UNIT – V

MULTIPLE INTEGRALS

5.1 Introduction

The mathematical modeling of any engineering problem which leads to the formation of differential equation of more than one variable has its solution by the integration in terms of those variables the need of the solution in an integral where many variables are involved motivated the study of integral calculus of several variables.

In this chapter all the basic concepts related to the methods to approach such integrals are discussed.

Double integration in Cartesian co – ordinates

Let $f(x, y)$ be a single valued function and continuous in a region R bounded by a closed curve C. Let the region R be subdivided in any manner into n sub regions $R_1, R_2, R_3, \cdots, R_n$ of areas $A_1, A_2, A_3, \cdots, A_n$.Let $(x_i,\ y_j)$ be any point in the sub regionR_i. Then consider the sum formed by multiplying the area of each sub – region by the value of the function $f(x, y)$ at any point of the sub – region and adding up the products which we denote

$$\sum_1^n f(x_i, y_j)A_i$$

The limit of this sum (if it exists) as $n \to \infty$ in such a way that each $A_i \to 0$ is defined as the double integral of $f(x, y)$ over the region R. Thus

$$\lim_{n \to \infty} \sum_1^n f(x_i, y_j)A_i = \iint_R f(x, y)\, dA$$

The above integral can be given as

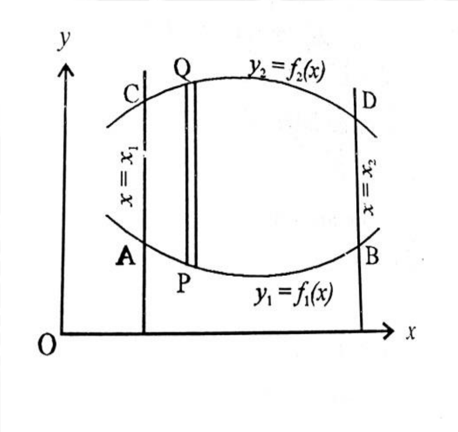
$$\iint_R f(x, y) dy dx \quad or \quad \iint_R f(x, y) dx dy$$

Evaluation of Double Integrals

To evaluate $\int_{y_0}^{y_1}\int_{x_0}^{x_1} f(x, y) dx\, dy$ we first integrate f(x, y) with respect to x partially, that is treating y as a constant temporarily, between x_0 and x_1. The resulting function got after the inner integration and substitution of limits will be function of y. Then we integrate this function of with respect to y between the limits y_0 and $y_1$as used.

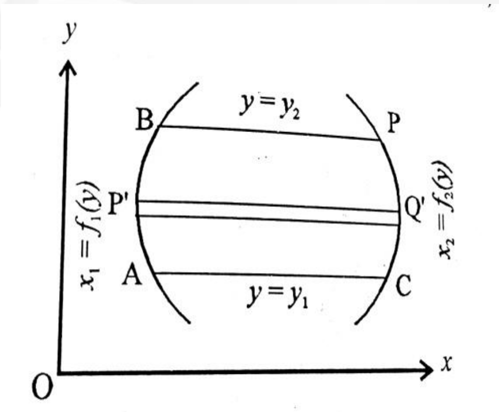
Region of Integration

Case (i) Consider the integral $\int_a^b \int_{f_1(x)}^{f_2(x)} f(x,y)dy\,dx$ Given that y varies from $y = f_1(x)$ to $y = f_2(x)$ x varies from $x = a\ to\ x = b$. We get the region R by $y = f_1(x),\ y = f_2(x),\ x = a\ ,\ x = b$. The points A, B, C, D are obtained by solving the intersecting curves. Here the region divided into vertical strips $(dy\,dx)$.



Case (ii) Consider the integral $\int_c^d \int_{f_1(y)}^{f_2(y)} f(x,y)dx\,dy$

Here varies from $x = f_1(y)$ to $x = f_2(y)$ and y varies from $y = c\ to\ y = d$ ∴ the region is bounded by $x = f_1(y),\ x = f_2(y),\ y = c\ ,\ y = d$. The points P, Q, R, S are obtained by solving the intersecting curves. Here the region divided into horizontal strips $(dx\,dy)$.



Problems based on Double Integration in Cartesian co-ordinates

Example:

Evaluate $\int_0^1 \int_1^2 x(x+y)dydx$

Solution:

$$\int_0^1 \int_1^2 x(x+y)dydx \quad = \int_0^1 \int_1^2 (x^2+xy)dydx$$

$$= \int_0^1 \left[x^2y + \frac{xy^2}{2}\right]_1^2 dx$$

$$= \int_0^1 \left[(2x^2+2x) - (x^2 + \frac{x}{2})\right] dx$$

$$= \int_0^1 \left[2x^2 + 2x - x^2 - \frac{x}{2})\right] dx$$

$$= \int_0^1 \left[x^2 + \frac{3}{2}x\right] dx$$

$$= \left[\frac{x^3}{3} + \frac{3}{2}\frac{x^2}{2}\right]_0^1 \quad = \left(\frac{1}{3} + \frac{3}{4}\right) - (0+0) \quad = \frac{13}{12}$$

Example:

Evaluate $\int_0^a \int_0^b xy(x-y)dydx$

Solution:

$$\int_0^a \int_0^b xy(x-y)dydx \quad = \int_0^a \int_0^b (x^2y - xy^2)dydx$$

$$= \int_0^a \left[\frac{x^2y^2}{2} - \frac{xy^3}{3}\right]_0^b dx$$

$$= \int_0^a \left[\left(\frac{b^2x^2}{2} - \frac{b^3x}{2}\right) - (0-0)\right] dx$$

$$= \left[\left(\frac{b^2x^3}{6} - \frac{b^3x^2}{6}\right)\right]_0^a$$

$$= \left(\frac{a^3b^2}{6} - \frac{a^2b^3}{6}\right) - (0-0)$$

$$= \frac{a^2b^2}{6}(a-b)$$

Example:

Evaluate $\int_2^a \int_2^b \frac{dxdy}{xy}$

Solution:

$$\int_2^a \int_2^b \frac{dxdy}{xy} \quad = \int_2^a \left[\frac{1}{y} logx\right]_2^b dy$$

$$=\int_2^a \frac{1}{y}(logb - log2)dy$$

$$= \int_2^a \frac{1}{y} log\left(\frac{b}{2}\right) dy \qquad \left[\because log\frac{a}{b} = loga - logb\right]$$

$$=log\frac{b}{2}\int_2^a \frac{1}{y} dy \quad = \ log\frac{b}{2}[\log y]_2^a$$

$$= \ log\frac{b}{2}[loga - log2] \quad =\left[log\frac{b}{2}\right]\left[log\frac{a}{2}\right]$$

Example:

Evaluate $\int_0^1 \int_2^3 (x^2 + y^2)dxdy$

Solution:

$$\int_0^1 \int_2^3 (x^2 + y^2)dxdy \ = \int_0^1 \left[\frac{x^3}{3} + y^2 x\right]_2^3 dy$$

$$=\int_0^1 \left[\left(\frac{3^3}{3} + 3y^2\right) - \left(\frac{2^3}{3} + 2y^2\right)\right] dy$$

$$=\int_0^1 \left[9 + 3y^2 - \frac{8}{3} - 2y^2\right] dy$$

$$=\int_0^1 \left[\frac{19}{3} + y^2\right] dy \quad =\left[\frac{19y}{3} + \frac{y^3}{3}\right]_0^1$$

$$=\left[\frac{19}{3} + \frac{1}{3}\right] \ = \ \frac{20}{3}$$

Example:

Evaluate $\int_0^3 \int_0^2 e^{x+y}\, dydx$

Solution:

$$\int_0^3 \int_0^2 e^{x+y}\, dydx \ = \int_0^3 \int_0^2 e^x\, e^y dydx \ =\left[\int_0^3 e^x\, dx\right]\left[\int_0^2 e^y\, dy\right]$$

$$=[e^x]_0^3 [e^y]_0^2 \quad = [e^3 - e^0][e^2 - e^0]$$

$$=[e^3 - 1][e^2 - 1]$$

Note: If the limits are variable, then check the given problem is in the correct form

Rule: (i) The limits for the inner integral are functions of , then the first integral is with respect to y

(ii) The limits for the inner integral are functions of , then the first integral is with respect to x

Example:

Evaluate $\int_0^a \int_0^{\sqrt{a^2-x^2}} dxdy$

Solution:

The given integral is in incorrect form

Thus the correct form is

$$\int_0^a \int_0^{\sqrt{a^2-x^2}} dydx = \int_0^a [y]_0^{\sqrt{a^2-x^2}} dx = \int_0^a \left[\sqrt{a^2-x^2}\right] dx$$

$$= \left[\frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2} sin^{-1}\frac{x}{a}\right]_0^a$$

$$= \left[\left(0 + \frac{a^2}{2} sin^{-1} 1\right) - (0+0)\right] \qquad \left[\because sin^{-1} 1 = \frac{\pi}{2}, sin^{-1} 0 = 0\right]$$

$$= \frac{a^2}{2}\left(\frac{\pi}{2}\right) = \frac{\pi a^2}{4}$$

Example:

Evaluate $\int_0^a \int_0^{\sqrt{a^2-x^2}} y(x^2+y^2) dxdy$

Solution:

The given integral is in incorrect form

Thus the correct form is

$$\int_0^a \int_0^{\sqrt{a^2-x^2}} y(x^2+y^2) dydx = \int_0^a \int_0^{\sqrt{a^2-x^2}} (x^2y+y^3) dydx$$

$$= \int_0^a \left[\frac{x^2y^2}{2} + \frac{y^4}{4}\right]_0^{\sqrt{a^2-x^2}} dx$$

$$= \int_0^a \left[\frac{x^2(a^2-x^2)}{2} + \frac{(a^2-x^2)^2}{4}\right] dx$$

$$= \int_0^a \left[\frac{a^2x^2}{2} - \frac{x^4}{2} + \frac{a^4}{4} + \frac{x^4}{4} - \frac{2a^2x^2}{4}\right] dx$$

$$= \left[\frac{a^2x^3}{6} - \frac{x^5}{10} + \frac{a^4x}{4} + \frac{x^5}{20} - \frac{2a^2x^3}{12}\right]_0^a$$

$$=\left[\frac{-x^5}{10}+\frac{a^4x}{4}+\frac{x^5}{20}\right]_0^a$$

$$=\left[\frac{-a^5}{10}+\frac{a^5}{4}+\frac{a^5}{20}\right]$$

$$=\frac{a^5}{5}$$

Example:

Evaluate $\int_0^1\int_x^{\sqrt{x}} xy(x+y)\,dxdy$

Solution:

The given integral is in incorrect form

Thus the correct form is

$$\int_0^1\int_x^{\sqrt{x}} xy(x+y)dydx = \int_0^1\int_x^{\sqrt{x}}(x^2y+xy^2)dydx$$

$$=\int_0^1\left[\frac{x^2y^2}{2}+\frac{xy^3}{3}\right]_x^{\sqrt{x}}dx$$

$$=\int_0^1\left[\left(x^2\frac{x}{2}+x\frac{x^{3/2}}{3}\right)-\left(x^2\frac{x^2}{2}+x\frac{x^3}{3}\right)\right]dx$$

$$=\int_0^1\left[\frac{x^3}{2}+\frac{x^{5/2}}{3}-\frac{5}{6}x^4\right]dx$$

$$=\left[\frac{x^4}{8}+\frac{x^{7/2}}{3(7/2)}-\frac{5}{6}\frac{x^5}{5}\right]_0^1$$

$$=\left(\frac{1}{8}+\frac{2}{21}-\frac{1}{6}\right)-(0+0-0) = \frac{3}{56}$$

Example:

Evaluate $\int_0^1\int_0^{\sqrt{1+x^2}}\frac{dxdy}{1+x^2+y^2}$

Solution:

The given integral is in incorrect form

Thus the correct form is

$$\int_0^1\int_0^{\sqrt{1+x^2}}\frac{dydx}{1+x^2+y^2} = \int_0^1\int_0^{\sqrt{1+x^2}}\frac{dydx}{\left(\sqrt{1+x^2}\right)^2+y^2}$$

$$=\int_0^1 \left[\frac{1}{\sqrt{1+x^2}} tan^{-1}\left(\frac{y}{\sqrt{1+x^2}}\right)\right]_0^{\sqrt{1+x^2}} dx$$

$$=\int_0^1 \left[\frac{1}{\sqrt{1+x^2}} tan^{-1}(1) - 0\right] dx \qquad \left[\because tan^{-1}(1) = \frac{\pi}{4}\right]$$

$$=\int_0^1 \frac{1}{\sqrt{1+x^2}} \frac{\pi}{4} dx \;=\; \frac{\pi}{4}\int_0^1 \frac{1}{\sqrt{1+x^2}}\, dx \qquad [tan^{-1}(0) = 0]$$

$$= \frac{\pi}{4}\left[log\left[x + \sqrt{1+x^2}\right]\right]_0^1$$

$$= \frac{\pi}{4}\log(1+\sqrt{2})$$

Example:

Evaluate $\int_0^4 \int_0^{x^2} e^{y/x}\, dydx$

Solution:

The given integral is in correct form

$$\int_0^4 \int_0^{x^2} e^{y/x}\, dydx = \int_0^4 \left[\frac{e^{y/x}}{1/x}\right]_0^{x^2} dx$$

$$=\int_0^4 \left[\left(\frac{e^x}{1/x}\right) - \left(\frac{1}{1/x}\right)\right] dx$$

$$=\int_0^4 [xe^x - x] dx \;=\int_0^4 x(e^x - 1)\, dx$$

$$=\left[x(e^x - x) - (1)\left(e^x - \frac{x^2}{2}\right)\right]_0^4 \qquad \text{(by Bernoulli's formula)}$$

$$=\left[4(e^4 - 4) - \left(e^4 - \frac{16}{2}\right) - (0-1)\right]$$

$$=4e^4 - 16 - e^4 + 8 + 1$$

$$=3e^4 - 7$$

Example:

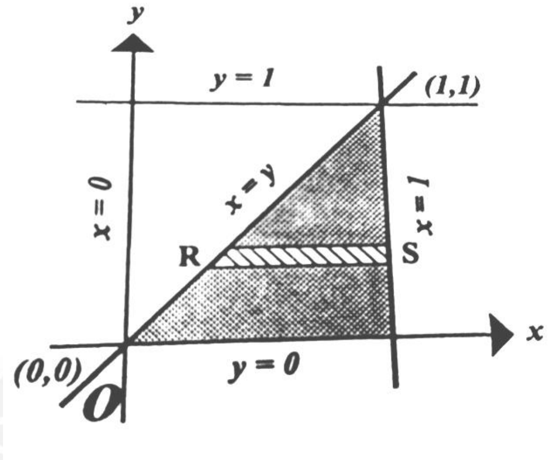
Sketch roughly the region of integration for $\int_0^1 \int_0^x f(x,y)\, dydx$

Solution:

Given $\int_0^1 \int_0^x f(x,y)\, dydx$

x varies from $x = 0$ to $x = 1$

y varies from $y = 0$ to $y = x$



Example:

Shade the region of integration $\int_0^a \int_{\sqrt{ax-x^2}}^{\sqrt{a^2-x^2}} dxdy$

Solution:

$\int_0^a \int_{\sqrt{ax-x^2}}^{\sqrt{a^2-x^2}} dydx$ is the correct form

x limit varies from $x = 0$ to $x = a$

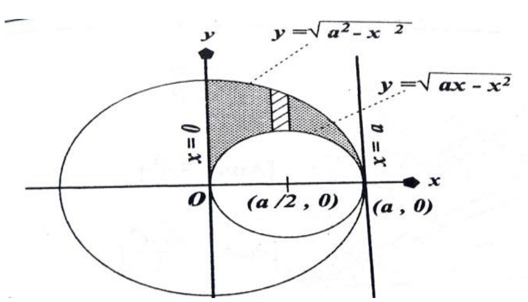
y limit varies from $y = \sqrt{ax - x^2}$ to $y = \sqrt{a^2 - x^2}$

i.e., $y^2 = ax - x^2$ to $y^2 = a^2 - x^2$

i.e., $y^2 + x^2 = ax$ to $y^2 + x^2 = a^2$

$x^2 + y^2 = ax$ is a circle with centre $\left(\frac{a}{2}, 0\right)$ and radius $\frac{a}{2}$

$x^2 + y^2 = a^2$ is a circle with centre $(0,0)$ and radius a



Exercise

Evaluate the following integrals

1. $\int_0^1 \int_0^{x^2} (x^2+y^2)\, dydx$ **Ans:** $\frac{26}{105}$

2. $\int_0^1 \int_x^1 (x^2+y^2)\, dxdy$ **Ans:** $\frac{1}{3}$

3. $\int_0^a \int_y^a \frac{x}{x^2+y^2}\, dxdy$ **Ans:** $\frac{\pi a}{4}$

4. $\int_1^2 \int_1^3 (xy^2)\, dxdy$ **Ans:** 13

5. $\int_0^3 \int_1^{\sqrt{4-y}} (x+y)\, dxdy$ **Ans:** $\frac{241}{60}$

6. $\int_0^1 \int_x^{\sqrt{2-x^2}} \frac{x}{\sqrt{x^2+y^2}}\, dydx$ **Ans:** $1-\frac{1}{\sqrt{2}}$

7. $\int_0^1 \int_0^x e^{x+y}\, dydx$ **Ans:** $\frac{1}{2}(e-1)^2$

8. $\int_{-1}^3 \int_{x^2}^{3x+3} dydx$ **Ans:** $\frac{32}{3}$

9. $\int_{-1}^2 \int_{x^2}^{x+2} dydx$ **Ans:** $\frac{9}{2}$

10. $\int_0^{a/\sqrt{2}} \int_0^y (y^2)\, dydx$ **Ans:** $\frac{a^4}{32}(\pi+2)$

5.1 Double integrals - Change of order of integration

Change of order of integration

Change of order of integration is done to make the evaluation of integral easier

The following are very important when the change of order of integration takes place

1. If the limits of the inner integral is a function of x (or function of y) then the first integration should be with respect to y (or with respect to x)
2. Draw the region of integration by using the given limits
3. If the integration is first with respect to x keeping y as a constant then consider the horizontal strip and find the new limits accordingly
4. If the integration is first with respect to y keeping x a constant then consider the vertical strip and find the new limits accordingly
5. After find the new limits evaluate the inner integral first and then the outer integral

Problems

Example:

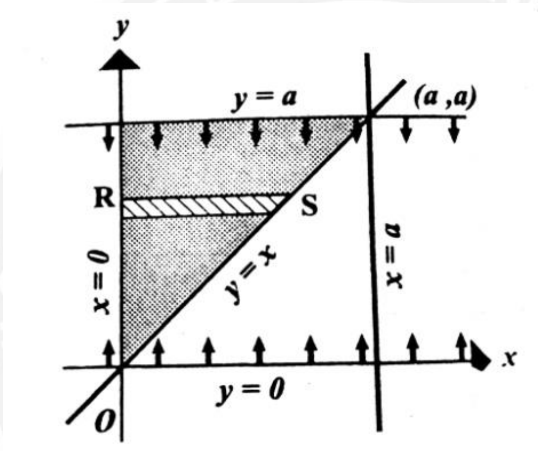
Change the order of integration in $\int_0^a \int_x^a f(x,y)dy\,dx$

Solution:

Given $y: x \to a$

$x: 0 \to a$

The region is bounded by $y = x, y = a, x = 0$ and $x = a$



x axis limit represents the horizontal strip and y axis limit represents vertical

$x: 0 \to y$

$y: 0 \to a$

By changing the order we get

$$\int_0^a \int_0^y f(x,y)dx\,dy$$

Example:

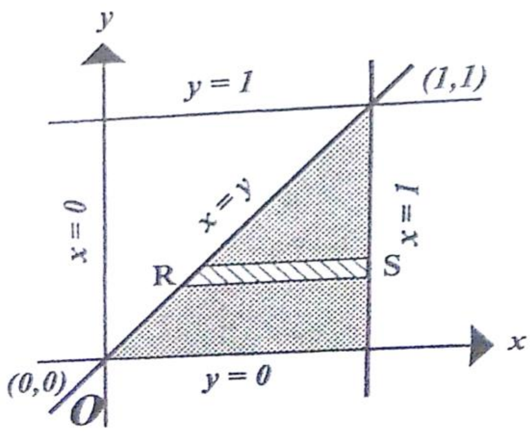
Change the order of integration $\int_0^1 \int_0^x f(x,y)dy\,dx$

Solution:

Given $y: 0 \to x$

$x: 0 \to 1$

The region is bounded by $y = 0, y = x, x = 0, x = 1$



$x: y \to 1$

$y: 0 \to 1$

By changing the order we get

$$\int_0^1 \int_y^1 f(x, y) dx\, dy$$

Example:

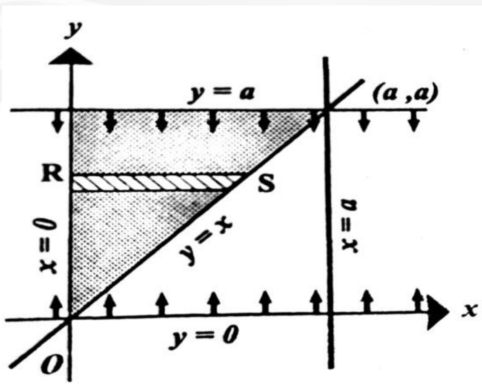
Change the order of integration and hence evaluate it $\int_0^a \int_x^a (x^2+y^2) dy\, dx$

Solution:

It is correct form, given order is $dydx$ given $y: x \to a$

$x: 0 \to a$

the region is bounded by $y = x, y = a\ x = 0$ and $x = a$



x axis limit represent the horizontal strip

y axis limit represents vertical path

changed order is $dx\ dy$

$x: 0 \rightarrow y$

$y: 0 \rightarrow a$

$$\int_0^a \int_0^y (x^2+y^2)dx\ dy = \int_0^a \left[\frac{x^3}{3} + y^2x\right]_0^y \text{dy}$$

$$= \int_0^a \left[\frac{y^3}{3} + y^3\right] \text{dy}$$

$$= \left[\frac{y^4}{12} + \frac{y^4}{4}\right]_0^a \quad = \frac{a^4}{12} + \frac{a^4}{4} = \frac{a^4}{3}$$

Example:

Change the order of integration for $\int_0^{4a} \int_{x^2/4a}^{2\sqrt{ax}} xy\ dy\ dx$

Solution:

It is correct form

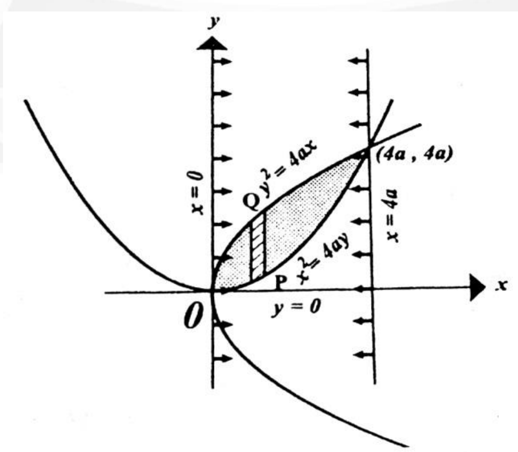
Given order is $dydx$

Given $y: \frac{x^2}{4a} \rightarrow 2\sqrt{ax}$

$x: 0 \rightarrow 4a$

The region is bounded by $x^2 = 4ay$, $y^2 = 4ax$

$x = 0$ and $x = 4a$



Changed order is $dxdy$ draw a horizontal strip

$x: \frac{y^2}{4a} \to 2\sqrt{ay}$

$y:\ 0\ \to 4a$

$$\int_0^{4a}\int_{y^2/4a}^{2\sqrt{ay}} xy\,dx\,dy\ =\ \int_0^{4a}\ \left[\frac{x^2y}{2}\right]_{\frac{y^2}{4a}}^{2\sqrt{ay}} dy$$

$$= \int_0^{4a}\ \{\frac{(2\sqrt{ay})^2y}{2} - [\frac{y^2}{4a}]^2\,\frac{y}{2}\}\ \ dy$$

$$= \ \int_0^{4a}\ \left[\left(\frac{4ay}{2}\right)y - \frac{y^5}{32a^2}\right]\ dy$$

$$= \left[\frac{4ay^3}{6} - \frac{y^6}{192a^2}\right]_0^{4a}$$

$$= \frac{4a(4a)^3}{6} - \frac{(4a)^6}{192a^2}$$

$$= \frac{128a^4}{3} - \frac{4096}{192}a^4$$

$$= \ \frac{64a^4}{3}$$

Example:

Change the order of integration of $\int_0^b\int_0^{\frac{a}{b}\sqrt{b^2-y^2}} xydxdy$ and hence evaluate it

Solution:

It is correct form

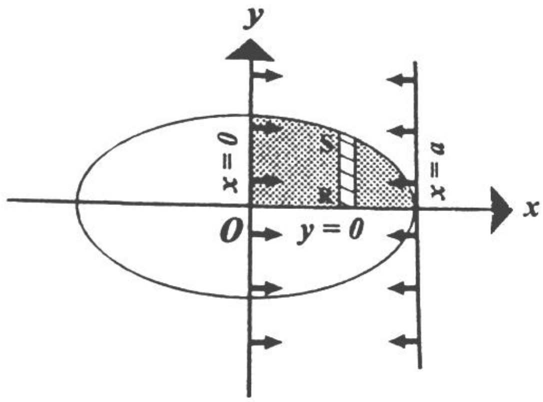
Given order is $dxdy$

Given $\quad x : 0 \to \frac{a}{b}\sqrt{b^2-y^2}$

$y:\ \ 0\ \to\ b$

The region is bounded by $x = 0, x = \frac{a}{b}\sqrt{b^2-y^2} \Rightarrow \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

$$y = 0, y = b$$



Changed order is $dydx$

Draw the vertical strip

$y : 0 \rightarrow \frac{b}{a}\sqrt{a^2 - x^2}$

$x : 0 \rightarrow a$

$$\int_0^a \int_0^{\frac{b}{a}\sqrt{a^2-x^2}} xy\, dy\, dx = \int_0^a \left[\frac{xy^2}{2}\right]_0^{\frac{b}{a}\sqrt{a^2-x^2}} dx$$

$$= \int_0^a \frac{\left[\frac{b}{a}\sqrt{a^2-x^2}\right]^2 x}{2}\, dx$$

$$= \frac{b^2}{2a^2}\int_0^a x(a^2 - x^2)\, dx$$

$$= \frac{b^2}{2a^2}\int_0^a (xa^2 - x^3)\text{dx}$$

$$= \frac{b^2}{2a^2}\left[\frac{a^2x^2}{2} - \frac{x^4}{4}\right]_0^a$$

$$= \frac{b^2}{2a^2}\left[\frac{a^4}{2} - \frac{a^4}{4}\right]$$

$$= b^2\left[\frac{a^4}{2a^2} - \frac{a^4}{4}\right] = b^2\left[\frac{a^2}{2} - \frac{a^4}{4}\right]$$

$$= \frac{a^2b^2}{8}$$

Example:

Change the order of integration and hence evaluate $\int_0^1 \int_{x^2}^{2-x} xy\, dy\, dx$

Solution:

It is correct form

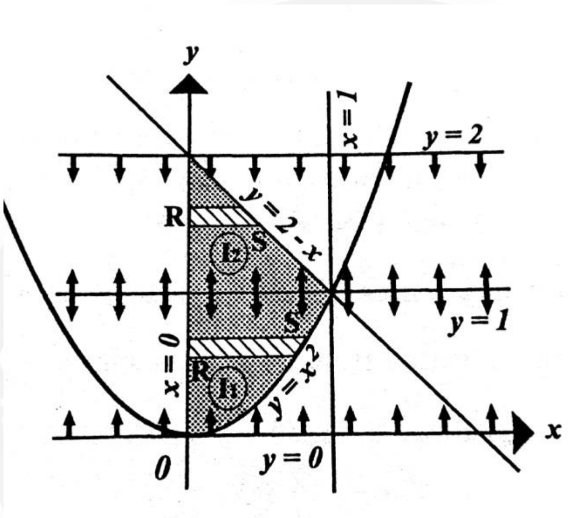
Given order is $dydx$

Given $y: x^2 \to 2 - x$

$x: 0 \to 1$

The region is bounded by $= x^2, y + x = 2$

$x = 0, x = 1$



Now divide the region in to two parts i.e. R_1 and R_2

Changed order is $dxdy$

Draw horizontal strip

For Region R_1

Limits are $x: 0 \to \sqrt{y}$

$y: 0 \to 1$

$$\int_0^1 \int_0^{\sqrt{y}} xy\, dx\, dy = \int_0^1 \left[\frac{x^2 y}{2}\right]_0^{\sqrt{y}} dy$$

$$= \int_0^1 \frac{(\sqrt{y})^2 y}{2} dy$$

$$= \int_0^1 \frac{y^2}{2} dy$$

$$= \left[\frac{y^3}{6}\right]_0^1$$

$$= 1/6$$

For region R_2

Limits are $x: 0 \rightarrow 2-y$

$y: 1 \rightarrow 2$

$$\int_1^2 \int_0^{2-y} xy\, dx\, dy = \int_1^2 \left[\frac{x^2 y}{2}\right]_0^{2-y} dy$$

$$= \int_1^2 \frac{(2-y)^2 y}{2}\, dy$$

$$= \int_1^2 \frac{(4-4y+y^2)y}{2}\, dy$$

$$= \frac{1}{2}\left[\frac{4y^2}{2} - \frac{4y^3}{3} + \frac{y^4}{4}\right]_1^2$$

$$= \frac{1}{2}\left[8 - \frac{32}{3} + 4 - 2 + \frac{4}{3} - \frac{1}{4}\right]$$

$$= \frac{5}{24}$$

$R = R_1 + R_2$

$$= \frac{1}{6} + \frac{5}{24}$$

$$= \frac{3}{8}$$

Example:

