LINEAR COMBINATIONS

Definition :

Let $v_1, v_2, \ldots, v_m$ be vectors of vector space V. The vector v in V is a linear combination of $v_1, \ldots, v_m$ if there exist scalars $a_1, \ldots, a_m$ such that v can be written as $v = a_1v_1 + a_2v_2 + \ldots + a_mv_m$

Span

Definition :

Let $v_1, v_2, \ldots, v_m$ be vector of vector space V. These vector span V if every vector in V can be expressed as a linear combination of them.

THE SYSTEM OF HOMOGENOUS EQUATIONS

The system of homogenous equations is $AX = 0$

$$\text{where } A = \begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ . & . & . & . \\ . & . & . & . \\ . & . & . & . \\ a_{n1} & a_{n2} & \ldots & a_{mn} \end{bmatrix}, X = \begin{bmatrix} x_1 \\ x_2 \\ . \\ . \\ . \\ x_n \end{bmatrix}, 0 = \begin{bmatrix} 0 \\ 0 \\ . \\ 0 \\ 0 \end{bmatrix}$$

Evidently $X = 0$ is a solution of $AX = 0$ in which $X = 0$, called trivial solution.

There are solutions to $AX = 0$ in which $X \neq 0$, called non-trivial solution.

Note: For $AX = 0$, there is more than one solution.

We have the following two theorems without proof.

Theorem 1 : The system of homogenous equations $AX = 0$ has trivial |ution

$(X = 0)$ if and only if $|A| \neq 0$

Theorem 2 :The system of homogenous equations $AX = 0$ has non-trivial ution

$(X \neq 0)$ if and only if $|A| = 0$.

Find the non-trivial solutions of the equations

$$\boldsymbol{x_1 + 2x_2 - x_3 = 0, 3x_1 + x_2 - x_3 = 0, 2x_1 - x_2 = 0}$$

Sol:

The system is equivalent to

$$AX = \begin{bmatrix} 1 & 2 & -1 \\ 3 & 1 & -1 \\ 2 & -1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\therefore |A| = \begin{vmatrix} 1 & 2 & -1 \\ 3 & 1 & -1 \\ 2 & -1 & 0 \end{vmatrix}$$

$$= 1(0 - 1) - 2(0 + 2) - 1(-3 - 2)$$

$$= -5 + 5 = 0$$

$$\begin{vmatrix} 1 & 2 \\ 3 & 1 \end{vmatrix} \neq 0$$

Hence rank of A is $r = 2$.

n = number of unknown = 3

Therefore, $n - r = 3 - 2 = 1$.

There is only one linearly independent non-zero solution.

Solving actually, by rule of cross multiplication, the equation

$x_1 + 2x_2 - x_3 = 0$

$3x_1 + x_2 - x_3 = 0$ we get,

$$\frac{x_1}{-2+1} = \frac{x_2}{-3+1} = \frac{x_3}{1-6}$$

$\frac{x_1}{-1} = \frac{x_2}{-2} = \frac{x_3}{-5} \frac{x_1}{1} = \frac{x_2}{2} = \frac{x_3}{5}$.

$$x_1 = 1, x_2 = 2, x_3 = 5$$

Solve the system of homogeneous equations

$x_1 + x_2 + 2x_3 = 0, 2x_1 - 3x_2 - x_3 = 0, -3x_1 + 2x_2 + 5x_3 = 0$

The system is equivalent to

$$AX = \begin{bmatrix} 1 & 1 & 2 \\ 2 & -3 & -1 \\ -3 & 2 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$|A| = \begin{vmatrix} 1 & 1 & 2 \\ 2 & -3 & -1 \\ -3 & 2 & 5 \end{vmatrix}$$

$$= 1(-15+2) - 1(10-3) + 2(4-9)$$

$$= -30 \neq 0$$

Therefore the system has a trivial solution

$$x_1 = 0, x_2 = 0, x_3 = 0$$

THE SYSTEM OF NON-HOMOGENOUS EQUATIONS

The system of non-homogenous equations is $AX = B$

$$\text{where } A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ 0 & \cdot & . & 0 \\ . & . & . & . \\ a_{n1} & a_{n2} & \dots & a_{mn} \end{bmatrix}, X = \begin{bmatrix} x_1 \\ x_2 \\ . \\ \vdots \\ x_n \end{bmatrix}, B = \begin{bmatrix} b_1 \\ b_2 \\ . \\ \vdots \\ b_n \end{bmatrix}$$

