UNIT – IV INTEGRAL CALCULUS

4.1. Definite and indefinite Integrals

Definite Integral

The integral which has definite value is called Definite Integral. In other words, when $\int g(x)dx = f(x) + C$, then $[f(b) - f(a)]$ is called the Definite Integral of $g(x)$ between the limits (or end values) a and b and denoted by the symbol $\int_a^b g(x)dx$,a is called the lower limit and b is called the upper limit and is denoted by $[f(x)]_a^b$

Thus $\int_a^b g(x)dx$=$[f(x)]_a^b$ = $[f(b) - f(a)]$

Theorem 1: If f is continuous on $[a, b]$, (or) if f has only a finite number of discontinuous, then f is integrable on $[a, b]$

i.e., $\int_a^b f(x)dx$ exists.

Theorem 2: If f is integrable on $[a, b]$ then $\int_a^b f(x)dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x$

$$\Delta x = \frac{b-a}{n} \text{ and } x_i = a + i\Delta x$$

Example :

Evaluate $\int_0^3 (x^2 - 2x)\, dx$ by using Riemann sum by taking right end points as the sample points.

Solution:

Take n subintervals, we have $\Delta x = \frac{b-a}{n} = \frac{3}{n}$

$$x_0 = 0, x_1 = \frac{3}{n}, x_2 = \frac{3}{n}, x_3 = \frac{3}{n}, \ldots, x_i = \frac{3i}{n}$$

Since we are using right end points.

$$\therefore \int_0^3 (x^2 - 2x)\, dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x = \lim_{n\to\infty} \sum_{i=1}^{n} f\left(\frac{3i}{n}\right)\left(\frac{3}{n}\right)$$

$$= \lim_{n\to\infty} \frac{3}{n} \sum_{i=1}^{n} \left[\left(\frac{3i}{n}\right)^2 - 2\left(\frac{3i}{n}\right)\right] = \lim_{n\to\infty} \frac{3}{n} \sum_{i=1}^{n} \left[\frac{9}{n^2} i^2 - \frac{6}{n} i\right]$$

$$= \lim_{n\to\infty} \frac{27}{n^3} \sum_{i=1}^{n} i^2 - \lim_{n\to\infty} \frac{18}{n^2} \sum_{i=1}^{n} i$$

$= \lim_{n\to\infty} \frac{27}{n^3}\left[\frac{n(n+1)(2n+1)}{6}\right] - \lim_{n\to\infty} \frac{18}{n^2}\frac{n(n+1)}{2}$

$= \lim_{n\to\infty} \frac{27}{6n^3} n^3 \left[1 + \frac{1}{n}\right]\left[2 + \frac{1}{n}\right] - \lim_{n\to\infty} \frac{9}{n^2} n^2 \left[1 + \frac{1}{n}\right]$

$= \left(\frac{27}{6}\right)(1)(2) - 9 = 9 - 9 = 0$

Example:

Evaluate the Riemann sum for f(x) = $x^3 - 6x$, taking the sample points to be right end points and a = 0, b = 3 and n= 6

Solution:

$$\Delta x = \frac{b-a}{n} = \frac{3-0}{6} = \frac{1}{2}$$

The right end points are 0.5, 1, 1.5, 2, 2.5 and 3

The Riemann sum is

$$R_6 = \sum_{i=1}^{6} f(x_i)\Delta x = \sum_{i=1}^{6} f(x_i)\left(\frac{1}{2}\right) = \frac{1}{2}\sum_{i=1}^{6} f(x_i)$$

$$= \frac{1}{2}[f(0.5) + f(1) + f(1.5) + f(2) + f(2.5) + f(3)]$$

$$= \frac{1}{2}[-2.875 - 5 - 5.625 - 4 + 0.625 + 9] = -3.9375$$

Example:

Use the definition of area to find an expression for the area under the curve of $f(x) = e^{-x}$ between $x = 0$, $x = 2$. Do not evaluate the limit.

Solution:

Given that $f(x) = e^{-x}, a = 0, b = 2$

$$\Delta x = \frac{b-a}{n} = \frac{2-0}{n} = \frac{2}{n}$$

$$x_i = a + i\Delta x = 0 + i\left(\frac{2}{n}\right)$$

Area under the curve $f(x) = e^{-x}$ between $x = 0$ and $x = 2$ is given by

$$A = \lim_{n\to\infty} R_n = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} f\left(\frac{2i}{n}\right)\left(\frac{2}{n}\right)$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} f\left(e^{-2i/n}\right)\left(\frac{2}{n}\right)$$

The Mid Point

The Riemann sum which is the approximation to a given integral using the midpoint is given by

$$\int_a^b f(x)dx \simeq \sum_{i=1}^{n} f(\overline{x_i})\Delta x$$

$$= \Delta x\left[f(\overline{x_1}) + \cdots + f(\overline{x_n})\right]$$

Where $\Delta x = \frac{b-a}{n}$ and $(\overline{x_i}) = \frac{1}{2}[x_{i-1} + x_i]$

