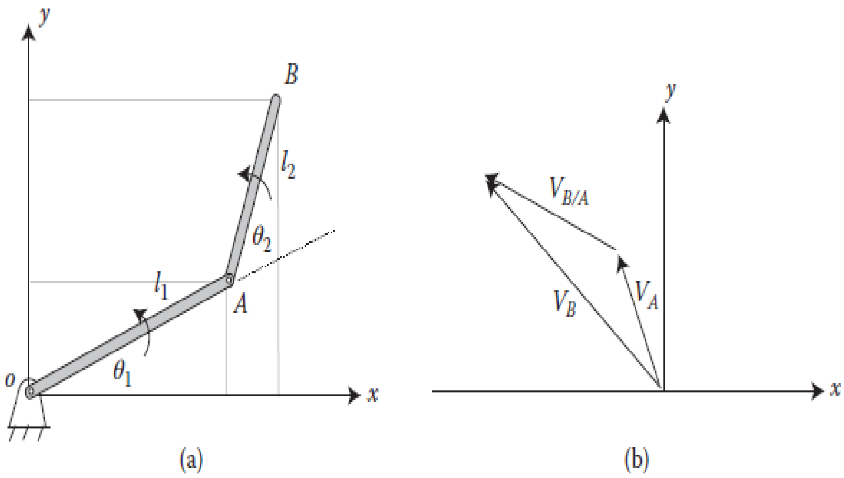
DIFFERENTIAL RELATIONSHIPS:

First, let's see what the differential relationships are. To do this, we will consider a simple 2-DOF mechanism, as shown in Figure, where each link can rotate independently. The rotation of the first link ($\theta 1$) is measured relative to the reference frame, whereas the rotation of the second link ($\theta 2$) is measured relative to the first link, as we might see in a D-H representation where the movements of each link of a robot are measured relative to a current frame attached to the previous link.



The velocity of point B can be calculated as follows:

$$\begin{aligned} \mathbf{v}_B &= \mathbf{v}_A + \mathbf{v}_{B/A} \\ &= l_1\dot{\theta}_1\,[\perp to\, l_1] + l_2\left(\dot{\theta}_1 + \dot{\theta}_2\right)[\perp to\, l_2] \\ &= -l_1\dot{\theta}_1 \sin\theta_1 \mathbf{i} + l_1\dot{\theta}_1 \cos\theta_1 \mathbf{j} - l_2\left(\dot{\theta}_1 + \dot{\theta}_2\right)\sin(\theta_1+\theta_2)\mathbf{i} + l_2\left(\dot{\theta}_1 + \dot{\theta}_2\right)\cos(\theta_1+\theta_2)\mathbf{j} \end{aligned}$$

Writing Eq. (5.1) in matrix form yields the following:

$$\begin{bmatrix} v_{B_x} \\ v_{B_y} \end{bmatrix} = \begin{bmatrix} -l_1 \sin\theta_1 - l_2 \sin(\theta_1+\theta_2) & -l_2 \sin(\theta_1+\theta_2) \\ l_1 \cos\theta_1 + l_2 \cos(\theta_1+\theta_2) & l_2 \cos(\theta_1+\theta_2) \end{bmatrix} \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix}$$

The left-hand side represents the x and y components of the velocity of point B. If the elements of the right-hand side of the equation are multiplied by the corresponding angular velocities of the two links, the velocity of point B can be found. Next, instead of deriving the components of the velocity directly from the velocity relationship, we try to find the same by differentiating the equations that describe the position of point B, as follows:

$$\begin{cases} x_B = l_1 \cos\theta_1 + l_2 \cos(\theta_1 + \theta_2) \\ y_B = l_1 \sin\theta_1 + l_2 \sin(\theta_1 + \theta_2) \end{cases}$$

Differentiating Eq. (5.3) with respect to the two variables θ_1 and θ_2 yields:

$$\begin{cases} dx_B = -l_1 \sin\theta_1 d\theta_1 - l_2 \sin(\theta_1 + \theta_2)(d\theta_1 + d\theta_2) \\ dy_B = l_1 \cos\theta_1 d\theta_1 + l_2 \cos(\theta_1 + \theta_2)(d\theta_1 + d\theta_2) \end{cases}$$

and in matrix form:

$$\underbrace{\begin{bmatrix} dx_B \\ dy_B \end{bmatrix}}_{\text{Differential motion of B}} = \underbrace{\begin{bmatrix} -l_1 \sin\theta_1 - l_2 \sin(\theta_1 + \theta_2) & -l_2 \sin(\theta_1 + \theta_2) \\ l_1 \cos\theta_1 + l_2 \cos(\theta_1 + \theta_2) & l_2 \cos(\theta_1 + \theta_2) \end{bmatrix}}_{\text{Jacobian}} \underbrace{\begin{bmatrix} d\theta_1 \\ d\theta_2 \end{bmatrix}}_{\text{Differential motion of joints}}$$

Similarly, in a robot with many degrees of freedom, the joint differential motions, or velocities, can be related to the differential motion, or velocity, of the hand using the same technique.