2.3 Differentiation Rules:

Derivatives of polynomials:

Formulae:

$\frac{d}{dx}(c) = 0$

$\frac{d}{dx}(x^n) = nx^{n-1}$

Example

Differentiate the following functions:

$\boldsymbol{a}$) $\boldsymbol{f(x) = \sqrt{30}}$

b) $\boldsymbol{f(x) = t^4}$

c) $\boldsymbol{f(x) = \sqrt[3]{x^2}}$

d) $\boldsymbol{f(x) = x^8 + 12x^5 - 4x^4 + 10x^3 - 6x + 7}$

e) $\boldsymbol{y = \frac{x^2+4x+3}{\sqrt{x}}}$

Solution:

a) Given $f(x) = \sqrt{30}$

$f'(x) = 0$

b) Given $f(x) = t^4$ i.e, y = t^4

$\frac{dy}{dt} = 4t^3$

c) $f(x) = \sqrt[3]{x^2} = (x^2)^{\frac{1}{3}} = x^{\frac{2}{3}}$

$f'(x) = \frac{2}{3}x^{\frac{2}{3}-1}$

$= \frac{2}{3}x^{\frac{-1}{3}}$

d) Given $f(x) = x^8 + 12x^5 - 4x^4 + 10x^3 - 6x + 7$

$f'(x) = 8x^7 + 12(5)x^4 - 4(4)x^3 + 10(3)x^2 - 6$

$= 8x^7 + 60x^4 - 16x^3 + 30x^2 - 6$

e) Given $y = \frac{x^2+4x+3}{\sqrt{x}}$

$y = x^2x^{\frac{-1}{2}} + 4xx^{\frac{-1}{2}} + 3x^{\frac{-1}{2}}$

$$y = x^{\frac{3}{2}} + 4x^{\frac{1}{2}} + 3x^{\frac{-1}{2}}$$

$$\frac{dy}{dx} = \frac{3}{2}x^{\frac{1}{2}} + 4.\frac{1}{2}x^{-\frac{1}{2}} + 3\left(\frac{-1}{2}\right)x^{\frac{-3}{2}}$$

$$= \frac{3}{2}\sqrt{x} + \frac{2}{\sqrt{x}} - \frac{3}{2}x^{\frac{-3}{2}}$$

Derivatives of exponential functions:

Formulae:

$$\frac{d}{dx}(e^x) = e^x$$

$$\frac{d}{dx}(a^x) = a^x \log a$$

Example

Differentiate the following functions:

a) $\boldsymbol{y = a^x}$ **b)** $\mathbf{y} = \boldsymbol{\frac{xe^x - 1}{x}}$

Solution:

a) Given $y = a^x = e^{\log a^x} = e^{x \log a} = e^{(\log a)x}$

$$y' = \frac{dy}{dx} = e^{(\log a)x}(\log a) = a^x \log a$$

b) $y = \boldsymbol{\frac{xe^x - 1}{x}}$

$$= e^x - \frac{1}{x} = e^x - x^{-1}$$

$$y' = \frac{dy}{dx} = e^x - (-1)\, x^{-2} \quad = e^x + \frac{1}{x^2}$$

Exercise:

I. Differentiate the following functions:

a) $y = 3x^4$ **Ans:** $y' = 12x^3$

b) $y = \sqrt{x}$ **Ans:** $y' = \frac{1}{2\sqrt{x}}$

c) $f(x) = x^{\sqrt{2}}$ **Ans:** $f'(x) = \sqrt{2}x^{\sqrt{2}-1}$

d) $y = ax^{2n} + 6x^n + c$ **Ans:** $f'(x) = 2nax^{2n-1} + nbx^{n-1}$

e) $y = \frac{x^3 + 4x^2 + 3}{x^2}$ **Ans:** $y' = 1 - 6x^{-3}$

II. Differentiate the following functions:

a) $y = e^x - x$ **Ans:** $\frac{dy}{dx} = e^x - 1$

b) $y = 2^x$ **Ans:** $y' = 2^x \log 2$

c) $y = e^{-x} - 7$ **Ans**: $y' = -e^{-x}$

d) $y = 3e^x + \frac{4}{\sqrt[3]{x}}$ **Ans:** $y' = 3e^x - \frac{4x^{\frac{-4}{3}}}{3}$

e) $y = e^{5x}$ **Ans:** $y' = 5e^{5x}$

The product and quotient rules:

Formulae:

Product Rule: $\frac{d}{dx}(uv) = uv' + vu'$

Quotient Rule: $\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\,u' - u\,v'}{v^2}$

Example

Find $f'(x)$ and $f''(x)$ of $f(x) = x^4 e^x$

Solution:

$u = x^4 \quad ; \quad v = e^x$

$u' = 4x^3 \; : \quad v' = e^x$

$f'(x) = x^4 e^x + 4e^x x^3$

$f''(x) = x^4 e^x + 4e^x x^3 + 4e^x x^3 + 12x^2 e^x$

Example

Differentiate the following functions:

a) $f(x) = e^x(x + x\sqrt{x})$ **b)** $f(x) = (\frac{1}{x^2} - \frac{3}{x^4})(x + 5x^3)$

Solution:

a) $f(x) = e^x(x + x\sqrt{x})$

$u = e^x \qquad v = x + x\sqrt{x}$

$u' = e^x \qquad v' = 1 + \frac{3}{2}x^{\frac{1}{2}}$

$f'(x) = e^x\ (1 + \frac{3}{2}x^{\frac{1}{2}}) + (x\ +\ x\sqrt{x})\ e^x$

b) $f(x) = (\frac{1}{x^2} - \frac{3}{x^4})(x + 5x^3)$

$u\ = \frac{1}{x^2} - \frac{3}{x^4}$ $\qquad v\ =\ x + 5x^3$

$u' = \frac{-2}{x^3} + \frac{12}{x^5}$ $\qquad v'\ =\ 1 + 15x^2$

$f'(x)\ = (\frac{1}{x^2} - \frac{3}{x^4})\ (1 + 15x^2) + (x + 5x^3)\ (\frac{-2}{x^3} + \frac{12}{x^5})$

$= \frac{1}{x^2} - \frac{3}{x^4} + 15 - \frac{45}{x^2} - \frac{2}{x^2} + \frac{12}{x^4} - 10 + \frac{60}{x^2}$

$= \frac{9}{x^4} + \frac{14}{x^2} + 5$

Example:

If $f(x) = \frac{x^2}{1+2x}$, then find $f'(x)$ and $f''(x)$

Solution:

Given $f(x) = \frac{x^2}{1+2x}$

$u = x^2$ $\qquad v\ =\ 1\ + 2x$

$u'\ =\ 2x$ $\qquad v' =\ 2$

$f'(x) = \frac{(1+2x)(2x)-x^2(2)}{(1+2x)^2}$

$= \frac{2x+ 4x^2 - 2x^2}{1+4x+ 4x^2}$

$f'(x) = \frac{2x\ +\ 2x^2}{1+4x+ 4x^2}$

Now,

$u\ =\ 2x\ +\ 2x^2 v\ =\ 1 + 4x +\ 4x^2$

$u'\ =\ 2\ +\ 4x \qquad v'\ =\ 4\ +\ 8x$

$f''(x) = \frac{(1+4x+ 4x^2)(2+4x) - (2x\ + 2x^2)(4+8x)}{(1+4x+ 4x^2)^2}$

$= \frac{2 + 8x + 8x^2 + 4x + 16x^2 + 16x^3 - 8x - 16x^2 - 8x^2 - 16x^3}{(1+4x+ 4x^2)^2}$

$f''(x) = \frac{2+4x}{(1+4x+ 4x^2)^2}$

Derivatives of trigonometric functions:

Formulae:

$\frac{d}{dx}(\sin x) = \cos x$ $\frac{d}{dx}(\cos x) = -\sin x$

$\frac{d}{dx}(\tan x) = \sec^2 x$ $\frac{d}{dx}(\operatorname{cosec} x) = -\operatorname{cosec} x \cot x$

$\frac{d}{dx}(\sec x) = \sec x . \tan x$ $\frac{d}{dx}(\cot x) = -\operatorname{cosec}^2 x$

Example :

