# UNIT - III

# FUNCTIONS OF SEVERAL VARIABLES

## 3.1 Introduction

The students have studied in the lower classes the concept of partial differentiation of a function of more than one variable. They were also exposed to Homogeneous functions of several variables and Euler's theorem associated with such functions. In this chapter, we discuss some of the applications of the concept of partial differentiation, which are frequently required in engineering problems.

**Functions of Two Variable**

If for every $x$ and $y$ a unique value $f(x,y)$ is associated, then $f$ is said to be a function of two independent variables $x$ and $y$. It is denoted by $z = f(x,y)$.

**Limits and Continuity**

**Limits**

A function $f(x,y)$ is said to tend to the limit $\ell$ as $x \to a$ and $y \to b$, if and only if the limit $\boldsymbol{\ell}$ is independent of the path followed by the point $(x,y)$ as $x \to a$ and $y \to b$.

Then, $\lim\limits_{\substack{x\to a\\ y\to b}} f(x,y) = \ell$

**Continuity**

A function $f(x,y)$ is said to be continuous at the point $(a,b)$ if $\lim\limits_{\substack{x\to a\\ y\to b}} f(x,y)$ exist and equal to $f(a,b)$.

If a function is continuous at all points of a region, then it is said to be continuous in that region.

A function which is not continuous at a point is said to be discontinuous at that point.

**Note**

Generally $\lim\limits_{x\to a}[\lim\limits_{y\to b} f(x,y)] = \lim\limits_{y\to b}[\lim\limits_{x\to a} f(x,y)]$

**Results**

1. If $\lim\limits_{\substack{x\to a\\ y\to b}} f(x,y) = \ell$ and $\lim\limits_{\substack{x\to a\\ y\to b}} g(x,y) = \text{m}$

Then (i) $\lim_{\substack{x\to a \\ y\to b}} [f(x,y) \pm g(x,y)] = \ell \pm m$

(ii) $\lim_{\substack{x\to a \\ y\to b}} [f(x,y).\, g(x,y)] = \ell . m$

(iii) $\lim_{\substack{x\to a \\ y\to b}} [f(x,y)/g(x,y)] = \ell / m$

2. If $(x,y)$ , $g(x,y)$ are continuous at $(a,b)$ then the following functions are continuous.

   (i) $f(x,y) \pm g(x,y)$

   (ii) $f(x,y).\, g(x,y)$ and

   (iii) $f(x,y)/g(x,y)$ provided $g(x,y) \neq 0$

**Example:**

**Evaluate** $\lim_{\substack{x\to 0 \\ y\to 0}} (x^2 + y^2)$

**Solution:**

$$\lim_{\substack{x\to 0 \\ y\to 0}} (x^2 + y^2) = \lim_{x\to 0} [\lim_{y\to 0} (x^2 + y^2)$$

$$= \lim_{x\to 0} x^2 = 0$$

**Example:**

**Evaluate** $\lim_{\substack{x\to \infty \\ y\to 2}} \frac{xy+5}{x^2+2y^2}$

**Solution:**

$$\lim_{\substack{x\to \infty \\ y\to 2}} \frac{xy+5}{x^2+2y^2} = \lim_{x\to\infty} [\lim_{y\to 2} \frac{xy+5}{x^2+y^2}]$$

$$= \lim_{x\to\infty} \frac{2x+5}{x^2+8}$$

$$= \lim_{x\to\infty} \frac{x(2+\frac{5}{x})}{x^2(1+\frac{8}{x^2})}$$

$$= \frac{2+\frac{5}{\infty}}{\infty(1+\frac{8}{\infty})}$$

$$= \frac{2+0}{\infty(1+0)} = \frac{2}{\infty} = 0$$

**Example:**

**Evaluate** $\lim\limits_{\substack{x\to 1\\ y\to 2}} \frac{x^2+y}{3x+y^2}$

**Solution:**

$$\lim_{\substack{x\to 1\\ y\to 2}} \frac{x^2+y}{3x+y^2} = \lim_{x\to 1}\left[\lim_{y\to 2} \frac{x^2+y}{3x+y^2}\right]$$

