TRANSPORTATION PROBLEM

Introduction

A special class of linear programming problem is **Transportation Problem**, where the objective is to minimize the cost of distributing a product from a number of **sources** to a number of **destinations** while satisfying both the supply limits and the demand requirement.

Problem:

Suppose a manufacturing company owns three factories (sources) and distribute his products to five different retail agencies (destinations). The following table shows the capacities of the three factories, the quantity of products required by the various retail agencies and the cost of shipping one unit of the product from each of the three factories to each of the five retail agencies.

	Retail Agency					
Factories	1	2	3	4	5	Capacity
1	1	9	13	36	51	50
2	24	12	16	20	1	100
3	14	33	1	23	26	150
Requirement	100	60	50	50	40	300

Usually the above table is referred as Transportation Table, which provides the basic information regarding the transportation problem. The quantities inside the table are known as transportation cost perunit of product. The capacity of the factories 1, 2, 3 is 50, 100 and 150 respectively. The requirement of the retail agency 1, 2, 3, 4, 5 is 100,60,50,50, and 40 respectively.

In this case, the transportation cost of one unitfrom factory 1 to retail agency 1 is 1,
from factory 1 to retail agency 2 is 9,
from factory 1 to retail agency 3 is 13, and so on.

A transportation problem can be formulated as linear programming problem using variables withtwo subscripts. Let

x_{11}=Amount to be transported from factory 1 to retail agency 1x_{12}= Amount to be transported from factory 1 to retail agency 2

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……..

x_{35}= Amount to be transported from factory 3 to retail agency 5.

Let the transportation cost per unit be represented by C_{11}, C_{12}, …..C_{35} that is $C_{11}=1$, $C_{12}=9$, and so on. Let the capacities of the three factories be represented by $a_1=50$, $a_2=100$, $a_3=150$.

Let the requirement of the retail agencies are $b_1=100$, $b_2=60$, $b_3=50$, $b_4=50$, and $b_5=40$.Thus, the problem can be formulated as

Minimize

$C_{11}x_{11}+C_{12}x_{12}+$………………………………………………………………….. $+C_{35}x_{35}$

Subject to:

$x_{11} + x_{12} + x_{13} + x_{14} + x_{15} = a1$	$x_{11} + x_{21} + x_{31} = b1$
$x_{21} + x_{22} + x_{23} + x_{24} + x_{25} = a2$	$x_{12} + x_{22} + x_{32} = b2$
$x_{31} + x_{32} + x_{33} + x_{34} + x_{35} = a3$	$x_{13} + x_{23} + x_{33} = b3$

	$x_{14} + x_{24} + x_{34} = b4$
$x_{11}, x_{12}, \ldots\ldots, x_{35} \geq 0$.	$x_{15} + x_{25} + x_{35} = b5$

Thus, the problem has 8 constraints and 15 variables. So, it is not possible to solve such a problem using simplex method. This is the reason for the need of special computational procedure to solve transportation problem.

Transportation Algorithm

The steps of the transportation algorithm are exact parallels of the simplex algorithm, they are:

Step 1: Determine a starting basic feasible solution, using any one of the following three methods

1. North West Corner Method
2. Least Cost Method
3. Vogel Approximation Method

Step 2: Determine the optimal solution using the following method

1. MODI (Modified Distribution Method) or UV Method.

Basic Feasible Solution of a Transportation Problem

The computation of an initial feasible solution is illustrated in this section with the help of the example1.1 discussed in the previous section. The problem in the example 1.1 has 8 constraints and 15 variables we can eliminate one of the constraints since $a_1 + a_2 + a_3 = b_1 + b_2 + b_3 + b_4 + b_5$. Thus now the problem contains 7 constraints and 15 variables. Note that any initial (basic) feasible solution has at most 7 non-zero X_{ij}. Generally, any basic feasible solution with m sources (such as factories) and n destination (such as retail agency) has at most m + n -1 non-zero X_{ij}. The special structure of the transportation problem allows securing a non-artificial basic feasible solution using one the following three methods.

1. North West Corner Method
2. Least Cost Method
3. Vogel Approximation Method

Generally the Vogel Approximation Method produces the **best** initial basic feasible solution, and the North West Corner Method produces the **worst**, but the North West Corner Method involves least computations.

North West Corner Method: The method starts at the North West (upper left) corner cell of the table (variable x_{11}).

