4.5 Integration of Rational functions by Partial fraction

Integration of Rational functions by Partial fraction

Let $f(x) = \frac{P(x)}{Q\,(x)}$ be any rational function where $P \; and \; Q$ are polynomials.

If $\deg P < \deg Q,$ then f is proper

If $\deg P \geq \deg Q$, then f is improper then to make them proper divide $P(x) \; by \; Q(x)$ by long division until a remainder $R(x)$ is obtained such that $\deg P < \deg Q$

Hence $\frac{P(x)}{Q\,(x)} = S(x) + \frac{R(x)}{Q\,(x)}$ (or) = Quotient + $\frac{Remainder}{Divisor}$

Where $S \; and \; R$ are also polynomials.

Case (i):

The denominator is a product of distinct linear factors

Example:

$$\frac{1}{(x+a)(x+b)} = \frac{A}{(x+a)} + \frac{B}{(x+b)}$$

Case (ii):

The denominator is a product of distinct linear factors, some of which are repeated.

Example:

$$\frac{1}{(x+a)(x+b)^2} = \frac{A}{(x+a)} + \frac{B}{(x+b)} + \frac{C}{(x+b)^2}$$

Case (iii):

The denominator contains irreducible quadratic factors, none of which is repeated.

Example:

$$\frac{1}{(x^2+a)(x^2+b)} = \frac{Ax+B}{(x^2+a)} + \frac{Cx+D}{(x^2+b)}$$

Example:

Evaluate $\int \frac{(x^2+1)}{(x^2-1)(2x+1)} \, dx$

Solution:

$$\frac{(x^2+1)}{(x^2-1)(2x+1)} = \frac{(x^2+1)}{(x-1)(x+1)(2x+1)}$$

$$= \frac{A}{(x-1)} + \frac{B}{(x+1)} + \frac{C}{(2x+1)}$$

$$(x^2+1) = A(x+1)(2x+1) + B\,(x-1)(2x+1) + C\,(x-1)(x+1)$$

Put $x = 1$, we get

$2 = A\,(2)\,(3)$

$A = \frac{1}{3}$

Put $x = -1$, we get

$2 = B\,(-2)\,(-1)$

$B = 1$

Put $x = 0$, we get

$1 = A - B - C$

$1 = \frac{1}{3} - 1 - C$

$C = -2 + \frac{1}{3} = \frac{-5}{3}$

$$\Rightarrow \frac{(x^2+1)}{(x^2-1)(2x+1)} = \frac{1}{3}\frac{1}{x-1} + \frac{1}{x+1} - \frac{5}{3}\frac{1}{2x+1}$$

$$\int \frac{(x^2+1)}{(x^2-1)(2x+1)}\,dx = \frac{1}{3}\int \frac{1}{x-1}\,dx + \int \frac{1}{x+1}\,dx - \frac{5}{3}\int \frac{1}{2x+1}\,dx$$

$$= \frac{1}{3}\log(x-1) + \log(x+1) - \frac{5}{3}\frac{\log(2x+1)}{2} + C$$

$$= \frac{1}{3}\log(x-1) + \log(x+1) - \frac{5}{6}\log(2x+1) + C$$

Example:

Evaluate $\int \frac{x^2+2x-1}{2x^3+3x^2-2x}\,dx$

Solution:

$$\frac{x^2+2x-1}{2x^3+3x^2-2x} = \frac{x^2+2x-1}{x\,(2x-1)(x+2)} = \frac{A}{x} + \frac{B}{2x-1} + \frac{C}{x+2}$$