$$= \lim_{x\to 1} \frac{x^2+2}{3x+4}$$

$$= \frac{1+2}{3+4} = \frac{3}{7}$$

**Example:**

**Evaluate** $\lim\limits_{\substack{x\to 0\\ y\to 2}} \frac{2xy+1}{x^2+y^2}$

**Solution:**

$$\lim_{\substack{x\to 0\\ y\to 2}} \frac{2xy+1}{x^2+y^2} = \lim_{x\to 0}\left[\lim_{y\to 2} \frac{2xy+1}{x^2+y^2}\right]$$

$$= \lim_{x\to 0} \frac{4x+1}{x^2+4}$$

$$= \frac{0+1}{0+4} = \frac{1}{4}$$

**Example:**

**Discuss the continuity of** $f(x,y) = \begin{cases} \frac{x}{\sqrt{x^2+y^2}}, & (x,y) \neq (0,0) \\ 2, & (x,y) = (0,0) \end{cases}$

**Solution:**

$$\lim_{\substack{x\to 0\\ y\to 0}} \frac{x}{\sqrt{x^2+y^2}} = \lim_{x\to 0}\left[\lim_{y\to 0} \frac{x}{\sqrt{x^2+y^2}}\right]$$

$$= \lim_{x\to 0} \frac{x}{\sqrt{x^2}}$$

$$= \frac{x}{x} = 1$$

$$\lim_{\substack{y\to 0\\ x\to 0}} \frac{x}{\sqrt{x^2+y^2}} = \lim_{y\to 0}\left[\lim_{x\to 0} \frac{x}{\sqrt{x^2+y^2}}\right]$$

$$= \lim_{y\to 0} \frac{0}{\sqrt{y^2}} \quad = 0$$

here $\lim_{\substack{x\to 0\\ y\to 0}} \frac{x}{\sqrt{x^2+y^2}} \neq \lim_{\substack{y\to 0\\ x\to 0}} \frac{x}{\sqrt{x^2+y^2}}$

∴ The function is discontinuous at the origin.

**Exercise:**

**Evaluate the following**:

1. $\lim_{\substack{x\to 2\\ y\to 3}} \frac{x^2+y^3}{2x^2y}$ **Ans:** $\frac{31}{24}$
2. $\lim_{\substack{x\to 0\\ y\to 0}} \frac{x-y}{x+2y}$ **Ans:** limit does not exist
3. $\lim_{\substack{x\to 1\\ y\to 1}} \frac{x(y-2)}{y(x-2)}$ **Ans:** 1
4. $\lim_{\substack{x\to \infty\\ y\to 3}} \frac{2xy-3}{x^3+4y^3}$ **Ans:** 0
5. Discuss the continuity of $f(x,y) = \begin{cases} \frac{x^2}{\sqrt{x^2+y^2}}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$ **Ans:** Discontinuous
6. Given $f(x,y) = \begin{cases} \frac{x^2-y^2}{x^2+y^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$ show that f is discontinuous at origin

## 3.1 Partial, Homogeneous and Euler's theorem

### Partial Differentiation

Let $u = f(x,y)$ be a function of two independent variables $x$ and $y$, then

Differentiating '$u$' with respect to '$x$' keeping '$y$' as a constant and it is denoted by $\frac{\partial u}{\partial x}$ or $u_x$, Similarly $\frac{\partial u}{\partial y}$ or $u_y$ means differentiating 'u' with respect to '$y$' keeping '$x$' as a constant.

$\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial y}$ are called first order partial derivatives.

