(ii) $E(X) = \left[\frac{d}{dt} M_X(t)\right]_{t=0} = \frac{1}{6}[e^t + 2e^{2t} + 3e^{3t} + 4e^{4t} + 5e^{5t} + 6e^{6t}]_{t=0}$

$$= \frac{1}{6}[1 + 2 + 3 + 4 + 5 + 6] = \frac{21}{6}$$

$$\Rightarrow E(X) = 3.5$$

$$E(X^2) = \left[\frac{d^2}{dt^2}[M_X(t)]\right]_{t=0}$$

$$= \frac{1}{6}[e^t + 4e^{2t} + 9e^{3t} + 16e^{4t} + 25e^{5t} + 36e^{6t}]$$

$$= \frac{1}{6}(1+4+9+16+25+36) = \frac{91}{6}$$

$$= 15.1$$

(iii) Variance of $X = E(X^2) - [E(X)]^2 = 15.1 - 12.25$

$$\sigma_X = 2.85$$

2. Find the moment generating function for the distribution

where $(X = x) = \begin{cases} \frac{2}{3}; x = 1 \\ \frac{1}{3}; x = 2 \\ 0 \text{ ; otherwise} \end{cases}$ **. Also find its mean & variance,**

Solution:

The probability distribution of X is given by

x	1	2
$p(x)$	2/3	1/3

$\Rightarrow M_X(t) = E[e^{tx}] = \sum_{x=1}^{2} e^{tx} p(x)$

$$= e^t p(X=1) + e^{2t} p(X=2) = e^t \frac{2}{3} + e^{2t} \frac{1}{3}$$

$\Rightarrow M_X(t) = \frac{1}{3}(2e^t + e^{2t})$

$\Rightarrow E(X) = M_X'(0)$

$$= \left[\frac{d}{dt}\left[\frac{1}{3}(2e^t + e^{2t})\right]\right]_{t=0}$$

$= \frac{1}{3}(2e^t + 2e^{2t})$

$\Rightarrow E(X) = \frac{4}{3}$

$\Rightarrow E(X^2) = M_X''(0) = \left[\frac{d^2}{dt^2}\left[\frac{1}{3}(2e^t + e^{2t})\right]\right]_{t=0}$

$= \left[\frac{d}{dt}\left[\frac{1}{3}(2e^t + 2e^{2t})\right]\right]_{t=0}$

$= \left[\frac{1}{3}(2e^t + 4e^{2t})_{t=0}\right] = \frac{6}{3} = 2$

Variance of $X = E(X^2) - [E(X)]^2 = 2 - \left(\frac{4}{3}\right)^2$

$\Rightarrow \text{Var}(X) = \frac{2}{9}$

3. Let X be a RV with PMF $P(x) = \left(\frac{1}{2}\right)^x$; $x = 1, 2, 3, \ldots$. Find MGF and hence find mean and variance of X.

Solution:

(i) $M_X(t) = E[e^{tX}]$

$= \sum_{x=1}^{\infty} e^{tx} p(x)$

$= \sum_{x=1}^{\infty} e^{tx} \left(\frac{1}{2}\right)^x$

$= \sum_{x=1}^{\infty} \left(\frac{e^t}{2}\right)^x$

$$= \left[\frac{e^t}{2} + \left(\frac{e^t}{2}\right)^2 + \left(\frac{e^t}{2}\right)^3 + \cdots\right]$$

$$= \frac{e^t}{2}\left(1 + \frac{e^t}{2} + \left(\frac{e^t}{2}\right)^2 + \cdots\right)$$

$$= \frac{e^t}{2}\left(1 - \frac{e^t}{2}\right)^{-1}$$

$$= \frac{e^t}{2}\frac{e^t}{2-e^t}$$

$$\Rightarrow M_X(t) = \frac{e^t}{2-e^t}$$

(ii) $E(X) = \left[\frac{d}{dt}[M_X(t)]\right]_{t=0}$

$$=\left[\frac{d}{dt}\left(\frac{e^t}{2-e^t}\right)\right]_{t=0}$$

$$= \left[\frac{(2-e^t)e^t - e^t(0-e^t)}{(2-e^t)^2}\right]_{t=0}$$

$$= \left[\frac{2e^t - e^{2t} + e^{2t}}{(2-e^t)^2}\right]_{t=0} \quad \because d\left(\frac{u}{v}\right) = \frac{vu' - uv'}{v^2}$$

$$= \left[\frac{2e^t}{(2-e^t)^2}\right]_{t=0} = \frac{2}{1}$$

$$\Rightarrow E(X) = 2$$

$$E(X^2) = \left[\frac{d^2}{dt^2}[M_X(t)]\right]_{t=0}$$

$$= \left[\frac{d}{dt}\left(\frac{2e^t}{(2-e^t)^2}\right)\right]_{t=0}$$

$$= \left[\frac{(2-e^t)^2 2e^t - 2e^t 2(2-e^t)(-e^t)}{(2-e^t)^4}\right]_{t=0} = \frac{2+4}{1} = 6$$