$$x^2+2x-1 = A(2x-1)(x+2) + Bx\,(x+2) + Cx\,(2x-1)$$

Put $x = 0$, we get

$-1 = A\,2$

$A = \frac{1}{2}$

Put $x = \frac{1}{2}$, we get

$\frac{1}{4} + 1 - 1 = B\left(\frac{1}{2}\right)\left(\frac{5}{2}\right)$

$\frac{1}{4} = \frac{5B}{4}$

$B = \frac{1}{5}$

Put $x = -2$, we get

$4 - 4 - 1 = C(2)(-5)$

$-1 = 10\,C$

$C = \frac{-1}{10}$

$$\Rightarrow \frac{x^2+2x-1}{2x^3+3x^2-2x} = \frac{1}{2}\left(\frac{1}{x}\right) + \frac{1}{5}\left(\frac{1}{2x-1}\right) - \frac{1}{10}\left(\frac{1}{x+2}\right)$$

$$\int \frac{x^2+2x-1}{2x^3+3x^2-2x} = \frac{1}{2}\int\left(\frac{1}{x}\right)dx + \frac{1}{5}\int\left(\frac{1}{2x-1}\right)dx - \frac{1}{10}\int\left(\frac{1}{x+2}\right)dx$$

$$= \frac{1}{2}\,log x + \frac{1}{5}\frac{\log(2x-1)}{2} - \frac{1}{10}\log(x+2) + C$$

$$= \frac{1}{2}\,log x + \frac{1}{10}\log\left(\frac{2x-1}{x+2}\right) + C$$

Example:

Evaluate $\int \frac{\boldsymbol{x^2}}{(\boldsymbol{x-1})^3(\boldsymbol{x-2})}\,\boldsymbol{dx}$

Solution:

$$\frac{x^2}{(x-1)^3(x-2)} = \frac{A}{x-2} + \frac{B}{x-1} + \frac{C}{(x-1)^2} + \frac{D}{(x-1)^3}$$

$$x^2 = A(x-1)^3 + B(x-1)^2(x-2) + C\,(x-1)(x-2) + D\,(x-2)$$

Put $x = 2$, get | Equating the coeffs of x^3 | Put x = 1, we get | Put x=0, we

On both sides | $1 = D(-1)$ | $0 = -A - 2B + 2C - 2D$

We get $4 = A$ | $0 = A + B$ | $D = -1$ | $2C = A + 2B + 2D$

$B = -4$ | $= 4 - 8 - 2$

$C = -3$

$$\Rightarrow \frac{x^2}{(x-1)^3(x-2)} = \frac{4}{x-2} - \frac{4}{x-1} - \frac{3}{(x-1)^2} - \frac{1}{(x-1)^3}$$

$$I = \int \frac{x^2}{(x-1)^3(x-2)}\,dx$$

$$= 4\int \frac{1}{x-2}\,dx - 4\int \frac{1}{x-1}\,dx - 3\int \frac{1}{(x-1)^2}\,dx - \int \frac{1}{(x-1)^3}\,dx$$

$$= 4\log(x-2) - 4\log(x-1) + 3\left(\frac{1}{x-1}\right) + \frac{1}{2(x-1)^2} + C$$

$$= 4\,log\left(\frac{x-2}{x-1}\right) + \frac{3}{x-1} + \frac{1}{2(x-1)^2} + C$$

Example:

Evaluate $\int \frac{\boldsymbol{1}}{\boldsymbol{x^2(x-1)}}\,\boldsymbol{dx}$

Solution:

Let $I = \int \frac{1}{x^2(x-1)}\,dx$

$$\frac{1}{x^2(x-1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{(x-1)} \quad \dots(1)$$

$$1 = Ax(x-1) + B(x-1) + Cx^2$$

Put $x = 0$, Put $x = 1$, we get Equating the Coefficients of x^2on both side

We get $1 = -B$ $1 = C$ $0 = A + C \Rightarrow A = -C$

$B = -1$ $A = -1$

(1) $\Rightarrow \frac{1}{x^2(x-1)} = \frac{-1}{x} - \frac{1}{x^2} + \frac{1}{(x-1)}$

$$I = \int \frac{1}{x^2(x-1)}\, dx = -\int \frac{1}{x}\, dx - \int \frac{1}{x^2}\, dx + \int \frac{1}{(x-1)} dx$$

$$= -logx + \frac{1}{x} + \log(x-1) + C = log\left(\frac{x-1}{x}\right) + \frac{1}{x} + C$$