Find the derivatives of the following functions:

a) y = x^3 sin x **b) y = $\frac{\cos x}{1-\sin x}$**

Solution:

a) $y = x^3 \sin x$

$y' = x^3 \cos x + \sin x(3x^2)$

$= x^3 \cos x + 3x^2 \sin x$

b) $y = \frac{\cos x}{1-\sin x}$

$y' = \frac{(1-\sin x)(-\sin x) - \cos x(-\cos x)}{(1-\sin x)^2}$

$= \frac{-\sin x + \sin^2 x + \cos^2 x}{(1-\sin x)^2}$

$= \frac{1-\sin x}{(1-\sin x)^2}$

$y' = \frac{1}{1-\sin x}$

Exercise

(a) Differentiate the following functions using product rule:

i) $f(x) = xe^x$ **Ans:** $f'(x) = e^x (x + 1)$

ii) $f(x) = (x^3 + 2x)\, e^x$ **Ans:** $f'(x) = e^x (x^3 + 3x^2 + 2x + 2)$

iii) $f(x) = (x - \sqrt{x})(x + \sqrt{x})$ **Ans:** $f'(x) = 2x - 1$

iv) $f(x) = (\frac{1}{x^2} - \frac{3}{x^4})(x + 5x^3)$ **Ans:** $f'(x) = 5 + \frac{14}{x^2} + \frac{9}{x^4}$

v) $f(x) = (x^2 + 1)(x + 5 + \frac{1}{x})$ **Ans:** $f'(x) = 3x^2 + 10x + 2 - \frac{1}{x^2}$

(b) Differentiate the following functions using quotient rule:

i) $f(x) = \frac{x^2+x-2}{x^3+6}$ **Ans:** $f'(x) = \frac{-x^4-2x^3+6x^2+12x+6}{(x^3+6)^2}$

ii) $f(x) = \frac{e^x}{x}$ **Ans:** $f'(x) = \frac{e^x(x-1)}{x^2}$

iii) $f(x) = \frac{1-xe^x}{x+e^x}$ **Ans:** $f'(x) = \frac{-(x^2e^x+e^{2x}+e^x+1)}{(x+e^x)^2}$

iv) $f(x) = \frac{2x}{2+\sqrt{x}}$ **Ans:** $f'(x) = \frac{4+\sqrt{x}}{(2+\sqrt{x})^2}$

(c) If $f(x) = \frac{x^2-1}{x^2+1}$, find $f'(x)$ and $f''(x)$ **Ans:** $f'(x) = \frac{4x}{(x^2+1)^2}$ $f''(x) = \frac{4(1-3x^2)}{(x^2+1)^3}$

(d) Differentiate the following trigonometric functions:

i) $y = \frac{\sec x}{1+\tan x}$ **Ans:** $y' = \frac{\sec x(\tan x-1)}{(1+\tan x)^2}$

ii) $y = x\, e^x \operatorname{cosec} x$ **Ans:** $y' = e^x \operatorname{cosec} x\,(x + 1 - x\cos x)$

iii) $f(x) = \sin x + \frac{1}{2}\cot x$ **Ans:** $f'(x) = \cos x - \frac{1}{2}\operatorname{cosec}^2 x$

iv) $f(x) = \frac{\sec x}{1+\sec x}$ **Ans:** $f'(x) = \cos x - \frac{1}{2}\operatorname{cosec}^2 x$

v) $y = x^3 \cos x$ **Ans:** $y' = 3x^2\cos x - x^3 \sin x$

Derivatives of inverse trigonometric functions:

Formulae:

$\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}$ $\frac{d}{dx}(\cos^{-1} x) = \frac{1}{\sqrt{1-x^2}}$

$\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}$ $\frac{d}{dx}(\cot^{-1} x) = \frac{-1}{1+x^2}$

$\frac{d}{dx}(\sec^{-1} x) = \frac{1}{x\sqrt{x^2-1}}$ $\frac{d}{dx}(\operatorname{cosec}^{-1} x) = \frac{-1}{x\sqrt{x^2-1}}$

Example :

Find $\frac{d}{dx}$ if (a) y = $\tan^{-1}\sqrt{x}$ b) y = $\sin^{-1}(x^2)$ c) y = $\sin^{-1}(e^x)$