Step -1: Allocate as much as possible to the selected cell, and adjust the associated amounts of capacity (supply) and requirement (demand) by subtracting the allocated amount.

Step -2: Cross out the row (column) with zero supply or demand to indicate that no further assignments can be made in that row (column). If both the row and column becomes zero simultaneously, cross out one of them only, and leave a zero supply or demand in the uncrossed out row (column).

Step -3: If exactly one row (column) is left uncrossed out, then stop. Otherwise, move to the cell to the right if a column has just been crossed or the one below if a row has been crossed out. Go to step -1.

Problem 1:

Determine the basic feasible solution by North West Corner Method

	Retail Agency					
Factories	1	2	3	4	5	Capacity
1	1	9	13	36	51	50
2	24	12	16	20	1	100
3	14	33	1	23	26	150
Requirement	100	60	50	50	40	300

The allocation is shown in the following table:

					Capacity
1 **5**	9	1	3	5	~~**50**~~
24 **5**	12 **50**	1	2	1	~~100~~ ~~50~~ ~~**150**~~
14 →	33 **10**	1 **50**	23 **50**	26 **40**	~~**140**~~ ~~**90**~~ ~~**40**~~
Requirement ~~**100**~~	~~**60**~~	~~**50**~~	~~**50**~~ ~~**50**~~	~~**40**~~ ~~**10**~~	

The arrows show the order in which the allocated (**bolded**) amounts are generated. The startingbasic solution is given as
$x_{11} = 50$, $x_{21} = 50$, $x_{22} = 50$ $x_{32} = 10$, $x_{33} = 50$, $x_{34} = 50$, $x_{35} = 40$

The corresponding transportation cost is

$$50 * 1 + 50 * 24 + 50 * 12 + 10 * 33 + 50 * 1 + 50 * 23 + 40 * 26 = 4420$$

Problem : 2

Find the basic feasible solution of the given transportation problem by applying North – West Corner rule:

Factory \ Warehouse	D	E	F	G	Capacity
A	42	48	38	37	160
B	40	49	52	51	150
C	39	38	40	43	190
Requirement	80	90	110	220	500

Solution.

We start from the North – West corner, i.e., the Factory A and Warehouse D. The quantity needed at the First Warehouse (Warehouse D) is 80, which is less than the quantity available (160) at theFirst Factory A. Therefore, a quantity equal to the warehouse D is to be allocated to the cell (A, D). Thus,the requirement of Warehouse D is met by Factory A. So, we cross out column 1 and reduce the capacityof Factory A by 80. Then we go to cell (A, E), which is North – West corner of the resulting matrix.

Now, the quantity needed at the second Warehouse (Warehouse E) is 90, which is greater than the quantityavailable (80) at the First Factory A. Therefore, we allocate a quantity equal to the capacity at Factory A,i.e., 80 to the cell (A, E). The requirement of Warehouse E is reduced to 10. The capacity of Factory A isexhausted and has to be removed from the matrix. Therefore, we cross out row 1 and proceed to cell (B, E).

Now, the quantity needed at the second Warehouse (Warehouse E) is 10, which is less than the quantity available at the Second Factory B, which is 150. Therefore, the quantity 10 equal to the requirement at Warehouse E is allocated to the cell (B, E). Hence, the requirement of Warehouse E is met and we cross out column 1. We reduce the capacity of Factory B by 10 and proceed to cell (B, F).

Again, the quantity needed at the Third Warehouse (Warehouse F) is 110. It is less than the quantity available at the Second Factory (Factory B), which is140. Therefore, a quantity equal to the requirement at Warehouse F is allocated to the cell (B, F). Since the requirement of Warehouse F is met, we cross outColumn 1 and reduce the capacity of Factory B by 110. Then we proceed to cell (B, G).Now, the quantity needed at the Fourth Warehouse (Warehouse G) is 220, which is greater than the quantity available at theSecond Factory (Factory B).Therefore, we allocate the quantity equal to the capacity of Factory B to the cell (B, G) so that the capacity of Factory B is exhausted and the requirement of Warehouse G is reducedto 190. Hence, we cross out Row 1 and proceed to cell (C, G).