Example:

Evaluate $\int \frac{10}{(x-1)(x^2+9)} dx$

Solution:

Let $I = \frac{10}{(x-1)(x^2+9)} dx$

$$\frac{10}{(x-1)(x^2+9)} = \frac{A}{x-1} + \frac{Bx+C}{x^2+9} \qquad ...(1)$$

$$10 = A\,(x^2+9) + (Bx+C)(x-1)$$

Put $x = 1$, We get

$10 = 10\,A$

A = 1

Equating the Coefficients of x^2 We get

$0 = A + B \Rightarrow B = -A$

$B = -1$

Equating the Coefficients of x,

$0 = -B + C \Rightarrow -B = -C$

$C = -1$

$$(1) \Rightarrow \frac{10}{(x-1)(x^2+9)} = \frac{1}{x-1} + \frac{-x-1}{x^2+9} = \frac{1}{x-1} - \left(\frac{x+1}{x^2+9}\right)$$

$$= \int \frac{1}{x-1}\, dx - \int \frac{x}{x^2+9}\, dx - \int \frac{1}{x^2+9} dx$$

$$= \log(x-1) - \frac{1}{2}\, log(x^2+9) - \frac{1}{3} tan^{-1}\left(\frac{x}{3}\right) + C$$

Example:

Evaluate $\int \frac{x^4-2x^2+4x+1}{x^3-x^2-x+1}\, dx$

Solution:

Let $I = \frac{x^4-2x^2+4x+1}{x^3-x^2-x+1}$

$$\begin{array}{r|l} & x+1 \\ \hline x^3-x^2-x+1 & x^4-0x^3-2x^2+4x+1 \\ & x^4-x^3-x^2+x \\ \hline & \quad x^3-x^2+3x+1 \\ & \quad x^3-x^2-x+1 \\ \hline & \qquad 4x \end{array}$$

$$\frac{x^4-2x^2+4x+1}{x^3-x^2-x+1} = x+1+\frac{4x+1}{x^3-x^2-x+1}$$

$$= x+1+\frac{4x+1}{(x-1)^2(x+1)}$$

$$[x^3-x^2-x+1 = (x-1)^2(x+1)]$$

$$\frac{4x}{(x-1)^2(x+1)} = \frac{A}{x-1}+\frac{B}{(x-1)^2}+\frac{C}{(x+1)}$$

$$\Rightarrow 4x = A(x+1)(x+1)+ B(x+1)+C(x+1)^2$$

Put $x = 1$, We get

$4 = 2\,B$

B = 2

Put $x = -1$, We get

$-4 = 4C$

$C = -1$

Equating the Coefficient of x^2 on, both sides , we get

$0 = A+C \Rightarrow A = -C$

$A = 1$

$$\Rightarrow \frac{x^4-2x^2+4x+1}{x^3-x^2-x+1} = (x+1)+\frac{1}{x-1}+\frac{2}{(x-1)^2}-\frac{1}{(x+1)}$$

$$I = \int (x+1)dx+\int \frac{1}{x-1}\,dx+\int \frac{2}{(x-1)^2}\,dx-\int \frac{1}{(x+1)}\,dx$$

$$= \frac{x^2}{2}+x+\log(x-1)-\frac{2}{x-1}-\log(x+1)+C$$

$$= \frac{x^2}{2}+x-\frac{2}{x-1}+\log\left(\frac{x-1}{x+1}\right)+C$$

Improper Integrals

The Integral I=$\int_a^b f(x)dx$ is said to be proper or definite only when the limits $a\ and\ b$ are finite and the integrand $f(x)$ is continuous in the interval $[a,b]$