Symbolically, if $u = u(x,y)$ then

$$\frac{\partial u}{\partial x} = \lim_{\Delta x \to 0} \frac{u(x+\Delta x, y) - u(x,y)}{\Delta x}$$

$$\frac{\partial u}{\partial y} = \lim_{\Delta y \to 0} \frac{u(x, y+\Delta y) - u(x,y)}{\Delta y}$$

**Rule's of partial differentiation:**

(i) Differential co-efficient of a sum:

If $u = v + w + \cdots$, where $v, w, \cdots$ are functions of $x, y, \cdots$ then

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial x} + \frac{\partial w}{\partial x} + \cdots$$

$$\frac{\partial u}{\partial y} = \frac{\partial v}{\partial y} + \frac{\partial w}{\partial y} + \cdots \text{ and so on.}$$

(ii) Differential co-efficient of a product:

If $u$ and $v$ are functions of $x, y, z$ etc, then

$$\frac{\partial (uv)}{\partial x} = u\,\frac{\partial v}{\partial x} + v\,\frac{\partial u}{\partial x}$$

$$\frac{\partial (uv)}{\partial y} = u\,\frac{\partial v}{\partial y} + v\,\frac{\partial u}{\partial y}$$

(iii) Differential co-efficient of a quotient:

If $u$ and $v$ are functions of $x, y, z$ etc, then

$$\frac{\partial}{\partial x}\left(\frac{u}{v}\right) = \frac{v\,\frac{\partial u}{\partial x} - u\frac{\partial v}{\partial x}}{v^2}$$

$$\frac{\partial}{\partial y}\left(\frac{u}{v}\right) = \frac{v\,\frac{\partial u}{\partial y} - u\frac{\partial v}{\partial y}}{v^2}$$

(iv)Derivative of a function:

If $u$ is a function of $t$ where $t$ is a function of the variables $x, y, z \cdots$ then

$$\frac{\partial u}{\partial x} = \frac{du}{dt} \times \frac{\partial t}{\partial x}$$

$$\frac{\partial u}{\partial y} = \frac{du}{dt} \times \frac{\partial t}{\partial y} \quad \text{and so on.}$$

**Successive partial Differentiation**

Let $u = f(x, y)$ be a function of two independent variables $x$ and $y$. Then $\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial y}$ will represent the first order partial derivative of '$u$' with respect to '$x$' and '$y$'. Here both $\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial y}$ are again in general a function of $x$ and $y$. Hence each of these partial derivatives may again be differentiated with respect to '$x$' and '$y$' respectively and it is denoted by $\frac{\partial^2 u}{\partial x^2}$ , $\frac{\partial^2 u}{\partial y^2}$ , $\frac{\partial^2 u}{\partial x \partial y}$

$u_{xx} = \frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right)$ = differentiating $\frac{\partial u}{\partial x}$ with respect to '$x$' keeping '$y$' as a constant.

$u_{yy} = \frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right)$ = differentiating $\frac{\partial u}{\partial y}$ with respect to '$y$' keeping '$x$' as a constant.

$u_{xy} = \frac{\partial^2 u}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right)$ = differentiating $\frac{\partial u}{\partial y}$ with respect to '$x$' keeping '$y$' as a constant.

**Note:**

$\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial x \partial y}$ or $u_{xy} = u_{yx}$

**Example:**

**If $u = (x - y)^4 + (y - z)^4 + (z - x)^4$, show that $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0$**

**Solution:**

Given $u = (x - y)^4 + (y - z)^4 + (z - x)^4$

$$\frac{\partial u}{\partial x} = 4(x - y)^3 + 4(z - x)^3(-1) \ldots (1)$$

$$\frac{\partial u}{\partial y} = 4(x - y)^3(-1) + 4(y - z)^3 \ldots (2)$$

$$\frac{\partial u}{\partial z} = 4(y - z)^3(-1) + 4(z - x)^3 \ldots (3)$$

$(1) + (2) + (3) \Rightarrow \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z}$

$= 4(x - y)^3 - 4(z - x)^3 - 4(x - y)^3 + 4(y - z)^3 - 4(y - z)^3 + 4(z - x)^3 =$