Thus, the allocations given using North – West corner rule are as shown in the following matrix along with the cost per unit of transportation:

Warehouse / Factory	D	E	F	G	Capacity
A	42 [80]	48 [80]	38	37	160
B	40	49 [10]	52 [11]	51 [30]	150
C	39	38	40	43 [190]	190
Requirement	80	90	110	220	500

Thus, the total transportation cost for these allocations

$= 42 \times 80 + 48 \times 80 + 49 \times 10 + 52 \times 110 + 51 \times 30 + 43 \times 190$

= 3360 + 3840 + 490 + 5720 + 1530 + 8170 =23110

Least Cost Method

The least cost method is also known as matrix minimum method in the sense we look for the row and the column corresponding to which C_{ij} is minimum. This method finds a better initial basic feasible solution by concentrating on the cheapest routes. Instead of starting the allocation with the northwest cell as in the North West Corner Method, we start by allocating as much as possible to the cell with the smallest unit cost. If there are two or more minimum costs then we should select the row and the columncorresponding to the lower numbered row. If they appear in the same row we should select the lower numbered column.

Problem : Find the optimal transportation cost of the following matrix using least cost method for finding the critical solution.

		Market					
		A	**B**	**C**	**D**	**E**	**Available**
	P	4	1	2	6	9	**100**
Factory	**Q**	6	4	3	5	7	**120**
	R	5	2	6	4	8	**120**
	Demand	**40**	**50**	**70**	**90**	**90**	

Soln.

Since $\sum a_i = \sum b_j = 150$, the given transportation problem is balanced.

The non – degenerate basic feasible solution is as shown in the following table.

The initial basic feasible solution is

	50	**50**		
10		**20**		**90**
30			**90**	

The initial transportation cost is Rs.1410 /-

For optimality,

-1	**50**	**50**	2	3	$U_1 = -1$
10	2	**20**	0	**90**	$U_2 = 0$
30	1	4	**90**	2	$U_3 = -1$
$V_1 = 6$	$V_2=2$	$V_3=3$	$V_4=5$	$V_5=7$	

Since $d_{11} = -1 < 0$, the current solution is not optimal.

Now let us form a new basic feasible solution by giving maximum allocation to the cell (i , j) for which d_{ij} is most negative by making an occupied cell empty. Here the cell (1,1) having the negative value $d_{11} = -1$. so we draw a closed loop consisting of horizontal and vertical lines beginning and ending at this cell (1,1) and other corners some occupied cells.

-1	**50**	**50**	2	3
10	2	**20**	0	**90**
30	1	4	**90**	2

Here the new basic feasible solution is in the following table

10	**50**	**40**		
		30		**90**
30			**90**	

By MODI method,

10	**50**	**40**	3	3	$U_1 = 0$
1	2	**30**	1	**90**	$U_2 = 1$
30	0	3	**90**	1	$U_3 = 1$
$V_1 = 4$	$V_2=1$	$V_3=2$	$V_4=3$	$V_5=6$	

Since all $d_{ij}>0$ with $d_{32}=0$, the current solution is optimal and there exists an alternative optimum solution. The transportation cost is Rs. 1400 /-.

Vogel Approximation Method (VAM):

VAM is an improved version of the least cost method that generally produces better solutions. The steps involved in this method are:

Step 1: For each row (column) with strictly positive capacity (requirement), determine a **penalty** by subtracting the **smallest** unit cost element in the row (column) from the next **smallest** unit cost element in the same row (column).

Step 2: Identify the row or column with the **largest penalty** among all the rows and columns. If the penalties corresponding to two or more rows or columns are equal we select the topmost row and the extreme left column.

Step 3: We select X_{ij} as a basic variable if C_{ij} is the **minimum cost** in the row or column with **largest penalty**. We choose the numerical value of X_{ij} as high as possible subject to the row and the column constraints. Depending upon whether a_i or b_j is the smaller of the two i^{th} row or j^{th} column is crossed out.

Step 4: The Step 2 is now performed on the uncrossed-out rows and columns until all the basic variableshave been satisfied.

Modified Distribution Method (MODI)

The Modified Distribution Method, also known as MODI method or u-v method, which provides a minimum cost solution (optimal solution) to the transportation problem. The following are the steps involved in this method.

Step 1: Find out the basic feasible solution of the transportation problem using any one of the three methods discussed in the previous section.

