INTRODUCTION

Random Process:

Consider a random experiment with a sample space S. If a time function X(t, s) is assigned to each outcome $s \in S$ and where $t \in T$, then the family of all such functions, denoted by $\{X(t, s)\}$, where $s \in S$, $t \in T$ is called a random process. In other words, a random process is a collection of random variables together with time.

Note: A random process is also called stochastic process.

Classification of Random Process:

Classify a random process according to the characteristic of T and the state space S. We shall consider only 4 cases based on T and S.

i) Continuous random process
ii) Continuous random sequence
iii) Discrete random process
iv) Discrete random sequence

Continuous random Process:

If both S and T are continuous, then the random process is called continuous Random process.

Continuous Random Sequence:

If S is continuous and T is discrete, then the random process is called continuous random sequence.

Discrete Random Process:

If S is discrete and T is continuous, then the random process is called discrete random process.

Discrete Random Sequence:

If both S and T are discrete, then the random process is called discrete random process.

Deterministic Random Process:

A random process is called a deterministic random process if all the future values are predicted from past observation.

Non Deterministic Random Process:

A random process is called a non - deterministic random process if the future values of any sample function cannot be predicted from the past observation.

Wide Sense Stationary Process (WSS);

A process $\{X(t)\}$ is said to be Wide Sense Stationary Process if

(i) $Mean = E[X(t)] = constant$

(ii) $Auto\ correlation\ \ R_{XX}(\tau) = E[X(t)X(t+\tau)]$ depends on τ

Note:

A WSS process is also called as Weak Sense Stationary Process.

A SSS process is also called a strongly stationary process.

For stationary process mean and variance are constants.

A random process, which is not stationary in any sense, is called evolutionary.

Formulae:

Wide Sense Stationary (WSS):

(i) $Mean = E[X(t)] = constant$

(ii) $Auto\ correlation\ \ R_{XX}(\tau) = E[X(t)X(t+\tau)]$ depends on τ

Stationary Process:

(i) $E[X(t)] = constant$

(ii) $Var[X(t)] = constant$

Strict Sense Stationary (SSS):

$$E[X^n(t)]\ is\ a\ constant\ for\ every\ n$$

Joint Wide Sense Stationary (JWSS):

(i) $E[X(t)] = constant$

(ii) $E[Y(t)] = constant$

(iii) $R_{XX}(t, t+\tau) = E[X(t)Y(t+\tau)]\ \ depends\ on\ \tau$

Mean Ergodic:

Time average, $\overline{X_T} = \frac{1}{2T}\int_{-T}^{T} X(t)dt$

$$E[X(t)] = \lim_{T\to\infty} \overline{X_T}$$

Correlation Ergodic:

$$\overline{X_T} = \frac{1}{2T}\int_{-T}^{T} X(t)\, X(t+\tau)dt$$

$$R_{XX}(t, t+\tau) = \lim_{T\to\infty} \overline{X_T}$$

If $Y(t) = X(t+a) - X(t-a)$, prove that $R_{YY}(\tau) = \langle 2R_{XX}(\tau) - R_{XX}(\tau + 2a) - R_{XX}(\tau - 2a)$

Solution:

Given $Y(t) = X(t+a) - X(t-a)$

$R_{YY}(t) = E[Y(t_1)Y(t_2)]$

$= E\left[\left(X(t_1+a) - X(t_1-a)\right)\left(X(t_2+a) - X(t_2-a)\right)\right]$

$= E[(X(t_1+a)(X(t_2+a) - X(t_1+a)X(t_2-a) - X(t_1-a)(X(t_2 \quad +a) + X(t_1-a)X(t_2-a))]$

$= E[X(t_1+a)(X(t_2+a)] - E[X(t_1+a)X(t_2-a)] - E[X(t_1 -a)(X(t_2+a)] + E[X(t_1-a)X(t_2-a)]$

$= R_{XX}(t_1+a, t_2+a) - R_{XX}(t_1+a, t_2-a) - R_{XX}(t_1-a, t_2+a) + R_{XX}(t_1-a, t_2-a)$

$= R_{XX}(t_1+a-t_2-a) - R_{XX}(t_1+a-t_2+a) - R_{XX}(t_1-a-t_2-a) + R_{XX}(t_1-a-t_2+a)$

$= R_{XX}(t_1-t_2) - R_{XX}(t_1-t_2+2a) - R_{XX}(t_1-t_2-2a) + R_{XX}(t_1-t_2)$

$= R_{XX}(\tau) - R_{XX}(\tau+2a) - R_{XX}(\tau-2a) + R_{XX}(\tau)$

$R_{YY}(\tau) = 2R_{XX}(\tau) - R_{XX}(\tau+2a) - R_{XX}(\tau-2a)$

The following formulas are very useful to solve problems under stationary process.

> If X is a RV with mean zero, then $\text{Var}(X) = E(X^2)$

> $1 + 2x + 3x^2 + \cdots = (1-x)^{-2}$

> $1 + 4x + 9x^2 + \cdots = (1+x)(1-x)^{-3}$

> If A and B are RV's and λ is a constant, then

$$E[A\cos\lambda t + B\sin\lambda t] = E(A)\cos\lambda t + E(B)\sin\lambda t$$

> $\therefore E(\cos\lambda\tau) = \cos\lambda\tau$, since λ and τ are constants.

STATIONARY PROCESS

Problems under Stationary process:

For a stationary process

(1) $E[X(t)]$ is a constant

(2) $\mathrm{Var}[X(t)]$ is a constant

1. The process $\{X(t)\}$ whose probability distribution unde certain conditions is given by $P[X(t)=n] = \begin{cases} \frac{(at)^{n-1}}{(1+at)^{(n+1)}}; n = 1,2,3,\dots. \\ \frac{at}{1+at}; n = 0 \end{cases}$

Show that it is not a stationary process (Evolutionary).

Solution:

n	0	1	2	3	...
$p_n(t)$	$\frac{at}{1+at}$	$\frac{1}{(1+at)^2}$	$\frac{at}{(1+at)^3}$	$\frac{(at)^2}{(1+at)^4}$	...

For a stationary process,

(1) $E[X(t)]$ is a a constant

(2) Var$[X(t)]$ is a constant

$$E[X(t)] = \sum_{n=0} np_n(t) = 0 + \frac{1}{(1+at)^2} + (2)\frac{at}{(1+at)^3} + (3)\frac{(at)^2}{(1+at)^4}..$$

$$= \frac{1}{(1+at)^2}\left[1 + 2\frac{at}{1+at} + 3\frac{(at)^2}{(1+at)^2} + \cdots \ldots \ldots\right]$$

$$= \frac{1}{(1+at)^2}\left[1 - \frac{at}{(1+at)}\right]^{-2}$$

$$= \frac{1}{(1+\mathrm{at})^2}\left[\frac{1+at-at}{(1+at)}\right]^{-2}$$

$$= \frac{1}{(1+\mathrm{at})^2}\left[\frac{1}{(1+at)}\right]^{-2}$$

$$= \frac{1}{(1+at)^2}(1+at)^2 = 1$$

$E[X\ (t)]\ =\ 1$ which is a constant

$$\mathrm{E}[\mathrm{X}^2(\mathrm{t})] = \sum_{n=1} n^2 p_n(t) = 0 + \frac{1}{(1+at)^2} + (4)\frac{at}{(1+at)^3} + (9)\frac{(at)^2}{(1+at)^4} + \cdots$$

$$= \frac{1}{(1+at)^2}\left[1 + 4\frac{at}{1+at} + 9\frac{(at)^2}{(1+at)^2} + \cdots\right]$$

$$= \frac{1}{(1+at)^2}\left(1 + \frac{at}{1+at}\right)\left[1 - \frac{at}{1+at}\right]$$

$$= \frac{1}{(1+at)^2}\left(\frac{1+2at}{1+at}\right)(1+at)^3$$

$\mathrm{E}[\mathrm{X}^2(\mathrm{t})]\ = 1 + 2at$, which is not a constant

Var [X(t)] $= E[X^2(t)] - [E[X(t)]]^2 = 1 + 2at - 1$

$= 2at$ which is not a constant.