**Example:**

**If $f(x,y) = log\sqrt{x^2+y^2}$, show that by $\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0$**

**Solution:**

Given $f = log\sqrt{x^2+y^2}$

$$\frac{\partial f}{\partial x} = \frac{1}{\sqrt{x^2+y^2}} \times \frac{1}{2\sqrt{x^2+y^2}} \times 2x$$

$$= \frac{x}{x^2+y^2}$$

$$\frac{\partial^2 f}{\partial x^2} = \frac{(x^2+y^2)1-x(2x)}{(x^2+y^2)^2}$$

$$= \frac{(y^2-x^2)}{(x^2+y^2)^2} \ldots(1)$$

$$\frac{\partial f}{\partial y} = \frac{1}{\sqrt{x^2+y^2}} \times \frac{1}{2\sqrt{x^2+y^2}} \times 2y$$

$$= \frac{y}{x^2+y^2}$$

$$\frac{\partial^2 f}{\partial y^2} = \frac{(x^2+y^2)1-y(2y)}{(x^2+y^2)^2}$$

$$= \frac{(x^2-y^2)}{(x^2+y^2)^2} \ldots(2)$$

$$(1)+(2) \Rightarrow \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = \frac{(y^2-x^2)}{(x^2+y^2)^2} + \frac{(x^2-y^2)}{(x^2+y^2)^2}$$

$$= \frac{(y^2-x^2+x^2-y^2)}{(x^2+y^2)^2} = 0$$

**Example:**

**If $r^2 = x^2+y^2$ then *show* that $\frac{\partial^2 r}{\partial x^2} + \frac{\partial^2 r}{\partial y^2} = \frac{1}{r}\left[\left(\frac{\partial r}{\partial x}\right)^2 + \left(\frac{\partial r}{\partial y}\right)^2\right]$**

**Solution:**

Given $r^2 = x^2 + y^2$

Differentiating partially with respect to '$x$'

$$2r\frac{\partial r}{\partial x} = 2x \quad \Rightarrow \frac{\partial r}{\partial x} = \frac{x}{r}$$

$$\frac{\partial^2 r}{\partial x^2} = \frac{r.1 - x.\frac{\partial r}{\partial x}}{r^2}$$

$$= \frac{r - x.\frac{x}{r}}{r^2}$$

$$= \frac{\frac{r^2-x^2}{r}}{r^2} = \frac{r^2-x^2}{r^3} \ldots (1)$$

Similarly $\frac{\partial^2 r}{\partial y^2} = \frac{r^2-y^2}{r^3} \ldots (2)$

$(1) + (2) \Rightarrow \frac{\partial^2 r}{\partial x^2} + \frac{\partial^2 r}{\partial y^2}$

$$\Rightarrow \frac{r^2-x^2}{r^3} + \frac{r^2-y^2}{r^3} = \frac{r^2-x^2+r^2-y^2}{r^3}$$

$$= \frac{2r^2-(x^2+y^2)}{r^3}$$

$$= \frac{2r^2-r^2}{r^3}$$

$$= \frac{r^2}{r^3} = \frac{1}{r} = \text{L.H.S}$$

$$\left(\frac{\partial r}{\partial x}\right)^2 = \left(\frac{x}{r}\right)^2 = \frac{x^2}{r^2}$$

Similarly $\left(\frac{\partial r}{\partial y}\right)^2 = \left(\frac{y}{r}\right)^2 = \frac{y^2}{r^2}$

$$\left(\frac{\partial r}{\partial x}\right)^2 + \left(\frac{\partial r}{\partial y}\right)^2 = \frac{x^2+y^2}{r^2} = \frac{r^2}{r^2} = 1 \quad (\because r^2 = x^2 + y^2)$$

$$\text{R.H.S} = \frac{1}{r}\left[\left(\frac{\partial r}{\partial x}\right)^2 + \left(\frac{\partial r}{\partial y}\right)^2\right]$$

$$= \frac{1}{r} \times 1 = \frac{1}{r}$$

L.H.S = R.H.S

$$\therefore \frac{\partial^2 r}{\partial x^2} + \frac{\partial^2 r}{\partial y^2} = \frac{1}{r}\left[\left(\frac{\partial r}{\partial x}\right)^2 + \left(\frac{\partial r}{\partial y}\right)^2\right]$$

## Homogeneous Function

Consider the expression $f(x, y) = a_0 x_n + a_1 x_{n-1} y + a_2 x_{n-2} y_2 + \cdots + a_n y_n \ldots (1)$

The degree of each term in the above expression is 'n'. Such an expression is called a homogeneous function of degree '$n$', Equation (i) can be written as $f(x, y) = x^n f\left(\frac{y}{x}\right)$.