(iii) Variance $= E(X^2) - [E(X)]^2 = 6 - 4$

$$\text{Var}(X) = 2$$

Problems under MGF of discrete random variable

$$M_X(t) = \int_{-\infty}^{\infty} e^{tx} f(x) dx$$

If X is a continuous RV and if $M_X(t)$ is known, then μ_r'

$$= r! \times \text{ coeff of } t^r \text{ in } M_X(t)$$

1. If a random variable "X" has the MGF, $M_X(t) = \frac{2}{2-t}$, find the variance of X.

Solution:

Given $M_X(t) = \frac{2}{2-t} = 2(2-t)^{-1}$

$$M_X{}'(t) = -2(2-t)^{-2}(-1)$$

$$= 2(2-t)^{-2}$$

$M_X{}'(t=0) = 2(2-0)^{-2} = \frac{2}{4} = \frac{1}{2}$

$M_X{}''(t) = -4(2-t)^{-3}(-1)$

$= 4(2-t)^{-3}$

$M_X''(t=0) = 4(2-0)^{-3} = \frac{4}{8} = \frac{1}{2}$

$Var(X) = E(X^2) - E[(X)]^2$

$= \frac{1}{2} - \frac{1}{4} = \frac{1}{4}$

2. Let X be a RV with PDF $f(x) = ke^{-2x}, x \geq 0$. Find (i) k, (ii) MGF, (iii) Mean and (iv) variance

Solution:

Given $f(x) = ke^{-2x}; 0 \leq x < \infty$

(i) To find k

$\Rightarrow \int_0^\infty f(x)dx = 1$

$\Rightarrow \int_0^\infty ke^{-2x}dx = 1$

$\Rightarrow k\left[\frac{e^{-2x}}{-2}\right]_0^\infty = 1$

$\Rightarrow \frac{k}{-2}(e^{-\infty} - 1) = 1$

$\Rightarrow \frac{k}{-2}(0-1) = 1$

$\Rightarrow \frac{k}{2} = 1$

$\Rightarrow k = 2$

(ii) $M_X(t) = E[e^{tx}]$

$$= \int_0^\infty e^{tx} f(x)dx$$

$$= 2\int_0^\infty e^{tx}e^{-2x}dx = 2\int_0^\infty e^{tx-2x}dx$$

$$= 2\int_0^\infty e^{-(2-t)x}dx = 2\left[\frac{e^{-(2-t)x}}{-(2-t)}\right]_0^\infty = 2\left(0+\frac{1}{2-t}\right)$$

$M_X(t) = \frac{2}{2-t}$

(iii) To find Mean and Variance

$$M_X(t) = \frac{2}{2\left(1-\frac{t}{2}\right)} = \left(1-\frac{t}{2}\right)^{-1} = 1 + \frac{t}{2} + \frac{t^2}{2^2} + \cdots$$

Coefficient of $t = \frac{1}{2}$ Coefficient of $t^2 = \frac{1}{2^2}$

Mean $E(X) = \mu_1' = 1! \times$ coefficient of $t \Rightarrow E(X) = \frac{1}{2}$

$E(X^2) = 2! \times$ coefficient of $t^2 = 2 \times \frac{1}{2^2} = \frac{1}{2}$

(iv) Variance $= E(X^2) - [E(X)]^2 = \frac{1}{2} - \frac{1}{4} = \frac{2-1}{4}$

$\text{Var}(X) = \frac{1}{4}$

3. Let X be a continuous RV with PDF $f(x) = Ae^{\frac{-x}{3}}; x \geq 0$. Find (i) A, (ii) MGF,(iii) Mean and (iv) variance

Solution:

Given $f(x) = Ae^{\frac{-x}{3}}; 0 \le x \le \infty$

(i) To find A

$\Rightarrow \int_0^\infty f(x)dx = 1$

$\Rightarrow \int_0^\infty Ae^{\frac{-x}{3}}dx = 1$

$\Rightarrow A\left[\frac{e^{\frac{-x}{3}}}{-\frac{1}{3}}\right]_0^\infty = 1$

$\Rightarrow -3A(0-1) = 1$

$\Rightarrow 3A = 1 \Rightarrow A = \frac{1}{3}$

$\therefore f(x) = \frac{1}{3}e^{\frac{-x}{3}}; 0 \le x \le \infty$

(ii) $M_X(t) = E[e^{tx}] = \int_0^\infty e^{tx}f(x)dx = \frac{1}{3}\int_0^\infty e^{tx}e^{\frac{-x}{3}}dx$