$\therefore$ $\{X(t)\}$ is not a stationary process.

2. Consider a random process A_1 and A_2 are independent random variables with $E(A_i) = a_i$ and Var$(A_i) = \sigma_i^2$ for $i = 1, 2$ Prove that the process $\{X(t)\}$ is evolutionary.

Solution:

Given $X(t) = A_1 + A_1 t$ where A_1 and A_2 are independent random variables with $E(A_i) = a_i$ and Var$(A_i) = \sigma_i^2$ for $i = 1,2$

For a stationary process

(1) $E[X(t)]$ is a constant

(2) Var$[X(t)]$ is a constant

$E[X(t)] = E[A_1 + A_1 t]$

$= E[A_1] + tE[A_2]$

$= a_1 + ta_2$

Which is not a constant.

Thus, the process {X(t)} is evolutionary.

3. Let $X(t) = B\sin \omega t$, where B is a random variable with mean and variance 1 and ω is a constant. Check whether $\{X(t)\}$ is a stationary or not

Solution:

Given $X(t) = B\sin \omega t$, where

B is a random variable with Mean=0 and Variance =1

Mean of $B = 0 \Rightarrow E(B) = 0$ ……..(i)

Variance of $B = 1 \Rightarrow E(B^2) = 1 \ldots \ldots \ldots$ (ii)

For a stationary process,

(1) $E[X(t)]$ is a constant

(2) Var$[X(t)]$ is a constant

(1) $E[X(t)] = E[B\sin \omega t]$

$= E[B]\sin \omega t$

$= 0$ From (i)

$\therefore E[X(t)]$ is a constant

(2) $E[X^2(t)] = E[B^2\sin^2 \omega t]$

$= E(B^2)\sin^2 \omega t$

$= \sin^2 \omega t$ which is not a constant From (ii)

$\text{Var}[X(t)] = E[X^2(t)] - [E[X(t)]^2] = \sin^2 \omega t$, which is not a constant.

Since the condition (2) for Stationary Process is not satisfied,

Hence $\{X(t)\}$ is not a Stationary Process.

4. Consider the random process $X(t) = \cos(t + \varphi)$ where φ is a random variable with density function $f(\varphi) = \frac{1}{\pi}$, where $-\frac{\pi}{2} < \varphi < \frac{\pi}{2}$. Check whether or not the process is stationary.

Solution:

$X(t) = \cos(t + \varphi)$ where φ is a random variable with

$f(\varphi) = \frac{1}{\pi}$, where $-\frac{\pi}{2} < \varphi < \frac{\pi}{2}$

For a stationary process,

(1) $E[X(t)]$ is a constant

(2) $\text{Var}[X(t)]$ is a constant

E $[X(t)] = E[\cos(t + \varphi)]$

$$= \int_{\frac{\pi}{2}}^{\frac{\pi}{2}} \cos(t + \varphi) f(\varphi) d\varphi$$

$$= \int_{\frac{\pi}{2}}^{\frac{\pi}{2}} \cos(t + \varphi) \frac{1}{\pi} d\varphi$$

$$= \frac{1}{\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos(t + \varphi) d\varphi$$

$$= \frac{1}{\pi}[\sin(t+\varphi)]_{-\frac{\pi}{2}}^{\frac{\pi}{2}}$$

$$= \frac{1}{\pi}\left[\sin\left(t+\frac{\pi}{2}\right) - \sin\left(t-\frac{\pi}{2}\right)\right]$$

$$= \frac{1}{\pi}\left[\sin\left(\frac{\pi}{2}+t\right) + \sin\left(\frac{\pi}{2}-t\right)\right]$$

$$\because \sin\left(\frac{\pi}{2}-\theta\right) = \sin\left(\frac{\pi}{2}+\theta\right) = \cos\theta$$

$$= \frac{1}{\pi}[\cos t + \cos t]$$

$$= \frac{1}{\pi}2\cos t$$

$E[X(t)] = \frac{1}{\pi}2\cos t$, which depends on t.

Since the condition (1) for Stationary Process is not satisfied,

$\{X(t)\}$ is not a Stationary Process.

5. Let $X(t) = \cos(\omega t+\theta)$, where θ is a random variable uniformly distributed over $(0, 2\pi)$. Prove that $\{X(t)\}$ is a stationary process of first order.

Solution:

Given: $X(t) = \cos(\omega t+\theta)$, where θ is random variable uniformly distributed over $(0,2\pi)$.

$$\therefore f_\theta(\theta) = \frac{1}{2\pi}; 0 < \theta < 2\pi$$

To prove $\{X(t)\}$ is a first order stationary process.

we have to prove $f_X(x;t)$ is independent of time.

To find $f_X(x;t)$:

We have $x = \cos(\omega t + \theta)$

$\Rightarrow \omega t + \theta = \pm \cos^{-1}[x]$

To find $f_X(x;t)$,

Take $x = X(t)$

$\Rightarrow \theta = -\omega t \pm \cos^{-1}[x] \quad \because \cos[\pm(\omega t + \theta)] = \cos(\omega t + \theta$

Let $\theta_1 = -\omega t - \cos^{-1} x$ and $\theta_2 = -\omega t + \cos^{-1} x$

$$\frac{d\theta_1}{dx} = 0 - \frac{-1}{\sqrt{1-x^2}} = \frac{1}{\sqrt{1-x^2}}$$

$$\frac{d\theta_1}{dx} = 0 + \frac{-1}{\sqrt{1-x^2}} = \frac{-1}{\sqrt{1-x^2}}$$

The first order density of $\{X(t)\}$ is given by

$$f_x(x,t) = \left|\frac{d\theta_1}{dx}\right| f_\theta(\theta_1) + \left|\frac{d\theta_2}{dx}\right| f_\theta(\theta_2)$$

$$= \left|\frac{1}{\sqrt{1-x^2}}\right| \frac{1}{2\pi} + \left|\frac{-1}{\sqrt{1-x^2}}\right| \frac{1}{2\pi}$$

$$= \frac{1}{2\pi}\frac{1}{\sqrt{1-x^2}} + \frac{1}{2\pi}\frac{1}{\sqrt{1-x^2}}$$

$$= \frac{2}{2\pi}\frac{1}{\sqrt{1-x^2}}$$

$$f_X(x,t) = \frac{1}{\pi}\frac{1}{\sqrt{1-x^2}}$$

To find the range of x :

We have $x = X(t) = \cos(\omega t + \theta)$.

Since the value of $\cos(\omega t + \theta)$ lies between -1 and $+1$, we have $-1 \leq x \leq 1$.

$$\therefore f_X(x,t) = \frac{1}{\pi}\frac{1}{\sqrt{1-x^2}}, -1 \leq x \leq 1 \ldots\ldots (1)$$

which is independent of time.

Hence, $\{X(t)\}$ is a stationary process of first order.

Problems on Wide Sense Stationary (WSS):

1. Show that the random process $X(t) = A\cos(\omega t + \theta)$ is WSS, where $A\ and\ \omega$ are constants and θ is uniformly distributed on the interval $(0, 2\pi)$

Solution:

Given, $X(t) = A\cos(\omega t + \theta)$

θ is uniformly distributed on the interval $(0, 2\pi)$

$$f(\theta) = \frac{1}{b-a}\ , a < \theta < b$$

$$f(\theta) = \frac{1}{2\pi}\ , 0 < \theta < 2\pi$$

To Prove $X(t)$ is WSS.