Types of Improper Integrals

There are two types of improper integrals

1. With infinite limits of integration
2. The integrand is discontinuous.

Type I (Infinite limits of integration)

1. $\int_a^{\infty} f(x)dx = \lim_{t\to\infty} \int_a^t f(x)dx$
2. $\int_{-\infty}^{b} f(x)dx = \lim_{t\to-\infty} \int_t^b f(x)dx$
3. $\int_{-\infty}^{\infty} f(x)dx = \int_{-\infty}^{a} f(x)dx + \int_a^{\infty} f(x)dx, 'a'$ is a real number.

Provided both the limits on right side exist.

Type II (Discontinuous of the integrand)

1. If f is discontinuous at b, then

$$\int_a^b f(x)dx = \lim_{t\to b^-} \int_a^t f(x)dx$$

2. If f is discontinuous at a, then

$$\int_a^b f(x)dx = \lim_{t\to a^+} \int_t^b f(x)dx$$

3. If f is discontinuous at c, in $[a, b]$ then

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$$

$$= \lim_{t\to c^-} \int_a^t f(x)dx + \lim_{t\to c^+} \int_t^b f(x)dx$$

Provided both the integral's on right exists.

Note:

The improper integral is said to be convergent if the limit exists and is divergent if the limit does not exist.

Example:

Determine whether the integral $\int_1^{\infty} \frac{1}{x} dx$ is convergent or divergent.

Solution:

The given integral is $\int_1^{\infty} \frac{1}{x} dx$

an improper integral , since upper limit of integration is infinite then,

$$\int_1^\infty \frac{1}{x}\,dx = \lim_{t\to\infty} \int_1^t \frac{1}{x}\,dx$$

$$= \lim_{t\to\infty} [logx]_1^t$$

$$= \lim_{t\to\infty} [logt - log1]$$

$$= \lim_{t\to\infty} [logt - 0] = \infty$$

The given integral is divergent and it diverges to ∞.

Example:

Determine whether the integral $\int_0^\infty \frac{1}{1+x^2}\,dx$ is convergent or divergent.

Solution:

The given integral is $\int_0^\infty \frac{1}{1+x^2}\,dx$ an improper integral, since upper limit of integration is infinite then,

$$\int_0^\infty \frac{1}{1+x^2}\,dx = \lim_{t\to\infty} \int_0^t \frac{1}{1+x^2}\,dx$$

$$= \lim_{t\to\infty} [tan^{-1}x]_0^t$$

$$= \lim_{t\to\infty} [tan^{-1}t - tan^{-1}0]$$

$$= \lim_{t\to\infty} tan^{-1}t$$

$$= tan^{-1}\infty = \frac{\pi}{2}$$

The given integral is convergent.

Example:

For what values of p the integral $\int_1^\infty \frac{1}{x^p}\,dx$ convergent?

Solution:

If $p \neq 1, \lim_{t\to\infty} \int_1^t \frac{1}{x^p}\,dx = \lim_{t\to\infty} \int_1^t x^{-p}dx$

$$= \lim_{t\to\infty} \left[\frac{x^{-p+1}}{-p+1}\right]_1^t$$

$$= \lim_{t\to\infty} \left[\frac{t^{-p+1}}{-p+1} - \frac{1}{-p+1}\right]$$

$$= \lim_{t\to\infty} \frac{1}{p-1}\left[1 - \frac{1}{t^{p-1}}\right]$$

$$= \begin{cases} \frac{1}{p-1}, p > 1, converges \\ \infty, \quad\quad p \leq 1, diverges \end{cases}$$

Example:

Evaluate $\int_1^\infty \frac{logx}{x} dx$

Solution:

Take $I = \int \frac{logx}{x} dx$

Put $u = logx \quad dv = \frac{1}{x}dx \quad\quad du = \frac{1}{x}dx \quad v = logx$

$$I = \int \frac{logx}{x} dx = (logx)^2 - \int logx \left(\frac{1}{x}\right) dx$$

$$I = (logx)^2 - I \Rightarrow 2I = (logx)^2 \Rightarrow I = \frac{1}{2}(logx)^2$$

$$\int_1^\infty \frac{logx}{x} dx = \lim_{t\to\infty} \int_1^t \frac{logx}{x} dx = \lim_{t\to\infty} \left(\frac{1}{2}(logx)^2\right)_1^t$$

$$= \lim_{t\to\infty} \left[\frac{1}{2}(logt)^2 - \frac{1}{2}(log1)^2\right]$$

$$= \lim_{t\to\infty} \left[\frac{1}{2}(logt)^2\right] = \infty \quad\quad [\log 1 = 0, \log\infty = \infty]$$

The given integral is divergent.

Example:

Evaluate $\int_{-\infty}^\infty xe^{-x^2} dx$

Solution:

Consider $\int xe^{-x^2} dx$

Put $u = x^2, \quad\quad du = 2xdx$

$$\int xe^{-x^2} dx = \int e^{-u} \frac{du}{2} = \frac{1}{2}\left[\frac{e^{-u}}{-1}\right]$$

$$= -\frac{1}{2}e^{-u} = -\frac{1}{2}e^{-x^2} \ldots (1)$$

$$\int_{-\infty}^\infty xe^{-x^2} dx = \int_{-\infty}^0 xe^{-x^2} dx + \int_0^\infty xe^{-x^2} dx \ldots(2)$$

Take $\int_{-\infty}^0 xe^{-x^2} dx = \lim_{t\to-\infty} \int_t^0 xe^{-x^2} dx = \lim_{t\to-\infty} \left[\frac{-1}{2}e^{-x^2}\right]_t^0 by\ (1)$

$$= \lim_{t\to-\infty} \left[\frac{-1}{2} + \frac{1}{2}e^{-t^2}\right] = \frac{-1}{2}$$

Take $\int_0^{\infty} xe^{-x^2} dx = \lim_{t\to\infty} \int_0^t xe^{-x^2} dx = \lim_{t\to\infty} \left[\frac{-1}{2} e^{-x^2}\right]_0^t$ by (1)

$$= \lim_{t\to\infty} \left[\frac{-1}{2} e^{-t^2} + \frac{1}{2}\right] = \frac{1}{2}$$

$$\therefore (2) \Rightarrow \int_{-\infty}^{\infty} xe^{-x^2} dx = \frac{-1}{2} + \frac{1}{2} = 0$$

Example:

Evaluate $\int_3^{\infty} \frac{1}{(x-2)^{3/2}} dx$

Solution:

Consider $\int \frac{1}{(x-2)^{3/2}} dx \ldots(1)$

Put $u = x - 2 \quad \Rightarrow du = dx$

$$(1) \Rightarrow \int \frac{1}{(x-2)^{3/2}} dx = \int \frac{1}{u^{3/2}} du = \int u^{-3/2}\, du = \frac{u^{-3/2+1}}{-3/2+1} = \frac{u^{-1/2}}{-1/2}$$

$$\frac{-2}{\sqrt{u}} = \frac{-2}{\sqrt{x-2}}$$

$$\int_3^{\infty} \frac{1}{(x-2)^{3/2}} dx = \lim_{t\to\infty} \left[\int_3^t \frac{1}{(x-2)^{3/2}} dx\right] = \lim_{t\to\infty} \left[\frac{-2}{\sqrt{x-2}}\right]_3^t$$

$$= \lim_{t\to\infty} \left[\left(\frac{-2}{\sqrt{t-2}}\right) - \left(\frac{-2}{\sqrt{1}}\right)\right]$$

$$= \lim_{t\to\infty} \left(\frac{-2}{\sqrt{t-2}}\right) + 2 = 0 + 2 = 2 \text{ (finite)}$$

The given integral $\int_3^{\infty} \frac{1}{(x-2)^{3/2}} dx$ is convergent.