**Note:**

A function $f(x, y)$ is said to be a homogeneous function in $x$ and $y$ of degree '$n$' if

$$f(tx, ty) = t^n f(x, y)$$

For example, $f(x, y) = \frac{x^3+y^3}{x-y}$

$$f(tx, ty) = \frac{t^3x^3+t^3y^3}{tx-ty}$$

$$= \frac{t^3(x^3+y^3)}{t(x-y)}$$

$$= \frac{t^2(x^3+y^3)}{x-y}$$

$$f(tx, ty) = t^2 f(x, y)$$

$\therefore$ $f(x, y)$ is a homogeneous function in of degree '2'.

**Euler's theorem on homogeneous function:**

If $u$ is a homogeneous function in $x$ and $y$ of degree '$n$' then $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nu$

**Proof:**

Given '$u$' is a homogeneous function of degree '$n$' in $x$ and $y$

$$\therefore u(x, y) = x^n f\left(\frac{y}{x}\right)$$

$$\frac{\partial u}{\partial x} = nx^{n-1} f\left(\frac{y}{x}\right) + x^n f'\left(\frac{y}{x}\right)\left(-\frac{y}{x^2}\right)$$

$$= nx^{n-1} f\left(\frac{y}{x}\right) - yx^{n-2} f'\left(\frac{y}{x}\right)$$

and $\frac{\partial u}{\partial x} = x^n f'\left(\frac{y}{x}\right)\left(\frac{1}{x}\right)$

$$= x^{n-1} f'\left(\frac{y}{x}\right)$$

Hence $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nx^n f\left(\frac{y}{x}\right) - yx^{n-1} f'\left(\frac{y}{x}\right) + yx^{n-1} f'\left(\frac{y}{x}\right)$

$$= nx^n f\left(\frac{y}{x}\right) = nu$$

$$\therefore \quad x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nu$$

**Note:**

(i) If '$u$' is a homogeneous function of three variables $x, y$ and $z$ of degree '$n$', then the Euler's theorem is $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = nu$

(ii)Euler's extension theorem is $x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = n(n-1)u$

**Example:**

**Verify Euler's theorem for the function $u = x^3 + y^3 + z^3 + 3xyz$**

**Solution:**

Given $u(x, y, z) = x^3+y^3+z^3+3xyz$

$$u(tx, ty, tz) = t^3x^3+t^3y^3+t^3z^3+3txtytz$$

$$= t^3( x^3+y^3+z^3+3xyz)$$

$$= t^3u(x, y, z)$$

$\therefore$ $u(x, y, z)$ is a homogeneous function in of degree '3'.

$\therefore$ Euler's theorem is $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = 3u$

Verification: Consider L.H.S = $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z}$

$$\frac{\partial u}{\partial x} = 3x^2+3yz$$

$$x\frac{\partial u}{\partial x} = 3x^3+3xyz \quad \dots.(1)$$

$$\frac{\partial u}{\partial y} = 3y^2+3xz$$

$$y\frac{\partial u}{\partial y} = 3y^3+3xyz \quad \dots(2)$$

$$\frac{\partial u}{\partial z} = 3z^2+3xy$$

$$z\frac{\partial u}{\partial z} = 3z^3+3xyz \quad \dots(3)$$

$(1) + (2) + (3) \Rightarrow$ L.H.S $= x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z}$

$$= 3x^3+3xyz + 3y^3+3xyz + 3z^3+3xyz$$

$$= 3x^3+ 3y^3 +3z^3+9xyz$$

$$= 3( x^3+y^3+z^3+3xyz) = 3u = \text{R.H.S}$$

Hence Euler's theorem is verified.