(i)$Mean = E[X(t)] = constant$

(ii) $Auto\ correlation\ \ R_{XX}(\tau) = E[X(t)X(t+\tau)]$ depends on τ

(i) $E[X(t)] = \int_{-\infty}^{\infty} X(t) f(\theta) d\theta$

$$= \int_0^{2\pi} A\cos(\omega t + \theta)\frac{1}{2\pi} d\theta$$

$$= \frac{A}{2\pi}\ [\sin(\omega t + \theta)]_0^{2\pi}$$

$$= \frac{A}{2\pi}[\sin(\omega t + 2\pi) - \sin(\omega t + 0)]$$

$$= \frac{A}{2\pi}[\sin\omega t - \sin\omega t] = 0$$

$E[X(t)] = 0\ is\ constant.$

(ii)$R_{XX}(\tau) = E[X(t)X(t+\tau)]$

$$= E[A\cos(\omega t + \theta)\, A\cos(\omega(t+\tau) + \theta)]$$

$$= E[A^2]E[\cos(\omega t + \theta)\cos(\omega(t+\tau) + \theta)]$$

$$= A^2\frac{1}{2}E[\cos(\omega t + \theta + \omega t + \omega\tau + \theta)\cos(\omega t + \theta - \omega t - \omega\tau - \theta)]$$

$$\cos(-\theta) = \cos\theta$$

$$\cos A \cos B = \frac{1}{2}[\cos(A+B) + \cos(A-B)]$$

$$= A^2 \frac{1}{2} E[\cos(2\omega t + 2\theta + \omega\tau)\cos(-\omega\tau)]$$

$$= \frac{A^2}{2} E[\cos(2\omega t + 2\theta + \omega\tau)\cos(\omega\tau)]$$

$$= \frac{A^2}{2}\left[\cos\ \omega\tau\ +\ \int_0^{2\pi} \cos(2\omega t + 2\theta + \omega\tau)\frac{1}{2\pi}d\theta\right]$$

$$= \frac{A^2}{2}\cos\ \omega\tau + \frac{A^2}{4\pi}\left[\frac{\sin(2\omega t + 2\theta + \omega\tau)}{2}\right]_0^{2\pi}$$

$$= \frac{A^2}{2}\cos\ \omega\tau + \frac{A^2}{8\pi}[\sin(2\omega t + \omega\tau + 4\pi) - \sin(2\omega t + \omega\tau)]$$

$$= \frac{A^2}{2}\cos\ \omega\tau + \frac{A^2}{8\pi}[\sin(2\omega t + \omega\tau) - \sin(2\omega t + \omega\tau)]$$

$$= \frac{A^2}{2}\cos\ \omega\tau + \frac{A^2}{8\pi}[0]$$

$$R_{XX}(\tau) = \frac{A^2}{2}\cos\ \omega\tau$$

Hence $X(t)$ is WSS process.

2.Show that the random process $X(t) = A\cos\ \lambda t\ + B\ sin\ \lambda t$ where λ is a constant, A and B are random variables, is WSS if (i) $E[A] = E[B] = 0$ (ii) $E[A^2] = E[B^2]$ and (iii) $E[AB] = 0$

Solution:

Given, $X(t) = A\cos\lambda t + B\sin\lambda t$

$E[A] = E[B] = 0$, $E[A^2] = E[B^2]$, $E[AB] = 0$

To Prove $X(t)$ is WSS.

(i)$Mean = E[X(t)] = constant$

(ii) $Auto\ correlation\ R_{XX}(\tau) = E[X(t)X(t+\tau)]$ depends on τ

(i)$E[X(t)] = E[A\cos\lambda t + B\sin\lambda t\ ]$

$$= E[A]\cos\lambda t + E[B]\sin\lambda t$$

$$= 0 * \cos\lambda t + 0 * \sin\lambda t$$

$$E[X(t)] = 0\ is\ constant.$$

(ii)$R_{XX}(\tau) = E[X(t)X(t+\tau)]$

$$= E[(A\cos\lambda t + B\sin\lambda t)(A\cos\lambda(t+\tau) + B\sin\lambda(t+\tau))]$$

$$= E[A^2\cos\lambda t\cos\lambda(t+\tau) + AB\cos\lambda t\sin\lambda(t+\tau)$$
$$+ AB\sin\lambda t\cos\lambda(t+\tau) + B^2\sin\lambda t\sin\lambda(t+\tau)]$$

$$= E[A^2\cos\lambda t\cos\lambda(t+\tau)] + E[AB\cos\lambda t\sin\lambda(t+\tau)] +$$
$$E[AB\sin\lambda t\cos\lambda(t+\tau)] + E[B^2\sin\lambda t\sin\lambda(t+\tau)]$$

$$= E[A^2\cos\lambda t\cos\lambda(t+\tau)] + E[B^2\sin\lambda t\sin\lambda(t+\tau)]$$

$$= E[A^2]\cos\lambda t\cos\lambda(t+\tau) + E[B^2]\sin\lambda t\sin\lambda(t+\tau)$$

$$= k\cos\lambda t\ \cos\lambda(t+\tau) + k\, sin\,\lambda t\, sin\,\lambda(t+\tau)$$

$$= k[\cos\lambda t\ \cos\lambda(t+\tau) + sin\,\lambda t\, sin\,\lambda(t+\tau)]$$

$$= k[\cos(\lambda t - \lambda t - \lambda\tau)]$$

$\cos(-\theta) = \cos\theta$ $\cos A\cos B + \sin A\sin B = \cos(A-B)$

$$= k[\cos(-\lambda\tau)]$$

$$= k[\cos(\lambda\tau)]$$

Hence $X(t)$ is WSS process.

3. Given a random variable y with characteristic function $\varphi(\omega) = E[e^{i\omega y}]$ and a random process $X(t) = \cos(\lambda t + y)$. Show that $X(t)$ is stationary in the wide sense if $\varphi(1) = \varphi(2) = 0$

Solution:

Given, $X(t) = \cos(\lambda t + y)$

$$\varphi(\omega) = E\left[e^{i\omega y}\right] = E[\cos\omega y + i\sin\omega y]$$

$$= E[\cos\omega y] + i\,\mathrm{E}[\sin\omega y]$$

Given, $\varphi(1) = 0$

$$\Rightarrow 0 = E[\cos\ y] + i\,\mathrm{E}[\sin\ y]$$

$$E[\cos y] = 0;\ \mathrm{E}[\sin y] = 0$$

Given, $\varphi(2) = 0$

$$\Rightarrow 0 = E[\cos\ 2y] + i\,\mathrm{E}[\sin 2y]$$

$$E[\cos\ 2y] = 0;\ \mathrm{E}[\sin\ 2y] = 0$$

To Prove $X(t)$ is WSS.