The system $AX = B$ is said to be consistent if it has a solution. Otherwise it is inconsistence.

Roaches' theorem :

The system $AX = B$ is consistent if and only if $r(A, B) = r(A)$

Note

- If $r(A, B) = r(A) =$ number of unknowns, then the system has unique solution.
- If $r(A, B) = r(A) <$ number of unknowns, then the system has an infinite number of solutions.
- If $\tau(A, B) \neq r(A)$, then the system has no solution.

Show that the equations $+y+z=6, x-y+2z=5\ 3x+y+z=8,$ and, $2x-2y+3z=7$ are consistent and solve them.

Sol:

The system of the given equations is

$$\begin{bmatrix}1 & 1 & 1\\ 1 & -1 & 2\\ 3 & 1 & 1\\ 2 & -2 & 3\end{bmatrix}\begin{bmatrix}x\\ y\\ z\end{bmatrix}=\begin{bmatrix}6\\ 5\\ 8\\ 7\end{bmatrix}$$

$$[A \quad , \quad B]=\begin{pmatrix}1 & 1 & 1 & 6\\ 1 & -1 & 2 & 5\\ 3 & 1 & 1 & 8\\ 2 & -2 & 3 & 7\end{pmatrix}$$

$$\sim\begin{pmatrix}1 & 1 & 1 & 6\\ 0 & -2 & 1 & -1\\ 0 & -2 & -2 & -10\\ 0 & -4 & 1 & -5\end{pmatrix}\begin{matrix}R_2\to R_2-R_1\\ R_3\to R_3-3R_1\\ R_4\to R_4-2R_1\end{matrix}$$

$$\begin{pmatrix}1 & 1 & 1 & 6\\ 0 & -2 & 1 & -1\\ 0 & 0 & 3 & 9\\ 0 & 0 & 1 & 3\end{pmatrix}\begin{matrix}R_3\to R_3-R_2\\ R_4\to R_4-2R_2\end{matrix}$$

$$[A \quad , \quad B]=\begin{pmatrix}1 & 1 & 1 & 6\\ 0 & -2 & 1 & -1\\ 0 & 0 & 3 & 9\\ 0 & 0 & 0 & 0\end{pmatrix} R_4\to 3R_4-R_3$$

$$\text{Now } A\sim\begin{bmatrix}1 & 1 & 1\\ 0 & -2 & 1\\ 0 & 0 & 3\\ 0 & 0 & 0\end{bmatrix}$$

$r(A) =$ number of non-zero rows of A

$= 3$

$r(A, B) =$ number of non-zero rows of $[A, B]$

$= 3$

Since $r(A, B) = r(A) = 3 =$ number of unknowns, the system is consistent unique solution.

$3z = 9$

$\therefore z = 3$

$-2y + z = -1$

$-2y + 3 = -1$

$-2y = -4$

$\therefore y = 2$

$x + y + z = 6$

$x + 2 + 3 = 6$

$\therefore x = 1$

Examine if the following system of equations is consistent and find the solution if it exists. The system of the given equations is $+y + z = 1, 2x -$

$\mathbf{2y + 3z = 1, x - y + 2z = 5,}$ **and,** $\mathbf{3x + y + z = 2}$

Sol: The system of the given equations is

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & -2 & 3 \\ 1 & -1 & 2 \\ 3 & 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 5 \\ 2 \end{bmatrix}$$