Solution:

a) y = $\tan^{-1}\sqrt{x}$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{1+(\sqrt{x})^2}\frac{1}{2\sqrt{x}}$$

$\Rightarrow \frac{dy}{dx} = \frac{1}{2\sqrt{x}} (\frac{1}{1+x})$

b) y = $sin^{-1}(x^2)$

$\Rightarrow \frac{dy}{dx} = \frac{1}{\sqrt{1-x^4}}(2x)$

$\Rightarrow \frac{dy}{dx} = \frac{2x}{\sqrt{1-x^4}}$

c) y = $cos^{-1}(e^x)$

$\Rightarrow \frac{dy}{dx} = \frac{-1}{\sqrt{1-(e^x)^2}} e^x$

$\Rightarrow \frac{dy}{dx} = \frac{-e^x}{\sqrt{1-e^{2x}}}$

The Chain Rule:

In Leibnitz notation, if $y = f(u)$ and $u = g(x)$ are both differentiable functions, then

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

The power rule combined with the chain rule:

If n is any real number and $u = g(x)$ is differentiable, then

$$\frac{d}{dx}(u^n) = nu^{n-1}\frac{du}{dx}$$

Example :

Find the derivative of y if

(a) y = $\sqrt{3x} + 4$ (b) y = tan (sin x) (c) y = $cos^{-1}(e^{2x})$ (d) y = log $(x^3 + 1)$ (e) y = sin (cos tan x)

Solution:

a) $y = \sqrt{3x} + 4$

$\Rightarrow \frac{dy}{dx} = \frac{1}{2\sqrt{3x}+4}(3)$

$\Rightarrow \frac{dy}{dx} = \frac{3}{2\sqrt{3x}+4}$

b) $y = tan\,(sin\,x)$

$\Rightarrow \frac{dy}{dx} = sec^2(sin\ x)\ .\ cos\ x$

$\Rightarrow \frac{dy}{dx} = \cos x\ sec^2(sin\ x)$

c) $y = cos^{-1}(e^{2x})$

$\Rightarrow \frac{dy}{dx} = \frac{-1}{\sqrt{1}-(e^{2x})^1}.2e^{2x}$

$\Rightarrow \frac{dy}{dx} = \frac{-2e^{2x}}{\sqrt{1}ie^{4x}}$

d) $y = \log(x^3+1)$

$\Rightarrow \frac{dy}{dx} = \frac{1}{x^3+1}(3x^2) = \frac{3x^2}{x^3+1}$

e) $y = sin\ (cos\ tan\ x)$

$\Rightarrow \frac{dy}{dx} = \cos(\cos\tan x)\ (-\sin\tan x)\ sec^2\ x$

$\Rightarrow \frac{dy}{dx} = -cos\ (cos\ tan\ x)\ sin\ (tan\ x)\ sec^2 x$

Exercise

1. Differentiate the following functions:

(i) $y = sin^{-1}(e^x)$ **Ans:** $y' = \frac{e^x}{\sqrt{1}- e^{2x}}$

(ii) $y = x\ sin^{-1}x + sec^{-1}x$ **Ans:** $y' = \frac{x}{\sqrt{1-x^2}} + sin^{-1}x + \frac{1}{x\sqrt{x^2}-1}$

(iii) $y = cos^{-1}(x^2)$ **Ans:** $y' = \frac{-2x}{\sqrt{1}-x^4}$

2. Find the derivative for the following functions:

(i) $y = \frac{3x-1}{2x+1}$ **Ans:** $y' = \frac{5}{(2x+1)^2}$

(ii) $y = \frac{e^x}{x+1}$ **Ans:** $y' = e^x[\frac{1}{1+x} - \frac{1}{(1+x)^2}]$

(iii) $y = sin\ (sin\ sin\ x)$ **Ans:** y' = cos (sin cos x) cos (sin x) cos x

(iv) $y = \sqrt{cos\sqrt{x}}$ **Ans:** $y' = \frac{-1}{4}\frac{sin\sqrt{x}}{\sqrt{x}\sqrt{cos\sqrt{x}}}$

(v) $y = 2^{sin\pi x}$ **Ans:** $y' = 2^{sin\pi}\pi \log 2 \cos\pi x$