Step 2: Introduce dual variables corresponding to the row constraints and the column constraints. If there are m origins and n destinations then there will be m+n dual variables. The dual variables corresponding to the row constraints are represented by u_i, i=1,2,…..m where as the dual variables corresponding to the column constraints are represented by v_j, j=1,2,…..n. The values of the dual variables are calculated from the equation given below

$$u_i + v_j = c_{ij} \text{ if } x_{ij} > 0$$

Step 3: Any basic feasible solution has m + n -1 $x_{ij} > 0$. Thus, there will be m + n -1 equation to determine m + n dual variables. One of the dual variables can be chosen arbitrarily. It is also to be noted that as the primal constraints are equations, the dual variables are unrestricted in sign.

Step 4: If x_{ij}=0, the dual variables calculated in Step 3 are compared with the c_{ij} values of this allocation as $c_{ij} - u_i - v_j$. If al $c_{ij} - u_i - v_j \geq 0$, then by the ***theorem of complementary slackness*** it can be shown thatthe corresponding solution of the transportation problem is optimum. If one or more $c_{ij} - u_i - v_j < 0$, we select the cell with the least value of $c_{ij} - u_i - v_j$ and allocate as much as possible subject to the row and column constraints.

Step 5: A fresh set of dual variables are calculated and repeat the entire procedure from Step 1 to Step 5.

Problem : 1

Solve the following transportation problem to maximum profit by Vogel's Approximation Method (VAM)

	A	B	C	D	Supply
I	40	25	22	33	100
II	44	35	30	30	30
III	38	38	28	30	70
Demand	40	20	60	30	

Soln.

Since the given problem is maximization type, first convert this into a minimization problem by subtracting the cost elements from the highest cost element in the given transportation problem.

i.e

	A	B	C	D	Supply
I	44-40=4	44-25=19	44-22=22	44-33=11	100
II	44-44=0	44-35=9	44-30=14	44-30=14	30
III	44-38=6	44-38=6	44-28=14	44-30=14	70
Demand	40	20	60	30	

Then the given problem becomes

	A	B	C	D	Supply
I	4	19	22	11	100
II	0	9	14	14	30
III	6	6	16	14	70
Demand	40	20	60	30	

This modified minimization problem is unbalanced,

	A	B	C	D	E	Supply
I	4	19	22	11	0	100
II	0	9	14	14	0	30
III	6	6	16	14	0	70
Demand	40	20	60	30	50	

The initial basic feasible solution is

	A	B	C	D	E
I	*10*		*60*	*30*	
II	*30*				
III		*20*	*0*		*50*

Now the number of non negative allocations at independent positions is m+n-1. We apply MODI method for optimal solution.

10	7	*60*	*30*	-6	$U_1 = 0$
30	1	-4	7	-2	$U_2 = -4$
8	*20*	*0*	9	*50*	$U_3 = -6$
$V_1 = 4$	$V_2 = 12$	$V_3 = 22$	$V_4 = 11$	$V_5 = 6$	

Since $d_{ij} < 0$, the current solution under the test is not optimal. Here $d_{15} = -6$ is the most negative value of d_{ij}. So we draw a closed path consisting of horizontal and vertical lines having its opposite corners at some occupied cells.

10		*60 (-θ)*	*30*	*(θ)*
30				
	20	*0 (θ)*		*50 (-θ)*

Hence the new basic feasible solution is

10		10	30	50
30				
	20	50		

Again we apply the modi method,

10	7	10	30	50	$U_1 = 0$
30	1	-4	7	4	$U_2 = -4$
8	20	50	9	6	$U_3 = -6$
$V_1 = 4$	$V_2 = 12$	$V_3 = 22$	$V_4 = 11$	$V_5 = 0$	

$d_{23} < 0$, the current solution is not optimal. We draw the vertical and horizontal lines.

$(\theta)10$	7	$10(-\theta)$	30	50
$(-\theta)30$	1	$-4(\theta)$	7	4
8	20	50	9	6

The new basic feasible solution is

20			30	50
20		10		
	20	50		

Again we apply MODI method

20	11	4	30	50	$U_1 = 0$
20	5	10	7	4	$U_2 = -4$
4	20	50	5	2	$U_3 = -2$
$V_1 = 4$	$V_2 = 8$	$V_3 = 18$	$V_4 = 11$	$V_5 = 0$	

Since All $d_{ij} > 0$, the current solution is optimal and unique.

The optimum cost is Rs.5130 /-.