(i)$Mean = E[X(t)] = constant$

(ii) $Auto\ correlation\ \ R_{XX}(\tau) = E[X(t)X(t+\tau)]$ depends on τ

(i)$E[X(t)] = E[\cos(\lambda t + y)\ \ ]$

$$= E[\cos\lambda t\ \ \cos y\ \ - \sin\lambda t\ sin\ y]$$

$\mathrm{Cos}(A+B) = \cos\ A\cos\ B - \sin A\sin B$
$\cos\ A\cos\ B = \frac{1}{2}[\cos(A+B) + \cos(A-B)]$

$$= \cos\lambda t\ \ E[\cos y\ ] - \sin\lambda t\, E[\sin y]$$

$$= \cos\lambda t * 0 - \sin\lambda t * 0$$

$E[X(t)] = 0\ is\ constant.$

(ii)$R_{XX}(\tau) = E[X(t)X(t+\tau)]$

$$= E[\cos\ (\lambda t + y)\ \cos\ (\lambda(t+\tau) + y)\ ]$$

$$= E[\cos\ (\lambda t + y)\ \cos\ (\lambda t + \lambda\tau + y)\ ]$$

$$= \frac{1}{2}E[\cos(\lambda t + y + \lambda t + \lambda\tau + y) + \cos(\lambda t + y - \lambda t - \lambda\tau - y)]$$

$$= \frac{1}{2} E[\cos(2\lambda t + 2y + \lambda\tau) + \cos(-\lambda\tau)]$$

$$= \frac{1}{2}\cos\lambda\tau + \frac{1}{2} E[\cos(2\lambda t + \lambda\tau)\cos 2y - \sin\,(2\lambda t + \lambda\tau)\sin 2y]$$

$$= \frac{1}{2}\cos\lambda\tau + \frac{1}{2}\cos(2\lambda t + \lambda\tau)\, E[\cos 2y] - \frac{1}{2}\sin\,(2\lambda t + \lambda\tau)\, E[\sin 2y]$$

$$= \frac{1}{2}\cos\lambda\tau + \frac{1}{2}(0)$$

$$R_{XX}(\tau) = \frac{1}{2}\cos\lambda\tau$$

Hence $\{X(t)\}$ is WSS process.

4. Show that the process $X(t) = Y\cos\omega t + Z\sin\omega t$ where Y and Z independent RV's which follows $N(0, \sigma^2)$ and ω is a constant, is wide sense stationary.

Solution:

Given $X(t) = Y\cos\omega t + Z\sin\omega t$, where Y and Z are independent

(i)$E(Y) = E(Z) = 0$

(ii) $E(YZ) = 0$

(iii) $E(Y^2) = E(Z^2) = \sigma^2$

To prove $\{X(t)\}$ is a WSS process,

(1) $E[X(t)]$ is a constant

(2) $R_{XX}(t_1, t_2)$ is a function of τ

(1) $E[X(t)] = E[Y\cos\omega t + Z\sin\omega t]$

$$= E(Y)\cos\omega t + E(Z)\sin\omega t$$

$$= 0 + 0 = 0 \text{ From (i)}$$

$\therefore E[X(t)]$ is a constant

(2) $R_{XX}(t_1, t_2) = E[X(t_1)X(t_2)]$

$$= E[(Y\cos\omega t_1 + Z\sin\omega t_1)(Y\cos\omega t_2 + Z\sin\omega t_2)]$$

$$= E[Y^2\cos\omega t_1\cos\omega t_2 + YZ\sin\omega t_2\cos\omega t_1 + ZY\sin\omega t_1\cos\omega t_2$$

$$+Z^2\sin\omega t_1\sin\omega t_2]$$

$$= E(Y^2)\cos\omega t_1\cos\omega t_2 + E(YZ)\sin\omega t_2\cos\omega t_1 + E(YZ)\sin\omega t_1\cos\omega t_2$$

$$+E(Z^2)\sin\omega t_1\sin\omega t_2$$

$$= \sigma^2\cos\omega t_1\cos\omega t_2 + 0 + 0 +$$

$\sigma^2\sin\omega t_1\sin\omega t_2$ From (ii) & (iii)

$$= \sigma^2[\cos\omega t_1\cos\omega t_2 + \sin\omega t_1\sin\omega t_2]$$

$$= \sigma^2\cos(\omega t_1 - \omega t_2)$$

$$= \sigma^2\cos[\omega(t_1 - t_2)]$$

$R_{XX}(t_1, t_2) = \sigma^2\cos\omega\tau$

$R_{XX}(t_1, t_2)$ is a function of τ.

Since the conditions (1) and (2) for WSS are satisfied.

Hence, $\{X(t)\}$ is a WSS Process.

5. If $X(t) = Y\cos t + Z\sin t$, where Y and Z are independent binary random variables each of which assumes the values -1 and $+2$ with probabilities $\frac{2}{3}$ and $\frac{1}{3}$ respectively. Prove that $\{X(t)\}$ is WSS.

Solution:

Given $X(t) = Y\cos t + Z\sin t$, where Y and Z are independent binary random variables. The probability distribution of Y and Z are given by

y	-1	2
P(y)	2/3	1/3

z	-1	2
P(z)	2/3	1/3

$$E(Y) = \sum yp(y) = (-1)\left(\frac{2}{3}\right) + 2\left(\frac{1}{3}\right)$$

$$E(Y) = 0$$

$$E(Y^2) = \sum y^2 p(y) = 1\left(\frac{2}{3}\right) + 4\left(\frac{1}{3}\right)$$

$$= \frac{2}{3} + \frac{4}{3}$$

$$= \frac{6}{3} = 2$$

Similarly, $E(Z) = 0, E(Z^2) = 2$.

Since, Y and Z are independent, $E(YZ) = E(Y)E(Z) = 0$

To prove $\{X(t)\}$ is a WSS process

(1) $E[X(t)]$ is a constant

(2) $R_{XX}(t_1, t_2)$ is a function of n of τ

(1) $E[X(\text{t})] = E[Y\cos t + Z\sin t]$

$$= E(Y)\cos t + E(Z)\sin t = 0$$

$\therefore E[X(t)]$ is a constant.

(2) $R_{XX}(t_1, t_2) = E[X(t_1)X(t_2)]$

$$= E[(Y\cos t_1 + Z\sin t_1)(Y\cos t_2 + Z\sin t_2)]$$

$$= E[Y^2\cos t_1\cos t_2 + YZ\sin t_2\cos t_1 + YZ\sin t_1\cos t_2 + Z^2\sin t_1\sin t_2]$$

$$= E(Y^2)\cos t_1\cos t_2 + E(YZ)\sin t_2\cos t_1 + E(YZ)\sin t_1\cos t_2$$

$$+E(Z^2)\sin t_1\sin t_2$$

$$= 2\cos t_1\cos t_2 + 2\sin t_1\sin t_2$$

$$= 2\cos(t_1 - t_2) = 2\cos\tau$$

$R_{XX}(t_1, t_2)$ is a function of τ.

Since the conditions (1) and (2) for WSS are satisfied.

Hence, $\{X(t)\}$ is a WSS Process.

6. Consider the process $X(t) = \sum_{i=1}^{n}(A_i(A_i\cos p_i t + B_i\sin p_i t)$ where A_i and B_i are uncorrelated R.v's with mean' 0 ' & variance σ_i^2. Prove that $\{X(t)\}$ is a WSS process.

Solution:

Given $X(t) = \sum_{i=1}^{n}(A_i\cos p_i t + B_i\sin p_i t)$

A_i and B_i are Rv's

Given Means of A_i and $B_i = 0 \Rightarrow E[A_i] = 0$ & $E[B_i] = 0$ and

$\text{Var}[A_i] = \text{Var}[B_i] = \sigma_i^2$

$\Rightarrow E[A_i^2] = E[B_i^2] = \sigma_i^2$

Also A_i and B_i are uncorrelated $\therefore E\left[A_i B_j\right] = 0$ for all i, j

$\therefore E\left[A_i A_j\right] = E\left[B_i B_j\right] = 0$ for $i \neq j$

To prove $\{X(t)\}$ is a WSS process

(1) $E[X(t)]$ is a constant

(2) $R_{XX}(t_1, t_2)$ is a function of τ.

(1) $E[X(t)] = E[\sum_{i=1}^{n}(A_i\cos p_i t + B_i\sin p_i t)]$

$= \sum_{i=1}^{n}[E(A_i)\cos p_i t + E(B_i)\sin p_i t] = 0$

$\therefore E[X(t)]$ is a constant.