**Example:**

**Verify Euler's theorem for the function $u = \left(x^{\frac{1}{2}} + y^{\frac{1}{2}}\right)(x^n + y^n)$**

**Solution:**

Given $u(x,y) = \left(x^{\frac{1}{2}} + y^{\frac{1}{2}}\right)(x^n + y^n)$

$$\text{U}(tx,ty) = \left(t^{\frac{1}{2}}x^{\frac{1}{2}} + t^{\frac{1}{2}}y^{\frac{1}{2}}\right)(t^n x^n + t^n y^n)$$

$$= t^{\frac{1}{2}}\left(x^{\frac{1}{2}} + y^{\frac{1}{2}}\right)t^n(x^n + y^n)$$

$$= t^{n+\frac{1}{2}}\left(x^{\frac{1}{2}} + y^{\frac{1}{2}}\right)(x^n + y^n$$

$$u(tx,ty) = t^{n+\frac{1}{2}}u(x,y)$$

$\therefore$ $u(x,y)$ is a homogeneous function in of degree '$n + \frac{1}{2}$'.

$\therefore$ Euler's theorem is $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = (n + \frac{1}{2})$u

Verification: Consider L.H.S $= x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}$

$$\frac{\partial u}{\partial x} = \frac{1}{2}x^{-\frac{1}{2}}(x^n + y^n) + nx^{n-1}(x^{\frac{1}{2}} + y^{\frac{1}{2}})$$

$$x\frac{\partial u}{\partial x} = \frac{1}{2}x^{\frac{1}{2}}(x^n + y^n) + nx^n\left(x^{\frac{1}{2}} + y^{\frac{1}{2}}\right) \ldots (1)$$

$$\frac{\partial u}{\partial y} = \frac{1}{2}y^{-\frac{1}{2}}(x^n + y^n) + ny^{n-1}(x^{\frac{1}{2}} + y^{\frac{1}{2}})$$

$$y\frac{\partial u}{\partial y} = \frac{1}{2}y^{\frac{1}{2}}(x^n + y^n) + ny^n\left(x^{\frac{1}{2}} + y^{\frac{1}{2}}\right) \ldots (2)$$

$(1) + (2) \Rightarrow$ L.H.S $= x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}$

$= \frac{1}{2}x^{\frac{1}{2}}(x^n + y^n) + nx^n(x^{\frac{1}{2}} + y^{\frac{1}{2}}) + \frac{1}{2}y^{\frac{1}{2}}(x^n + y^n) +$

$ny^n(x^{\frac{1}{2}} + y^{\frac{1}{2}})$

$= \frac{1}{2}(x^{\frac{1}{2}} + y^{\frac{1}{2}})(x^n + y^n) + n(x^{\frac{1}{2}} + y^{\frac{1}{2}})(x^n + y^n)$

$= (n + \frac{1}{2})(x^{\frac{1}{2}} + y^{\frac{1}{2}})(x^n + y^n)$

$= (n + \frac{1}{2})u$ = R.H.S

Hence Euler's theorem is verified.

**Example:**

**Verify Euler's theorem for the function $u = sin^{-1}\left(\frac{x}{y}\right) + tan^{-1}\left(\frac{y}{x}\right)$**

**Solution:**

Given $u(x,y) = sin^{-1}\left(\frac{x}{y}\right) + tan^{-1}\left(\frac{y}{x}\right)$

$$u(tx,ty) = sin^{-1}\left(\frac{tx}{ty}\right) + tan^{-1}\left(\frac{ty}{tx}\right)$$

$$= t^0\left[sin^{-1}\left(\frac{x}{y}\right) + tan^{-1}\left(\frac{y}{x}\right)\right]$$

$$u(tx,ty) = t^0\, u(x,y)$$

$\therefore$ $u(x,y)$ is a homogeneous function in of degree '0'.