$= \frac{1}{3}\int_0^\infty e^{tx-\frac{x}{3}}dx = \frac{1}{3}\int_0^\infty e^{-\left(\frac{1}{3}-t\right)x}dx = \frac{1}{3}\left[\frac{e^{-\left(\frac{1}{3}-t\right)x}}{-\left(\frac{1}{3}-t\right)}\right]_0^\infty$

$= \frac{1}{3}\left[0 + \frac{1}{\frac{1}{3}-t}\right] = \frac{1}{3}\frac{1}{\frac{1-3t}{3}}$

$= (1-3t)^{-1}$

(iii) To find mean and variance:

$M_X(t) = (1-3t)^{-1}$

$$= 1 + 3t + 9t^2 + 27t^3 + \cdots$$

coefficient of $t = 3$

coefficient of $t^2 = 9$

$E(X) = 1!$ X coefficient of t in $M_X(t) = 1$ X 3

Mean = 3

$E(X) = 2!$ X coefficient of t^2 in $M_X(t)$

$$= 2 \times 9 = 18$$

(iv)Variance = $E(X^2) – [E(X)]^2$ = 18-9

$$Var\ (X) = 9$$

4. Let X be a continuous random variable with the pdf

$$\boldsymbol{f(x) = \begin{cases} x & ; 0 < x < 1 \\ 2 - x & ; 1 < x < 2 \\ 0 & ; \text{elsewhere} \end{cases}}$$ **. Find (i) MGF, (ii) Mean and variance.**

Solution:

Since X is defined in the region $0 < x < 2, X$ is a continuous RV.

$$M_X(t) = E[e^{tX}] = \int_0^2 e^{tx} f(x) dx$$

$$= \int_0^1 e^{tx} x dx + \int_1^2 e^{tx} (2 - x) dx$$

$$= \int_0^1 x e^{tx} dx + \int_1^2 (2-x) e^{tx} dx$$

$$= \left[x\left(\frac{e^{tx}}{t}\right) - 1\left(\frac{e^{tx}}{t^2}\right)\right]_0^1 + \left[(2-x)\frac{e^{tx}}{t} - (-1)\frac{e^{tx}}{t^2}\right]_1^2$$

$$= \left[1\left(\frac{e^{t}}{t}\right) - 1\left(\frac{e^{t}}{t^2}\right) - \left(\frac{-1}{t^2}\right)\right] + \left[0 + \frac{e^{2t}}{t^2} - \frac{e^{t}}{t} - \frac{e^{t}}{t^2}\right]$$

$$= \left[\frac{e^{t}}{t} - \frac{e^{t}}{t^2} + \frac{1}{t^2} + \frac{e^{2t}}{t^2} - \frac{e^{t}}{t} - \frac{e^{t}}{t^2}\right] = \frac{1}{t^2} - \frac{2e^{t}}{t^2} + \frac{e^{2t}}{t^2}$$

$$M_X(t) = \frac{1 - 2e^{t} + e^{2t}}{t^2}$$

To find Mean and Variance:

$$M_X(t) = \frac{1}{t^2}\left[1 - 2\left(1 + \frac{t}{1!} + \frac{t^2}{2!} + \frac{t^3}{3!} + \frac{t^4}{4!} + \cdots\right)\right.$$

$$\left. + \left(1 + \frac{2t}{1!} + \frac{2^2t^2}{2!} + \frac{2^3t^3}{3!} + \frac{2^4t^4}{4!} + \cdots\right)\right]$$

$$\mu_r' = r! \times \text{coefficient of } t^r$$

Coefficient of $t = -\frac{2}{3!} + \frac{2^3}{31} = \frac{-2}{6} + \frac{8}{6} = 1$

Coefficient of $t^2 = -\frac{2}{4!} + \frac{2^4}{4!} = \frac{14}{24} = \frac{7}{12}$

$$\mu_1' = 1! \times \text{coefficient of } t$$

$$\mu_1' = 1$$

Mean $= 1$

$\mu_2' = 2! \times$ coefficient of t^2; $\mu_2' = 2 \times \frac{7}{12} = \frac{7}{6}$

Variance $= \mu_2' - (\mu_1')^2 = \frac{7}{6} - 1 = \frac{1}{6}$

5. Let X be a continuous random variable with PDF $f(x)\frac{1}{2a}; -a < x < a$. Then find the M.G.F of X.

Solution:

Let X is a continuous random variable defined in $-a < x < a$.

$$M_x(t) = E[e^{tx}]$$

$$= \int_{-a}^{a} e^{tx} f(x) dx$$

$$= \int_{-a}^{a} e^{tx} \frac{1}{2a} dx$$

$$= \frac{1}{2a}\left(\frac{e^{tx}}{t}\right)^{a} \quad e^x - e^{-x} = 2\sin hx$$

$$= \frac{1}{2a}\left(\frac{e^{ta} - e^{-ta}}{t}\right)$$

$$= \frac{1}{2a}\frac{2\sinh at}{t}$$

$$M_X(t) = \frac{\sinh at}{at}$$