2) The ACF of $\{X(t)\}$ is given by $R(\tau) = E[X(t_1)X(t_2)]$

$= E\left[\sum_{i=1}^{n}(A_i \cos p_i t_1 + B_i \sin p_i t_1)\sum_{j=1}^{n}\left(A_j \cos p_j t_2 + B_j \sin p_j t_2\right)\right.$

$= E\left[\sum_{i=1}^{n}\sum_{j=1}^{n}(A_i \cos p_i t_1 + B_i \sin p_i t_1)\left(A_j \cos p_j t_2 + B_j \sin p_j t_2\right)\right]$

$=E[\sum_{i=1}^{n}\sum_{j=1}^{n}\left[\left[A_iA_j \cos p_i t_1 \cos p_j t_2 + A_iB_j \cos p_i t_1 \sin p_j t_2 +\right.\right.$

$A_jB_i \sin p_i t_1 \cos p, t_2 + B_iB_j \sin p_i t_1 \sin p_j t_2)]$

$=\sum_{i=1}^{n}\sum_{j=1}^{n}\left[\left[E(A_iA_j) \cos p_i t_1 \cos p_j t_2 + E(A_iB_j) \cos p_i t_1 \sin p_j t_2 +\right.\right.$

$E(A_jB_i) \sin p_i t_1 \cos p, t_2 + E(B_iB_j) \sin p_i t_1 \sin p_j t_2)]$

$=\sum_{i=1}^{n}\sum_{j=1}^{n}[E(A_iA_j) \cos p_i t_1 \cos p_j t_2 + E(B_iB_j) \sin p_i t_1 \sin p_j t_2)]$

$=\sum_{i=1}^{n}[E(A_i)^2 \cos p_i t_1 \cos p_j t_2 + E(B_i)^2 \sin p_i t_1 \sin p_j t_2)]$

$= \sum_{=1}^{n}\sigma_i^2(\cos p_i t_1 \cos p_i t_2 + \sin p_i t_1 \sin p_i t_2) = \sum_{i=1}^{n}\sigma_i^2 \cos(p_i t_1 - p_i t_2)$

$R(\tau) = \sum_{i=1}^{n}\sigma_i^2 \cos p_i \tau$

$R_{XX}(t_1, t_2)$ is a function of τ.

Since the conditions (1) and (2) for WSS are satisfied.

Hence, $\{X(t)\}$ is a WSS Process.

7. Let $X(t) = B\sin(100t + \theta)$, where B and θ are independent RV' s such that θ is uniform distributed over $(-\pi, \pi)$ and B has mean '0' and variance

‘1’. Find mean and auto correlation function of {X(t)}

Solution:

Given $X(t) = B\sin(100t + \theta)$, where B and θ are independent RV' s.

θ is uniform distributed over $(-\pi, \pi)$.

$$f(\theta) = \frac{1}{2\pi}, -\pi < \theta < \pi$$

Mean of $B = 0 \Rightarrow E[B] = 0$

variance of B= $1 \Rightarrow E[B^2] = 1$

The mean of $X(t)$ is given by

$$E[X(t)] = E[B \sin(100t + \theta)]$$

$$= E[B]E[\sin(100t + \theta)$$

$$= 0 \times E[\sin(100t + \theta)]$$

$$E[X(t)] = 0$$

Mean of $X(t) = 0$

The ACF of $\{X(t)\}$ is given by

$$R_{XX}(t_1, t_2) = E[X(t_1)X(t_2)]$$

$$= E[B\sin(100t_1 + \theta)B\sin(100t_2 + \theta)]$$

$$= E[B^2\sin(100t_1 + \theta)\sin(100t_2 + \theta)]$$

$$= E[B^2]E[\sin(100t_1 + \theta)\sin(100t_2 + \theta)]$$

$$= 1 \times \frac{1}{2}E[\cos(100t_1 + \theta - 100t_2 - \theta) - \cos(100t_1 + \theta + 100t_2 + \theta)]$$

$$= \frac{1}{2}\left[E[\cos(100t_1 - 100t_2) - \cos(100t_1 + 100t_2 + 2\theta)]\right]$$

$$= \frac{1}{2}\left[E(\cos 100\tau - \cos(100t_1 + 100t_2 + 2\theta)]\right]$$

$$= \frac{1}{2}\cos 100\tau - \frac{1}{2}E[\cos(100t_1 + 100t_2 + 2\theta)]$$

$$= \frac{1}{2}\cos 100\tau - \frac{1}{2}\int_{-\pi}^{\pi} \cos(100t_1 + 100t_2 + 2\theta) f(\theta)\, d\theta$$

$$= \frac{1}{2}\cos 100\tau - \frac{1}{2}\int_{-\pi}^{\pi} \cos(100t_1 + 100t_2 + 2\theta)\frac{1}{2\pi}\, d\theta$$

$$= \frac{1}{2}\cos 100\tau - \frac{1}{4\pi}\left[\frac{\sin(100t_1+100t_2+2\theta)}{2}\right]_{-\pi}^{\pi}$$

$$= \frac{1}{2}\cos 100\tau - \frac{1}{8\pi}[\sin(100t_1 + 100t_2 + 2\pi) - \sin(100t_1 + 100t_2 + 2\pi)]$$

$$= \frac{1}{2}\cos 100\tau - \frac{1}{8\pi}(0) \quad = \frac{1}{2}\cos 100\tau$$

Auto Correlation function of X(t) = $\frac{1}{2}\cos 100\tau$

8. Let $X(t) = A\cos \lambda t + B\sin \lambda t$, where λ is a constant and A&B are independent RV 's with mean $'0'$ & variance 1. Prove that $\{X(t)\}$ is covariance stationary.

Solution:

Given $X(t) = A\cos \lambda t + B\sin \lambda t$, where A&B are RV's and λ is a constant.

Given $E(A) = E(B) = 0$.

Also given A and B are independent RV's.

$\therefore E[AB] = E[A]E[B] = 0$

Also given $\text{Var}(A) = \text{Var}(B) = 1$

$\Rightarrow E[A^2] = E[B^2] = 1$

To prove $\{X(t)\}$ is a covariance stationary process

(1) $E[X(t)]$ is a constant

(2) $C_{XX}(t_1, t_2)$ is a function of τ

(1) E[X(t)] $= E[A\cos\lambda t + B\sin\lambda t]$

$= E[A]\cos\lambda t + E[B]\sin\lambda t$

$E[X(t)] = 0$

$\therefore E[X(t)]$ is a constant.

(2) $C_{XX}(t_1, t_2) = E[X(t_1)X(t_2)] - E[X(t_1)]E[X(t_2)]$

$= E[(A\cos\lambda t_1 + B\sin\lambda t_1)(A\cos\lambda t_2 + B\sin\lambda t_2)] - 0 \times 0$

$= E[A^2\cos\lambda t_1\cos\lambda t_2 + AB\cos\lambda t_1\sin\lambda t_2 +$

$AB\sin\lambda t_1\cos\lambda t_2 + B^2\sin\lambda t_1\sin\lambda t_2]$

$= E[A^2]\cos\lambda t_1\cos\lambda t_2 + E[AB]\cos\lambda t_1\sin\lambda t_2 + E[AB]\sin\lambda t_1\cos\lambda t_2 +$

$E[B^2]\sin\lambda t_1\sin\lambda t_2$

$=\cos\lambda t_1\cos\lambda t_2 + 0 + 0 + \sin\lambda t_1\sin\lambda t_2$

$= \cos\lambda(t_1 - t_2)$

$$C_{XX}(t_1, t_2) = \cos \lambda\tau$$

$\therefore C_{XX}(t_1, t_2)$ is a function of τ

Since the conditions (1) and (2) for Covariance Stationary Process are {X(t)} is a covariance stationary process.