$\therefore$ Euler's theorem is $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}$ = 0.u = 0

Verification: Consider L.H.S = $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}$

$$\frac{\partial u}{\partial x} = \frac{1}{\sqrt{1-\frac{x^2}{y^2}}} \times \frac{1}{y} + \frac{1}{1+\frac{y^2}{x^2}} \times \left(-\frac{y}{x^2}\right)$$

$$= \frac{1}{\sqrt{y^2+x^2}} - \frac{y}{x^2+y^2}$$

$$x\frac{\partial u}{\partial x} = \frac{x}{\sqrt{y^2+x^2}} - \frac{xy}{x^2+y^2} \ldots (1)$$

$$\frac{\partial u}{\partial y} = \frac{1}{\sqrt{1-\frac{x^2}{y^2}}} \times \left(\frac{-x}{y^2}\right) + \frac{1}{1+\frac{y^2}{x^2}} \times \frac{1}{x}$$

$$= -\frac{x}{y\sqrt{y^2+x^2}} + \frac{x}{x^2+y^2}$$

$$y\frac{\partial u}{\partial y} = -\frac{x}{\sqrt{y^2+x^2}} + \frac{xy}{x^2+y^2} \quad \dots (2)$$

$$(1) + (2) \Rightarrow \quad L.H.S = x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}$$

$$= \frac{x}{\sqrt{y^2+x^2}} - \frac{xy}{x^2+y^2} - \frac{x}{\sqrt{y^2+x^2}} + \frac{xy}{x^2+y^2}$$

= 0 = R.H.S

Hence Euler's theorem is verified.

**Example:**

**If $= \frac{x}{y} + \frac{y}{z} + \frac{z}{x}$, show that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = 0$**

**Solution:**

Given $u(x, y, z) = \frac{x}{y} + \frac{y}{z} + \frac{z}{x}$

$$u(tx, ty, tz) = \frac{tx}{ty} + \frac{ty}{tz} + \frac{tz}{tx}$$

$$= t^0\left(\frac{x}{y} + \frac{y}{z} + \frac{z}{x}\right)$$

$$u(tx, ty, tz) = t^0\, u(x, y, z)$$

$\therefore$ $u(x, y, z)$ is a homogeneous function in of degree '0'.

$\therefore$ By Euler's theorem $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = 0.u = 0$

**Example:**

**If $u = tan^{-1}\left(\frac{x^3+y^3}{x-y}\right)$ then prove that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = sin2u$**

**Solution:**

Given $u(x, y) = tan^{-1}\left(\frac{x^3+y^3}{x-y}\right)$

$$tanu = \frac{x^3+y^3}{x-y}$$

Let $z = tanu = \frac{x^3+y^3}{x-y}$

Consider $z(x,y) = \frac{x^3+y^3}{x-y}$

$$z(tx,ty) = \frac{t^3x^3+t^3y^3}{tx-ty} = t^2 z(x,y)$$

$\therefore$ $z(x,y)$ is a homogeneous function in of degree '2'.

$\therefore$ By Euler's theorem $x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = 2.z$

Put $z = tanu$

$$x\frac{\partial(tanu)}{\partial x} + y\frac{\partial(tanu)}{\partial y} = 2tanu$$

$$xsec^2u\frac{\partial u}{\partial x} + ysec^2u\frac{\partial u}{\partial y} = 2tanu$$

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 2\frac{sinu}{cosusec^2u}$$

$$= \frac{2\frac{sinu}{cosu}}{\frac{1}{cos^2u}}$$

$$= 2sinu\,cosu$$

$$= sin2u$$

Hence proved.