The augmented matrix is given by

$$[A, B] = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 2 & -2 & 3 & 1 \\ 1 & -1 & 2 & 5 \\ 3 & 1 & 1 & 2 \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & -4 & 1 & -1 \\ 0 & -2 & 1 & 4 \\ 0 & -2 & -2 & -1 \end{pmatrix} \begin{matrix} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - R_1 \\ R_4 \to R_4 - 3R_1 \end{matrix}$$

$$\sim \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & -4 & 1 & -1 \\ 0 & 0 & 1 & 9 \\ 0 & 0 & -5 & -1 \end{pmatrix} \begin{matrix} R_3 \to 2R_3 - R_2 \\ R_4 \to 2R_4 - R_2 \end{matrix}$$

$$[A, B] \sim \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & -4 & 1 & -1 \\ 0 & 0 & 1 & 9 \\ 0 & 0 & 0 & 44 \end{pmatrix} R_4 \to R_4 + 5R_3$$

$\sim (A) =$ number of non-zeru rows of $[A, B]$

$= 4$

Now $A \sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & 4 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

$r(A) =$ number of non-zero rows of A

$= 3$

$r(A, B) =$ number of non-zero rows of $[A, B]$

$= 3$

Since $r(A, B) \neq r(A)$, the system is inconsistent and has no solution.

Solve the system of equations if consistent

$$\boldsymbol{x_1 + 2x_2 - x_3 - 5x_4 = 4}$$

$$\boldsymbol{x_1 + 3x_2 - 2x_3 - 7x_4 = 5}$$

$$\boldsymbol{2x_1 - x_2 + 3x_3 = 3}$$

Sol: The system of the given equations is

$$\begin{bmatrix} 1 & 2 & -1 & -5 & 4 \\ 1 & 3 & -2 & -7 & 5 \\ 2 & -1 & 3 & 0 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

The augmented matrix is given by

$$[A,B] = \begin{bmatrix} 1 & 2 & -1 & -5 & 4 \\ 1 & 3 & -2 & -7 & 5 \\ 2 & -1 & 3 & 0 & 3 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 2 & -1 & -5 & 4 \\ 0 & 1 & -1 & -2 & 1 \\ 0 & -5 & 5 & 10 & -5 \end{bmatrix} \begin{matrix} R_2 \to R_2 - R_1 \\ R_3 \to R_3 - 2R_1 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 2 & -1 & -5 & 4 \\ 0 & 1 & -1 & -2 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} R_3 \to R_3 + 5R_2$$

$r(A, B) =$ number of non-zero rows of $[A, B]$

$= 2$

$$A \sim \begin{bmatrix} 1 & 2 & -1 & -5 \\ 0 & 1 & -1 & -2 \end{bmatrix} \begin{matrix} 4 \\ 1 \end{matrix}$$

$r(A) =$ number of non-zero rows of A

$= 2$

$(A, B) = r(A) = 2 <$ number of unknowns $= 4$,

The system is consistent and has many solution.

To find the solutions

we have,

$$x_1 + 2x_2 - x_3 - 5x_4 = 4 \text{ ...(1)}$$

and

$$x_2 - x_3 - 2x_4 = 1\text{......(2)}$$

As there are 2 equations, we can solve for only two unknown. Hence other two variables are treated as parameters

Let $x_3 = k_1$, $x_4 = k_2$

$$(2) \Rightarrow x_2 - k_1 - 2k_2 = -1$$

$$x_2 = k_1 + 2k_2 + 1$$

$$(1) \Rightarrow x_1 + 2(k_1 + 2k_2 + 1) - k_1 - 5k_2 = 4$$

$$x_1 + 2k_1 + 4k_2 + \underline{2} - k_1 - 5k_2 = 4$$

$$x_1 + k_1 - k_2 = 2$$

$$x_1 = 2 - k_1 + k_2$$

∴ The given system possess a two parameters family of solution.