Problems under SSS process:

For a SSS process, $E[X^n(t)]$ $\boldsymbol{is\ a\ constant\ for\ every\ n}$.

1.Verify whether the sine wave $X(t) = Y \cos \omega t$, where Y is a random variable uniformly distributed over $(0, 1)$ is a SSS process or not.

Solution:

Given, $X(t) = Y \cos \omega t$, where Y is a random variable uniformly distributed over (0,1)

$$\therefore f_Y(y) = 1\ ;0 < y < 1$$

For a SSS process, $E[X^n(t)]$ $is\ a\ constant\ for\ every\ n.$

$$E[Y] = \int_0^1 y\, f_Y(y)\, dy$$

$$= \int_0^1 y\, dy = \left[\frac{y^2}{2}\right]_0^1 = \frac{1}{2}$$

$$E[X(t)] = E[Y \cos \omega t\,]$$

$$= E[Y] \cos \omega t$$

$= \frac{1}{2} \cos \omega t$

$E[X(t)]$ is not a constant.

Hence $\{X(t)\}$ is not a SSS process.

2. Consider random process X((t) defined by $X(t) = Ucost + Vsint$, where U and V are independent random variables each of which assumes the values -2 and 1 with probabilities 1/3and 2/3 respectively. Show that {X(t)} is a wide sense stationary process and not a strict sense stationary process(SSS)

Solution:

Given $X(t) = Ucost + Vsint$ where U and V are R.V'S with the following probability distributions

u	-2	1
P(u)	1/3	2/3

v	-2	1
P(v)	1/3	2/3

$$E(U) = \sum up(u) = \left(-2 \times \frac{1}{3}\right) + \left(1 \times \frac{2}{3}\right) = -\frac{2}{3} + \frac{2}{3} = 0$$

$$E(U^2) = \sum u^2 p(u) = \left(4 \times \frac{1}{3}\right) + \left(1 \times \frac{2}{3}\right) = \frac{4}{3} + \frac{2}{3} = 2$$

$$E(U^3) = \sum u^3 p(u) = \left(-8 \times \frac{1}{3}\right) + \left(1 \times \frac{2}{3}\right) = -\frac{8}{3} + \frac{2}{3} = -2$$

Similarly $E(V) = 0, E(V^2) = 2, E(V^3) = -2$

Since U and V are independent R.V'S , follow that

$$E(UV) = E(U)E(V) = 0 \times 0 = 0$$

$$E(U^2V) = E(U^2)E(V) = 2 \times 0 = 0$$

$$E(UV^2) = E(U)E(V^2) = 0 \times 2 = 0$$

To Prove $X(t)$ is WSS.

(i)$Mean = E[X(t)] = constant$

(ii) $Auto\ correlation\ \ R_{XX}(\tau) = E[X(t_1)X(t_2)]$ depends on τ

(i)$E[X(t)] = E[U\cos t\ + V\ sint\ ]$

$= E[U]\cos t\ + E[V]\sin\ t$

$= 0 + 0 = 0$

$E[X(t)]\ is\ a\ constant.$

(ii)$R_{XX}(t_1, t_2) = E[X(t_1)X(t_2)]$

$= E[(U\cos t_1\ + V\ sin\ t_1)(U\cos\ t_2\ + V\ sint_2)]$

$= E[U^2\cos\ t_1\ \cos\ t_2 + UV\ \cos\ t_1\ sin\ t_2 + UV\ sin\ t_1\ \cos\ t_2 +$
$V^2 sin\ t_1\ sin\ t_2]$

$= E[U^2\cos\ t_1\ \cos\ t_2\ ] + E[UV\ \cos t_1\ sin\ t_2] + E[UV\ sin\ t_1\ \cos\ t_2\ ] +$
$E[V^2 sin\ t_1\ sin\ t_2]$

$= E[U^2\cos\ t_1\ \cos\ t_2\ ] + E[V^2 sin\ t_1\ sin\ t_2]$

$= E[U^2]\cos\ t_1\ \cos t_2 + E[V^2] sin\ t_1\ sin\ t_2$

$= 2\cos\ t_1\ \cos\ t_2\ + 2\ sin\ t_1\ sin\ t_2$

$= 2[\cos\ \ t_1\ \cos\ \ t_2 + \ sin\ t_1\ sin\ \ t_2]$

$= 2[\cos(\ t_1 - \ t_2)]$

$\cos(-\theta) = \cos\theta$

$\cos\ A\cos\ B + \sin A\sin B = \cos(A - B)$

$= 2[\cos\ \tau]$

Since the conditions (1) and (2) for WSS are satisfied , $X(t)$ is WSS process.

To check {X(t)} is Strict Sense Stationary

$E[X^3(t)] = E[(U\cos t\ + V\ sint)^3]$

$= E(U^3cos^3t + 3U^2Vcos^2tsint + 3UV^2costsin^2t + V^3sin^3t)$

$= E(U^3)cos^3t + 3E(U^2V)cos^2tsint + 3E(UV^2)costsin^2t + E(V^3)sin^3t$

$= -2cos^3t + 0 + 0 - 2sin^3t = -2(cos^3t + sin^3t)$

Which depends on t.

Hence {X(t)} is not a strict sense stationary process.

Cross Correlation Function:

Let {X(t)} and {Y(t)}be two random processes. Then cross correlation function of X(t) and Y(t) is $R_{XY}(t_1, t_2\ ) = E[X(t_1)Y(t_2)]$

Jointly WSS Process:

Let {X(t)} and {Y(t)}be two random processes. Then the function of $X(t)$ and $Y(t)$ is said to be JWSS process if

(i)$E[X(t)] = constant$

(ii)$E[Y(t)] = constant$

(iii)$R_{XY}(t_1, t_2\ ) is\ a\ function\ of\ \tau$

Problems under Joint Wide Sense Stationary:

1.If $X(t) = 5\cos(10t + \theta)$ and $Y(t) = 20\sin(10t + \theta)$, where θ is uniformly distributed over $(0, 2\pi)$. Prove $\{X(t)\}$ and $\{Y(t)\}$ are JWSS.

Solution:

Given $(t) = 5\cos(10t + \theta), Y(t) = 20\sin(10t + \theta)$, where θ is a RV uniform distributed over $(0, 2\pi)$

$$f(\theta) = \frac{1}{2\pi}\ , 0 < \theta < 2\pi$$

To prove $\{X(t)\}$ and $\{Y(t)\}$ is a JWSS process.

(i)$E[X(t)] = constant$

(ii)$E[Y(t)] = constant$

(iii)$R_{XY}(t, t+\tau\,) is\ a\ function\ of\ \tau$

(i)$E[X(t)] = E[5\cos(10t+\theta)]$

$= 5\,E[\cos(10t+\theta)]$

$= 5\int_0^{2\pi} \cos(10t+\theta)\,\frac{1}{2\pi}\,d\theta$

$= \frac{5}{2\pi}\int_0^{2\pi} \cos(10t+\theta)d\theta$

$= \frac{5}{2\pi}[\sin(10t+\theta)]_0^{2\pi}$

$= \frac{5}{2\pi}[\sin(10t+2\pi) - \sin(10t)]$

$= \frac{5}{2\pi}[\sin 10t - \sin 10\,t] = 0$

$E[X(t)] = 0\ is\ a\ constant$

(ii)$E[Y(t)] = E[20\sin(10t+\theta)]$

$= 20\,E[\sin(10t+\theta)]$

$= 20\int_0^{2\pi} \sin\,(10t+\theta)\,\frac{1}{2\pi}\,d\theta$

$= \frac{20}{2\pi}\int_0^{2\pi} \sin\,(10t+\theta)d\theta$

$= \frac{10}{\pi}[-\ cos(10t+\theta)]_0^{2\pi}$

$= \frac{10}{\pi}[-\ \cos(10t+2\pi) + \cos(10t)]$

$$= \frac{10}{\pi}[-\cos 10t + \cos 10t] = 0$$

$E[Y(t)] = 0\ is\ a\ constant$

(iii)$R_{XY}(\tau) = E[X(t)Y(t+\tau)]$

$$= E[5\cos(10t+\theta)20\sin(10(t+\tau)+\theta)]$$

$$= E[5\cos(10t+\theta)20\sin(10t+10\tau+\theta)]$$

$$= \frac{100}{2}E[\sin(10t+\theta+10t+10\tau+\theta) - \sin(10t+\theta-10t-10\tau-\theta)]$$