**Example:**

**If $u = sin^{-1}\left(\frac{x+y}{\sqrt{x}+\sqrt{y}}\right)$, then prove that $x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\partial y} + y^2\frac{\partial^2 u}{\partial y^2} = \frac{-sinucos2u}{4cos^3u}$**

**Solution:**

Given $u(x,y) = sin^{-1}\left(\frac{x+y}{\sqrt{x}+\sqrt{y}}\right)$

$$sinu = \frac{x+y}{\sqrt{x}+\sqrt{y}}$$

Let $z = sinu = \frac{x+y}{\sqrt{x}+\sqrt{y}}$

Consider $z(x,y) = \frac{tx+ty}{\sqrt{t}\sqrt{x}+\sqrt{t}\sqrt{y}}$

$$z(tx,ty) \quad = \quad t^{\frac{1}{2}}(\frac{x+y}{\sqrt{x}+\sqrt{y}}) \quad = \quad t^{\frac{1}{2}}z(x,y)$$

$\therefore$ $z(x,y)$ is a homogeneous function in of degree $\frac{1}{2}$.

$\therefore$ By Euler's theorem, $x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} \;=\; \frac{1}{2}$ z

Put $z = \; sinu \quad \Rightarrow x\frac{\partial(sinu)}{\partial x} + y\frac{\partial(sinu)}{\partial y} \;=\; \frac{1}{2}$sinu

$$xcosu\frac{\partial u}{\partial x} + ycosu\frac{\partial u}{\partial y} \;=\; \frac{1}{2}\text{sinu}$$

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} \;=\; \frac{1}{2}\frac{sinu}{cosu}$$

$$= \frac{1}{2}tanu \;=\; f(u)$$

By Euler's extension theorem

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\partial y} + y^2\frac{\partial^2 u}{\partial y^2} = \; f(u)[f'(u) - 1]$$

$$= \frac{1}{2}tanu(\frac{1}{2}sec^2u - 1)$$

$$= \frac{1}{2}\frac{sinu}{cosu}(\frac{1}{2cos^2u} - 1)$$

$$= \frac{1}{2}\frac{sinu}{cosu}\;(\frac{1-2cos^2u}{2cos^2u})$$

$$= \frac{1}{4}\frac{sinu(-cos2u)}{cos^3u}$$

$$= \frac{-sinucos2u}{4cos^3u}$$

Hence proved.

**Example:**

**If $\boldsymbol{u \;=\; (x-y)f\left(\frac{y}{x}\right)}$, then find $\boldsymbol{x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\partial y} + y^2\frac{\partial^2 u}{\partial y^2}}$ [AU May 2001,Dec2014]**

**Solution:**

Given $u(x,y) = \; (x-y)f\left(\frac{y}{x}\right)$

$$u(tx, ty) = (tx - ty) f\left(\frac{ty}{tx}\right)$$

$$= t^1 u(x, y)$$

$\therefore$ $u(x, y)$ is a homogeneous function in of degree '1'.

$\therefore$ By Euler's theorem, $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 1.u = u = f(u)$

By Euler's extension theorem

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = f(u)[f'(u) - 1]$$

$$= u(1 - 1) = u.0 = 0$$

**Exercise:**

1. Verify Euler's theorem in the following cases.

(i) $z = ax^2 + 2hxy + by^2$

(ii) $u = x^3 + y^3 + z^3 + 3xyz$

(iii) $u = e^{\frac{x}{y}} sin\left(\frac{x}{y}\right) + e^{\frac{y}{x}} cos\left(\frac{y}{x}\right)$

(iv) $u = x^3 cos\left(\frac{y}{x}\right)$

2. If $u = log(x^2+y^2) + tan^{-1}\left(\frac{y}{x}\right)$,show that $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 2$

3. If $= \frac{x^2+y^2}{\sqrt{x}+\sqrt{y}}$, prove that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{3}{2}u$

4. If $u = log(x^2+y^2+z^2)$ show that that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 2$

5. If $u = sin^{-1}\left(\frac{x+y}{\sqrt{x}+\sqrt{y}}\right)$, then prove that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{1}{2}tanu$

6. If $u = tan^{-1}\left(\frac{x^3+y^3}{x-y}\right)$, then prove that $x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = 2cos3usinu$

7. Given $u(x, y) = x^2 tan^{-1}\left(\frac{y}{x}\right) - y^2 tan^{-1}\left(\frac{y}{x}\right)$ find the value of $x^2u_{xx} + 2xyu_{xy} + y^2u_{yy}$.