LINEAR COMBINATION

Definition : Let V be a vector space over F and $v_1, v_2, \ldots\ldots v_n \in V$. Any vector of the form

$$\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n$$

where $\alpha_1, \alpha_2, \ldots, \alpha_n \in F$, is called a linear combination of the vectors $v_1, v_2, \ldots$,

If $w_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$ and $w_2 = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$, what is the linear combination $w_1 y_1 + w_2 y_2$?

Sol:

$$w_1y_1 + w_2y_2 = \begin{pmatrix}1\\0\\1\end{pmatrix}y_1 + \begin{pmatrix}1\\2\\0\end{pmatrix}y_2$$

$$= \begin{pmatrix}y_1\\0\\y_1\end{pmatrix} + \begin{pmatrix}y_2\\2y_2\\0\end{pmatrix}$$

$$= \begin{pmatrix}y_1 + y_2\\2y_2\\y_1\end{pmatrix}$$

In R^3, determine whether $(5, 1, -5)$ is expressed as a line combination of $(1, -2, -3)$ and $(-2, 3, -4)$.

Sol: Given $v = (5,1,-5), v_1 = (1,-2,-3)$ and $v_2 = (-2,3,-4)$

The linear combination of v_1 and v_2 is

$$v = a_1v_1 + \alpha_2v_2$$

$$(5,1,-5) = a_1(1,-2,-3) + \alpha_2(-2,3,-4) \dots (1)$$

$$= (\alpha_1, -2\alpha_1, -3\alpha_1) + (-2\alpha_2, 3\alpha_2, -4\alpha_2)$$

$$= (\alpha_1 - 2\alpha_2, -2\alpha_1 + 3\alpha_2, -3\alpha_1 - 4\alpha_2)$$

From the equivalent system of equations by setting corresponding components equal to each other and then reduce to echelon form

$$a_1 - 2\alpha_2 = 5 \dots (2)$$

$$-2\alpha_1 + 3\alpha_2 = 1 \dots (3)$$

$$-3\alpha_1 - 4a_2 = -5 \dots .(4)$$

sol ve(2) and (3)

$$(1) \times 2 \Rightarrow 2\alpha_1 - 4\alpha_2 = 10$$

$$(3) \Rightarrow -2\alpha_1 + 3\alpha_2 = 1$$

$$\alpha_2 = -11$$

$(3) \Rightarrow \alpha_1 - 2(-11) = 5$

$$a_1 = -17$$

Substitute the values in (1), we get

$$(5,1,-5) = -17(1,-2,-3) - 11(-2,3,-4)$$

$(5,1,-5) = (5,1,95)$, which is false

$\therefore v$ is not a linear combination of v_1 and v_2

In R^3, determine whether $(1,7,-4)$ is expressed as a linear ;ombination of $u = (1,-3,2)$ and $v = (2,-1,1)$ in R^3.

Sol: We wish to write

$$(1,7,-4) = \alpha_1 u + \alpha_2 v$$

$$= \alpha_1(1,-3,2) + \alpha_2(2,-1,1) \ldots (1)$$

$$= (\alpha_1 + 2\alpha_2, -3\alpha_1 - \alpha_2, 2\alpha_1 + \alpha_2)$$

From the equivalent system of equations by setting corresponding component equal to each other, and then reduce to echelon form

$$\alpha_1 + 2\alpha_2 = 1 \ldots (2)$$

$$-3\alpha_1 - \alpha_2 = 7 \ldots (3)$$

$$2\alpha_1 + \alpha_2 = -4 \ldots \ \ldots (4)$$

***V*erify $2x^3 - 2x^2 + 12x - 6$ is a linear combination of $x^3 - 2x^2 - 5x - 3$ and $3x^3 - 5x^2 - 4x - 9$ in $P_3(R)$.**