Example:

Evaluate $\int_0^2 \frac{1}{\sqrt{x}} dx$

Solution:

Here, infinite discontinuity occurs at x=0

$$\therefore \int_0^2 \frac{1}{\sqrt{x}} dx = \lim_{t\to 0^+} \int_t^2 x^{-1/2}\, dx$$

$$= \lim_{t\to 0^+} \left[\frac{x^{1/2}}{1/2}\right]_t^2$$

$$= \lim_{t\to 0^+} \left[2\sqrt{x}\right]_t^2$$

$$= \lim_{t \to 0^+} \left[2\sqrt{2} - 2\sqrt{t}\right]$$

$$= 2\sqrt{2} \text{ (finite)}$$

The given integral $\int_0^2 \frac{1}{\sqrt{x}} dx$ is convergent.

Example:

Evaluate $\int_0^3 \frac{1}{x-1} dx$

Solution:

Here, infinite discontinuity occurs at $x = 1$

$$\therefore \int_0^3 \frac{1}{x-1} dx = \int_0^1 \frac{1}{x-1} dx + \int_1^3 \frac{1}{x-1} dx$$

Take $\int_0^1 \frac{dx}{x-1} = \lim_{t \to 1^-} \int_0^t \frac{1}{x-1} dx = \lim_{t \to 1^-} [\log(x-1)]_0^t$

$$= \lim_{t \to 1^-} \log(t-1) = -\infty$$

$\int_0^1 \frac{1}{x-1} dx$ is divergent.

$\Rightarrow \int_1^3 \frac{1}{x-1} dx$ is also divergent.

The given integral $\int_0^3 \frac{1}{x-1} dx$ isdivergent

Example:

Evaluate $\int_2^5 \frac{1}{\sqrt{x-2}} dx$

Solution:

The infinite discontinuity occurs at $x = 2$

$$\therefore \int_2^5 \frac{1}{\sqrt{x-2}} dx = \lim_{t \to 2^+} \int_2^5 \frac{1}{\sqrt{x-2}} dx$$

$$= \lim_{t \to 2^+} \left[2\sqrt{x-2}\right]_t^5$$

$$= \lim_{t \to 2^+} (2\sqrt{3} - 2\sqrt{t-2})$$

$$= 2\sqrt{3} \text{ (finite)}$$

The given integral $\int_2^5 \frac{1}{\sqrt{x-2}} dx$ is convergent.

Example:

Evaluate $\int_0^3 \frac{1}{(x-1)^{2/3}}$ dx

Solution:

Here infinite discontinuity occurs at $x = 1$

1) $\int_0^3 \frac{1}{(x-1)^{2/3}} dx = \int_0^1 \frac{1}{(x-1)^{2/3}} dx + \int_1^3 \frac{1}{(x-1)^{2/3}} dx$ ...(1)

Take $\int_0^1 \frac{1}{(x-1)^{2/3}} dx = \lim_{t\to 1^-} \int_0^t \frac{1}{(x-1)^{2/3}} dx$

$$= \lim_{t\to 1^-} \left[3(x-1)^{1/3}\right]_0^t$$

$$= \lim_{t\to 1^-} \left[3(t-1)^{1/3} + 3\right]$$

$$= 3$$

Take $\int_1^3 \frac{1}{(x-1)^{2/3}} dx = \lim_{t\to 1^+} \int_t^t \frac{1}{(x-1)^{2/3}} dx$

$$= \lim_{t\to 1^+} \left[3(x-1)^{1/3}\right]_t^3$$

$$= \lim_{t\to 1^+} \left[3\left[2^{1/3} - (t-1)^{1/3}\right]\right]$$

$$= 3\left(2^{1/3}\right)$$

(1)⇒ $\int_0^3 \frac{1}{(x-1)^{2/3}} dx = 3 + 3\left(2^{1/3}\right)$

$$= 3 \left[1 + 2^{1/3}\right]$$

Comparison test for improper integrals

Let $\int_a^b f(x)dx$ be an improper integral.