$$= 50E[\sin(20t+10\tau+2\theta) - \sin(-10\tau)]$$

$$= 50E[\sin(20t+10\tau+2\theta) + \sin(10\tau)]$$

$$= 50\sin 10\tau + \frac{50}{2\pi}\int_0^{2\pi} \sin(20t+10\tau+2\theta)d\theta$$

$$= 50\sin 10\tau + \frac{25}{\pi}\left[\frac{-\cos(20t+10\tau+2\theta)}{2}\right]_0^{2\pi}$$

$$= 50\sin 10\tau + \frac{25}{2\pi}[-\cos(20t+10\tau+4\pi) + \cos(20t+10\tau)]$$

$$= 50\sin 10\tau + \frac{25}{2\pi}[-\cos(20t+10\tau) + \cos(20t+10\tau)]$$

$$= 50\sin 10\tau$$

$R_{XX}(t, t+\tau\ )is\ a\ function\ of\ \tau$

Since the conditions (i), (ii) and (iii) for JWSS are satisfied, $\{X(t)\}$ and $\{Y(t)\}$ are JWSS processes.

2. Two random processes are obtained by $X(t) = A\ cos\omega_0 t + Bsin\omega_0 t\ and\ Y(t) = B\ cos\omega_0 t - Asin\omega_0 t$. Show that X(t) and Y(t) are JWSS if A and B are uncorrelated random variables with zero mean and same variances and ω_0 is a constant.

Solution:

Given $X(t) = A\ cos\omega_0 t + Bsin\omega_0 t\ and\ Y(t) = B\ cos\omega_0 t - Asin\omega_0 t$

Where A and B are random variables with mean zero.

∴ E (A) = E (B) = 0……..(1)

Also given A and B have same variances

$(i.e)\ Var(A) = Var(B)$

∴ E (A^2) = E (B^2) = 0……..(2)

Also given A and B are uncorrelated

E(AB) = E(A) E(B) = 0……………(3)

To Prove {X(t)} and {Y(t)} is a JWSS process.

(i)$E[X(t)] = constant$

(ii)$E[Y(t)] = constant$

(iii)$R_{XY}(t_1, t_2\ ) is\ a\ function\ of\ \tau$

(i) $E[X(t)] = E[A\ cos\omega_0 t + Bsin\omega_0 t]$

$= E[A]\ cos\omega_0 t + E[B] sin\omega_0 t]$

= 0 (by 1)

$E[X(t)] = 0\ is\ a\ constant$

(ii) $E[Y(t)] = \mathrm{E}[B\ cos\omega_0 t - Asin\omega_0 t]$

$= \mathrm{E}[B]\ cos\omega_0 t - E[A] sin\omega_0 t]$

= 0 (by 1)

$E[Y(t)] = 0\ is\ a\ constant$

(iii) $R_{XY}(t_1, t_2) = E[X(t_1)Y(t_2)]$

$= E[(A\ cos\omega_0 t_1 + Bsin\omega_0 t_1)(B\ cos\omega_0 t_2 - Asin\omega_0 t_2)$

$= E[AB\ cos\omega_0 t_1\ cos\omega_0 t_2 - A^2\ cos\omega_0 t_1 sin\omega_0 t_2 + B^2\ sin\omega_0 t_1\ \ cos\omega_0 t_2 - ABsin\omega_0 t_1\ sin\omega_0 t_2]$

$= E[AB]\ cos\omega_0 t_1\ cos\omega_0 t_2 - E[A^2]\ cos\omega_0 t_1 sin\omega_0 t_2 + E[B^2]\ sin\omega_0 t_1\ \ cos\omega_0 t_2 - E[AB] sin\omega_0 t_1\ sin\omega_0 t_2]$

$= 0 - E[A^2]\ cos\omega_0 t_1 sin\omega_0 t_2 + E[B^2]\ sin\omega_0 t_1\ \ cos\omega_0 t_2 - 0$

$= -E[A^2]\ cos\omega_0 t_1 sin\omega_0 t_2 + E[B^2]\ sin\omega_0 t_1\ \ cos\omega_0 t_2$

$= E[A^2](sin\omega_0 t_1\ \ cos\omega_0 t_2 -\ cos\omega_0 t_1 sin\omega_0 t_2)$

$= E[A^2] sin\omega_0(t_1 - t_2)$

$= E[A^2] sin\omega_0\tau$, which is a function of τ

$$\therefore R_{XY}(t_1, t_2) is\ a\ function\ of\ \tau$$

$\sin A \cos B - \cos A \sin B = \sin(A - B)$

Since the conditions (1), (2) and (3) for JWSS are satisfied , $X(t) and\ Y(t) are$ is JWSS process.

Problems under Ergodic Process:

1. **Let $X(t) = A$, where A is a random variable. Prove that $\{X(t)\}$ is not a mean ergodic.**

Solution:

Given $X(t) = A$, where A is a random variable

To Prove $\{X(t)\}$ is a mean ergodic, we have to prove

$$E[X(t)] = \lim_{T\to\infty} \overline{X_T}$$

The ensemble mean of $\{X(t)\}$ is given by,

$$E[X(t)] = E[A] - - - - - - (1)$$

The time average is given by,

$$\overline{X_T} = \frac{1}{2T}\int_{-T}^{T} X(t)dt = \frac{1}{2T}\int_{-T}^{T} A\, dt$$

$$= \frac{A}{2T}\int_{-T}^{T} dt = \frac{A}{2T}\,[t]_{-T}^{T} = \frac{A}{2T}(2T) = A$$

$$\therefore \overline{X_T} = A$$

$$\lim_{T\to\infty} \overline{X_T} = A ----(2)$$

From $(1) and\ (2)$

$$E[X(t)] \neq \lim_{T\to\infty} \overline{X_T}$$

$\therefore \{X(t)\}$ is not mean Ergodic.

2. A random process has sample functions of the form $X(t) = A\cos(\omega t + \theta)$, where ω is constant and A is a random variable with mean zero and variance one and θ is also a random variable that is uniformly distributed between 0 and 2π. Assume that the random variables A and θ are independent. Prove that X(t) is a mean ergodic process?

Solution:

Given $X(t) = A\cos(\omega t + \theta)$, where A is a random variable with mean zero .

$$\therefore E(A) = 0, E(A^2) = 1$$

θ is uniformly distributed between 0 and 2π

$$f(\theta) = \frac{1}{2\pi}; 0 < \theta < 2\pi$$

To Prove {X(t)} is Mean Ergodic.

we have to prove

$$E[X(t)] = \lim_{T\to\infty} \overline{X_T}$$

The ensemble mean of $\{X(t)\}$ is given by,

$$E[X(t)] = E[A\cos(\omega t + \theta)]$$

$$= E[A]\cos(\omega t + \theta)\ since\ A\ and\ \theta\ are\ independent\ R.V'S$$

$$= 0 \ldots\ldots\ldots\ldots\ldots..(1)$$

The time average is given by,

$$\overline{X_T} = \frac{1}{2T}\int_{-T}^{T} X(t)dt$$

$$= \frac{1}{2T}\int_{-T}^{T} A\cos(\omega t + \theta)dt$$

$$= \frac{A}{2T}\int_{-T}^{T} \cos(\omega t + \theta)dt$$

$$= \frac{A}{2T}\left[\frac{\sin(\omega t+\theta)}{\omega}\right]_{-T}^{T}$$

$$\overline{X_T} = \frac{A}{2T\omega}[\sin(\omega T + \theta) - \sin(-\omega T + \theta)]$$

$$\lim_{T\to\infty} \overline{X_T} = \lim_{T\to\infty} \frac{A}{2T\omega}[\sin(\omega T + \theta) - \sin(-\omega T + \theta)]$$

$$= 0\ldots\ldots\ldots.(2)$$

From (1) *and* (2) , $E[X(t)] = \lim_{T\to\infty} \overline{X_T}$

$\therefore \{X(t)\}$ is a mean Ergodic Process.