Sol: $P(x) = 2x^3 - 2x^2 + 12x - 6, Q(x) = x^3 - 2x^2 - 5x - 3$

and $R(x) = 3x^3 - 5x^2 - 4x - 9$

We wish to write $P(x) = \alpha_1 Q(x) + a_2 R(x)$, with α_1 and α_2 as unknown scalars. Thus

$$2x^3 - 2x^2 + 12x - 6$$

$$= a_1(x^3 - 2x^2 - 5x - 3) + a_2(3x^3 - 5x^2 - 4x - 9) \ldots (1)$$

$$2x^3 - 2x^2 + 12x - 6$$

$$= (\alpha_1 + 3a_2)x^3 + (-2\alpha_1 - 5\alpha_2)x^2 + (-5\alpha_1 - 4\alpha_2)x + (-3a_1 - 9\alpha_2)$$

Equating the co-efficient on both sides, we get

$$a_1 + 3a_2 = 2 \ldots (2)$$

$$-2\alpha_1 - 5\alpha_2 = -2 \ldots (3)$$

$$-5\alpha_1 - 4\alpha_2 = 12 \ldots (4)$$

$$-3\alpha_1 - 9\alpha_2 = -6 \ldots (5)$$

Solve (2) and (3)

$$(2) \times 2 \Rightarrow 2\alpha_1 + 6\alpha_2 = 4$$

Adding $(3) \Rightarrow \frac{-2\alpha_1 - 5\alpha_2 = -2}{\alpha_2 = 2}$

From (2), we get $\alpha_1 + 3(2) = 2$

$$\alpha_1 = 2 - 6$$

$$\therefore \alpha_1 = -4.$$

From (4), $-5\alpha_1 - 4\alpha_2 = 12$

$$-5(-4) - 4(2) = 12$$

20-8=12

12=12

(4) holds good.

From (5), $-3\alpha_1 - 9\alpha_2 = -6$

$$-3(-4) - 9(2) = -6$$

12-18=-6

(5) holds good.

∴ P(x) is a linear combination of Q(x) and R(x).

Seample (44) Verify $3x^3 - 2x^2 + 7x + 8$ is a linear combination of $x^3 - 2x^2 - 5x - 3$ and $3x^3 - 5x^2 - 4x - 9$ in $P_3(R)$

Sol: $P(x) = 3x^3 - 2x^2 + 7x + 8, Q(x) = x^3 - 2x^2 - 5x - 3$

and $R(x) = 3x^3 - 5x^2 - 4x - 9$

We wish to write $P(x) = \alpha_1 Q(x) + \alpha_2 R(x)$, with α_1 and α_2 as unknown scalars. Thus

$$3x^3 - 2x^2 + 7x + 8$$
$$= \alpha_1(x^3 - 2x^2 - 5x - 3) + \alpha_2(3x^3 - 5x^2 - 4x - 9) \ldots (1)$$

$$3x^3 - 2x^2 + 7x + 8$$
$$= (\alpha_1 + 3\alpha_2)x^3 + (-2\alpha_1 - 5\alpha_2)x^2 + (-5\alpha_1 - 4\alpha_2)x + (-3\alpha_1 - 9\alpha_2)$$

Equating the co-efficient on both sides, we get

$$\alpha_1 + 3\alpha_2 = 3 \ldots (2)$$

$$-2\alpha_1 - 5\alpha_2 = -2 \ldots (3)$$

$$-5\alpha_1 - 4\alpha_2 = 7 \ldots (4)$$

$$-3\alpha_1 - 9\alpha_2 = 8 \ldots (5)$$

Solve (2) and (3)

$$(2) \times 2 \Rightarrow 2\alpha_1 + 6\alpha_2 = 6$$

$$(3) \Rightarrow -2\alpha_1 - 5\alpha_2 = -2$$

Adding $\alpha_2 = 4$

From (2), we get $\alpha_1 + 3(4) = 3$

$$\alpha_1 = 3 - 12$$

$$\therefore \alpha_1 = -9$$

From (4), $-5\alpha_1 - 4\alpha_2 = 7$

$$(-9) - 4(4) = 7$$

$$45 - 16 = 7$$

$$29 = 7$$

(4) does not holds good.

∴ P(x) cannot be written as a linear combination of Q(x) and R(x).