i) If there exists a $g(x)$ such that $|f(x)| \le g(x)$ for all x in $[a,b]$ and $\int_a^b g(x)dx$ converges then $\int_a^b f(x)dx$ also converges.

ii) If there exists function $g(x)$ such that $f(x) \ge |g(x)|$ for all x in $[a,b]$ and $\int_a^b g(x)dx$ diverges then $\int_a^b f(x)dx$ also diverges.

Limit form of comparison Tests.

Let $f(x) > 0$ and $g(x) > 0$and $\lim_{x\to\infty} \frac{f(x)}{g(x)} = k$ where $k \neq 0$

Then, the improper integrals $\int_a^\infty f(x)dx$ and $\int_a^\infty g(x)dx$ converge or diverge together.

If $k = 0$, only the convergence of $\int_a^\infty g(x)dx$ implies that of $\int_a^\infty f(x)dx$

Absolute Convergence

The improper integral $\int_a^b f(x)dx$ is said to be absolutely convergent if $\int_a^b |f(x)|dx$ is convergent.

Note:

1) The same definition holds for $\int_a^\infty f(x)dx$ also
2) When the improper integral changes sign within the limits of the integration, then the above test is applied.

Example:

Discuss the convergence of $\int_1^\infty \frac{xtan^{-1}x}{\sqrt{4+x^3}}\ dx$

Solution:

Let $f(x) = \frac{xtan^{-1}x}{\sqrt{4+x^3}} = \frac{tan^{-1}x}{\sqrt{x}\sqrt{1+4x^{-3}}}$ and $g(x) = \frac{1}{\sqrt{x}}$

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{tan^{-1}x}{\sqrt{1+4x^{-3}}}$$

$$= \frac{\pi}{2}$$

Hence, by comparision test 2, the integrals $\int_1^\infty f(x)dx$ and $\int_1^\infty g(x)dx$ converge or diverge together, Now $\int_1^\infty g(x)dx$ is divergent.

$\therefore\ \int_1^\infty f(x)dx$ is also divergent.

Example :

Discuss the convergence of $\int_1^\infty \frac{sinx}{x^4} dx$

Solution:

$$\left|\int_1^\infty \frac{sinx}{x^4} dx\right| \le \int_1^\infty \left|\frac{sinx}{x^4}\right| dx \le \int_1^\infty \frac{dx}{x^4}$$

$\Rightarrow$ convergent

$\int_1^\infty \frac{sinx}{x^4} dx$ is absolutely convergent and hence convergent.

Example:

Test the convergence of $\int_0^\infty e^{-x^2}\ dx$

Solution:

The given integral $\int_0^\infty e^{-x^2}\ dx$is an improper integral of first kind and the integral can be written as $\int_0^\infty e^{-x^2}\ dx = \int_0^1 e^{-x^2}\ dx + \int_1^\infty e^{-x^2}\ dx$

The first integral in the right hand side $\int_0^1 e^{-x^2}\ dx$ is proper integral. So it is enough to check the second one.

We have that,

$$x \geq 1$$

$$x^2 \geq x$$

$$-x^2 \leq -x$$

$$e^{-x^2} \leq e^{-x}$$

$$\int_1^\infty e^{-x^2}\ dx \leq \int_1^\infty e^{-x}\ dx$$

$$= \lim_{b\to\infty} \int_1^b e^{-x}\ dx \qquad = \lim_{b\to\infty} [-e^{-x}]_1^b$$

$$= \lim_{b\to\infty} [e^{-1} - e^{-b}]$$

$$= [e^{-1} - 0] = \frac{1}{e}$$

Hence by comparison test the given integral is convergent.