Change the order of integration in $\int_0^1 \int_y^{2-y} xy\, dx\, dy$ and hence evaluates

Solution:

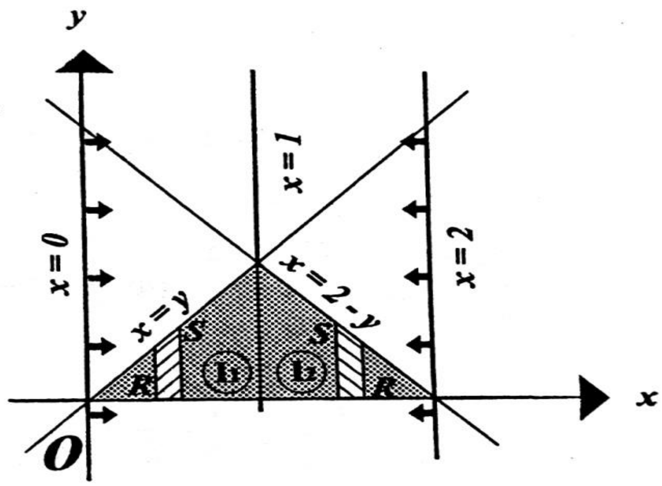
It is correct form

$x: y \rightarrow 2-y$

$y: 0 \rightarrow 1$

The region is bounded by $x = y$, $x + y = 2$

$y = 0$, $y = 1$



Now divide the region into two parts ie. R_1 and R_2

Changed order is $dydx$

Draw horizontal strip

For region R_1

Limits are $x: 0 \to 1$

$y: 0 \to x$

$$\int_0^1 \int_y^{2-y} xy\, dx\, dy = \int_0^1 \int_0^x xy\, dy\, dx$$

$$= \int_0^1 \left[\frac{xy^2}{2}\right]_0^x dx$$

$$= \int_0^1 \left[\frac{x^3}{3-0}\right] dx$$

$$= \frac{1}{2}\int_0^1 x^3\ dx = \frac{1}{2}\left[\frac{x^4}{4}\right]_0^1$$

$$= \frac{1}{8}[x^4]_0^1 = \frac{1}{8}[1-0]$$

$$= \frac{1}{8}$$

For region R_2

$x: 1 \to 2$

$y: 0 \to 2 - x$

$$\int_1^2 \int_0^{2-x} xy\, dy\, dx = \int_1^2 \left[\frac{xy^2}{2}\right]_0^{2-x} dx$$

$$= \int_1^2 \left[\frac{x\,(2-x)^2}{2} - 0\right] dx$$

$$= \frac{1}{2}\int_1^2 \frac{x(4+x^2-4x)}{2}\; dx$$

$$= \frac{1}{2}\int_1^2 (4x + x^3 - 4x^2)\, dx$$

$$= \frac{1}{2}\left[4\frac{x^2}{2} + \frac{x^4}{4} - 4\frac{x^3}{3}\right]_1^2$$

$$= \frac{1}{2}\left[2x^2 + \frac{x^4}{4} - 4\frac{x^3}{3}\right]_1^2$$

$$= \frac{1}{2}\left[\left(8 + \frac{16}{4} - \frac{4}{3}(8)\right) - \left(2 + \frac{1}{4} - \frac{4}{3}\right)\right]$$

$$= \frac{1}{2}\left[8 + 4 - \frac{32}{3} - 2 - \frac{1}{4} + \frac{4}{3}\right]$$

$$= \frac{1}{2}\left[\frac{5}{12}\right] = \frac{5}{24}$$

$\Rightarrow$ R = $R_1 + R_2$

$$= \frac{1}{8} + \frac{5}{24}$$

$$= \frac{1}{3}$$

Example:

Change the order of integration $\int_0^\infty \int_x^\infty \frac{e^{-y}}{y}\, dy\, dx$ and hence evaluate it

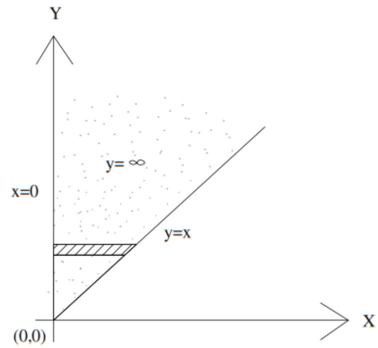
Solution:

It is correct form

Given order is $dy\, dx$

Given $y : x \to \infty$

$x : 0 \to \infty$



Changed order is $dxdy$

Draw a horizontal strip

$x : 0 \to y$

$y : 0 \to \infty$

$$\int_0^\infty \int_0^y \frac{e^{-y}}{y} dx\, dy = \int_o^\infty \left[e^{-y} \frac{x}{y}\right]_0^y dy$$

$$= \int_o^\infty e^{-y}\, dy = \left[\frac{e^{-y}}{1}\right]_0^\infty$$

$$= -[e^{-\infty} - e^0] = 1$$

Example:

Change the order of integration I = $\int_0^a \int_y^a \frac{x}{x^2+y^2} dx\, dy$ **and the evaluate it**

Solution:

It is correct form

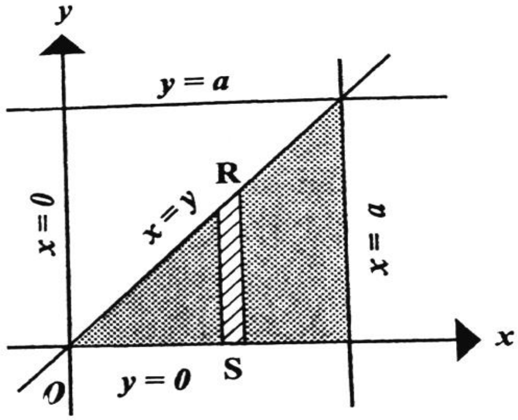
Given order $dxdy$

$x : y \to a$

$y : 0 \to a$

The region is bounded by $x = y , x = a$

$$y = 0 , y = a$$



Changed order is $dy\,dx$

Draw a vertical strip

$y:\ 0 \rightarrow x$

$x:\ 0 \rightarrow\ a$

$$\int_0^a \int_0^x \frac{x}{x^2+y^2}\,\mathrm{dy\,dx} = \int_0^a x\left[\frac{tan^{-1}}{x}\left(\frac{y}{x}\right)\right]_0^x dx$$

$$= \int_0^a [tan^{-1}\left(\frac{x}{x}\right) - tan^{-1}0]\,dx$$

$$= \int_0^a \frac{\pi}{4}\,dx$$

$$= \left[\frac{\pi}{4}x\right]_0^a$$

$$= \frac{\pi}{4}a$$

Example:

Evaluate $\int_0^a \int_0^{\sqrt{a^2-x^2}} xy\,dy\,dx$ by changing the order of integration

Solution:

It is correct form

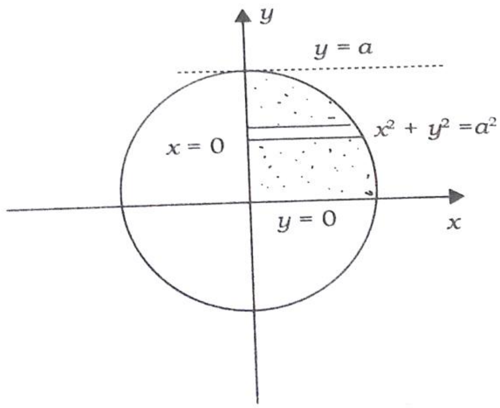
Given order $dy\,dx$

Given $y:\ 0 \rightarrow \sqrt{a^2-x^2}$

$x:\ 0 \rightarrow\ a$

the region is bounded by $y = 0,\ y = \sqrt{a^2-x^2}$

$$x = 0 \; , \; x = a$$



changed order $dxdy$

Draw horizontal strip

$$x : 0 \to \sqrt{a^2 - y^2}$$

$$y : 0 \to a$$

$$\int_0^a \int_0^{\sqrt{a^2-x^2}} xy\, dy\, dx = \int_0^a \int_0^{\sqrt{a^2-y^2}} xy\, dx\, dy$$

$$= \int_0^a y \left[\frac{x^2}{2}\right]_0^{\sqrt{a^2-y^2}} dy$$

$$= \frac{1}{2}\int_0^a y(a^2\text{-}y^2)\, dy$$

$$= \frac{1}{2}\left[\frac{a^2y^2}{2} - \frac{y^4}{4}\right]_0^a$$

$$= \frac{1}{2}\left[\frac{a^2}{2} - \frac{a^4}{4}\right]$$

$$= \frac{a^4}{8}$$

Exercise

Change the order of integration and hence evaluate the following

1.$\int_{-a}^a \int_0^{\sqrt{a^2}-x^2} (x^2 + y^2)\, dy\, dx$ **Ans:** $\frac{\pi a^4}{4}$

2. $\int_0^a \int_0^{2\sqrt{ax}} x^2\; dy\, dx$ **Ans:** $\frac{4}{7}a^4$

3. $\int_0^4 \int_{\frac{x^2}{4}}^{2\sqrt{x}} dy \quad dx$ **Ans:** $\frac{16}{3}$

4. $\int_0^a \int_{a-\sqrt{a^2-y^2}}^{a+\sqrt{a^2}-y^2} dx\, dy$ **Ans:** $\frac{1}{2}$

5. $\int_0^\infty \int_0^y ye^{-\frac{y^2}{x}}$ **Ans:** $\frac{1}{2}$

6. $\int_0^1 \int_y^{2-y} xy \; dxdy$ **Ans:** $\frac{1}{3}$

7. $\int_0^1 \int_y^{2-x} \frac{x}{y} \; dydx$ **Ans:** $\log 4 - 1$

8. $\int_1^3 \int_0^{6/x} x^2 \; dy\, dx$ **Ans:** 24

9. $\int_0^a \int_y^a \frac{x}{\sqrt{x^2+y^2}} \; dx\, dy$ **Ans:** $\frac{a^2}{2} \log(1 + \sqrt{2)}$

10. $\int_1^4 \int_{2/y}^{2\sqrt{y}} dx \quad dy$ **Ans:** $\frac{28}{3} - 2 \log 4$