Correlation Ergodic Process:

Let {X(t)} be a random process. The ensemble auto correlation function is

$R_{XX}(t_1, t_2) = E[X(t_1)X(t_2)]$

The time auto correlation function is $\overline{X_T} = \frac{1}{2T}\int_{-T}^{T} X(t)\, X(t+\tau\,)dt$

A process {X(t)} is said to be correlation ergodic if $R_{XX}(t_1, t_2) = \lim_{T\to\infty} \overline{X_T}$

Problems under Correlation Ergodic Process:

1. **Given a WSS random process $\{X(t)\} = 10\cos(100t+\theta)$, where θ is uniformly distributed over $(-\pi, \pi)$. Prove that $X(t)$ is a correlation ergodic.**

Solution:

Given $\{X(t)\} = 10\cos(100t+\theta)$

$\Rightarrow f(\theta) = \frac{1}{2\pi}\,, -\pi < \theta < \pi$

$R_{XX}(t, t+\tau\,) = E[X(t)X(t+\tau)]$

$= E[10\cos(100t+\theta)\, 10cos\,(100(t+\tau)+\theta)]$

$= \frac{100}{2} E[\cos(100t+\theta)\, cos\,(100t+100\tau+\theta)]$

$= 50\, E[\cos(100t+\theta+100t+100\tau+\theta)+\cos\,(100t+\theta-100t-$

$100\tau-\theta)]$

$= 50\, E[\cos(200t+2\theta+100\tau)+\cos\,(-100\tau)]$

$= 50\, E[\cos(200t+2\theta+100\tau)+\cos\,(100\tau)]$

$= 50\cos\,(100\tau)+\frac{50}{2\pi}\int_0^{2\pi}\cos\,(200t+100\tau+2\theta)d\theta$

$= 50\cos\,(100\tau)+\frac{25}{\pi}\left[\frac{\sin\,(200t+100\tau+2\theta)}{2}\right]_0^{2\pi}$

$= 50\cos\ 100\tau+\frac{25}{2\pi}\,[\sin\,(200t+100\tau+4\pi)-\sin(200t+100\tau-0)$

$= 50\cos\ 100\tau+\frac{25}{2\pi}\,[\sin\,(200t+100\tau)-\sin(200t+100\tau)]$

$$R_{XX}(t,t+\tau\,)=50\cos\ 100\tau$$

Let $\overline{X_T}=\frac{1}{2T}\int_{-T}^{T}X(t)\,X(t+\tau\,)dt$

$$=\frac{1}{2T}\left[\int_{-T}^{T}50\cos(100\tau)dt+50\ \int_{-T}^{T}\cos(200t+100\tau+2\theta)dt\right]$$

$$=\frac{1}{2T}\left\{[50\cos(100\tau)\,t]_{-T}^{T}+50\left[\frac{\sin\,(200t+100\tau+2\theta)}{200}\right]_{T}^{T}\right\}$$

$= \frac{1}{2T}\Big\{[50\cos(100\tau)\,(T-(-T)] + \frac{50}{200}[\sin\,(200t + 100\tau + 2\theta) -$

$\sin\,(200(-T) + 100\tau + 2\theta)]\Big\}$

$$= \frac{1}{2T}\Big\{[50\cos(100\tau)\,(2T)] + \frac{1}{4}[\sin\,(200t + 100\tau + 2\theta) - \sin\,(-200T + 100\tau + 2\theta)]\Big\}$$

$$\lim_{T\to\infty} \overline{X_T} = \lim_{T\to\infty} \frac{1}{2T}\Big\{[50\cos(100\tau)\,(2T)] + \frac{1}{4}[\sin\,(200t + 100\tau + 2\theta) - \sin\,(-200T + 100\tau + 2\theta)]\Big\}$$

$$= \lim_{T\to\infty} \frac{1}{2T} 50\cos(100\tau)\,(2T) + \lim_{T\to\infty} \frac{1}{8T}[\sin\,(200t + 100\tau + 2\theta) - \sin\,(-200T + 100\tau + 2\theta)]$$

$$= \lim_{T\to\infty} 50\cos(100\tau) + \lim_{T\to\infty} \frac{1}{8T}[\sin\,(200t + 100\tau + 2\theta) - \sin\,(-200T + 100\tau + 2\theta)]$$

$$= 50\cos 100\tau + 0$$

$$= 50\cos 100\tau$$

$$R_{XX}(t, t+\tau\,) = \lim_{T\to\infty} \overline{X_T}$$

Hence $X(t)$ is a correlation ergodic.

2. Find the ACF of the periodic time function $X(t) = A\, sin\omega t$.

Solution:

Since periodic time function X(t) is given, we use time auto correlation function.

The ACF of the process is given by $R_{XX}(t_1, t_2) = \lim_{T\to\infty} \overline{X_T}$

To find $\lim_{T\to\infty} \overline{X_T}$

$$\overline{X_T} = \frac{1}{2T}\int_{-T}^{T} X(t)\, X(t+\tau)dt$$

$$= \frac{1}{2T}\int_{-T}^{T} A\, sin\omega t\; A\sin(\omega t + \omega\tau)dt$$

$$= \frac{A^2}{2T}\int_{-T}^{T} sin\omega t\; \sin(\omega t + \omega\tau)dt$$

$$= \frac{A^2}{4T}\int_{-T}^{T} [\cos(\omega t - \omega t - \omega\tau) - \cos(\omega t + \omega t + \omega\tau)]dt$$

$$= \frac{A^2}{4T}\int_{-T}^{T} [\cos(-\omega\tau) - \cos(2\omega t + \omega\tau)]dt$$

$$= \frac{A^2}{4T}\left[cos\omega\tau(t)_{-T}^{T} - \left(\frac{\sin(2\omega t + \omega\tau)}{2\omega}\right)_{-T}^{T}\right]$$

$$= \frac{A^2}{4T}\left[cos\omega\tau(2T) - \frac{\sin(2\omega T + \omega\tau)}{2\omega} + \frac{\sin(-2\omega T + \omega\tau)}{2\omega}\right]$$

$$= \frac{A^2}{4T}\ cos\omega\tau(2T) + \frac{A^2}{4T}\left[- \frac{\sin(2\omega T + \omega\tau)}{2\omega} + \frac{\sin(-2\omega T + \omega\tau)}{2\omega}\right]$$

The ACF of the process is given by

$$R_{XX}(t_1, t_2) = \lim_{T \to \infty} \overline{X_T}$$

$$= \frac{A^2}{2}\ cos\omega\tau + \lim_{T \to \infty} \frac{A^2}{4T}\left[- \frac{\sin(2\omega T + \omega\tau)}{2\omega} + \frac{\sin(-2\omega T + \omega\tau)}{2\omega}\right]$$

$$R_{XX}(t_1, t_2) = \frac{A^2}{2}\ cos\omega\tau$$