= midpoint of $[x_{i-1}, x_i]$

The Fundamental theorem of Calculus

Part 1: If f is continuous on $[a, b]$ then the function g is defined by

$$g(x) = \int_a^x f(t)dt\,; \quad a \le x \le b$$

is continuous on $[a, b]$ and differentiable on $(a\,, b), and \;\; g'(x) = f(x)$

The Fundamental theorem of Calculus

Part 2: If f is continuous on $[a, b]$ then $\int_a^b f(x)dx = F(b) - F(a)$

Where F is any anti derivative of f, that is, a function such that $F' = f$

Example :

Find the derivative of the following

(i) $\boldsymbol{g(x) = \int_0^x (t^2 + 1)\, dt}$

Solution:

Given $g(x) = \int_0^x (t^2 + 1)\, dt$

$\therefore \; g'(x) = (x^2 + 1)$ $\quad$ ($\because f(t) = t^2 + 1$ is continuous by FTC1)

(ii) $h(x) = \int_1^{e^x} \log t \, dt$

Solution:

Given $h(x) = \int_1^{e^x} \log t \, dt$

Put $u = e^x \Rightarrow du = e^x \, dx \Rightarrow \frac{du}{dx} = e^x$

$$\frac{dh}{dx} = \frac{dh}{du}\frac{du}{dx}$$

$$= \frac{d}{du}\left[\int_1^u \log t \, dt\right] e^x = \log u \,(e^x) = \log(e^x)\, e^x = xe^x$$

(iii) $f(x) = \int_0^{\tan x} \sqrt{t + \sqrt{t}} \, dt$

Solution:

Given $f(x) = \int_0^{\tan x} \sqrt{t + \sqrt{t}} \, dt$

Put $u = tanx \Rightarrow du = sec^2 x \, dx \Rightarrow \frac{du}{dx} = sec^2 x$

$$\frac{df}{dx} = \frac{df}{du}\frac{du}{dx}$$

$$= \frac{d}{du}\left[\int_0^u \sqrt{t + \sqrt{t}} \, dt\right] sec^2 x = \sqrt{u + \sqrt{u}}\, sec^2 x$$

$$= \sqrt{tan\, x + \sqrt{tan\, x}}\, sec^2 x$$

Example :

Evaluate $\int_3^6 \frac{1}{x} dx$ by fundamental theorem of calculus

Solution:

The function $f(x) = \frac{1}{x}$ is continuous in $3 \le x \le 6$.

By fundamental theorem of calculus part II, Anti derivative $F(x) = \log x$

$$\int_3^6 \frac{1}{x} dx = [logx]_3^6 = log6 - log3$$

$$= log\left(\frac{6}{3}\right) = log2$$

Example:

Find the derivative of the following

(i) $\int_{-1}^{2} (x^3 - 2x) dx$

Solution:

Given $f(x) = x^3 - 2x$ is continuous in $-1 \leq x \leq 2$

By FTC 2, Anti derivative $F(x) = \frac{x^4}{4} - \frac{2x^2}{2} = \frac{x^4}{4} - x^2$

$\int_{-1}^{2}(x^3 - 2x)dx = F(b) - F(a) = F(2) - F(-1)$

$$= \left[\frac{2^4}{4} - 2^2\right] - \left[\frac{(-1)^4}{4} - (-1)^2\right] = \frac{3}{4}$$

(ii) $\int_{1/\sqrt{3}}^{\sqrt{3}} \frac{8}{1+x^2}\, dx$

Solution:

Given $f(x) = \frac{8}{1+x^2}$ is continuous in the given interval.

By FTC 2, Anti derivative $F(x) = 8tan^{-1}\, x$

$$\int_{1/\sqrt{3}}^{\sqrt{3}} \frac{8}{1+x^2}\, dx = F(b) - F(a) = F(\sqrt{3}) - F\left(\frac{1}{\sqrt{3}}\right)$$

$$= 8\, tan^{-1}\left(\sqrt{3}\right) - 8\, tan^{-1}\left(\frac{1}{\sqrt{3}}\right)$$

$$= 8\left(\frac{\pi}{3}\right) - 8\left(\frac{\pi}{6}\right) = \frac{4}{3}\pi$$

(iii) $\int_{1}^{9} \frac{x-1}{\sqrt{x}}\, dx$

Solution:

Given $f(x) = \frac{x-1}{\sqrt{x}} = \sqrt{x} - \frac{1}{\sqrt{x}} = x^{1/2} - x^{-1/2}$ is continuous in the given interval.

By FTC 2, Anti derivative $F(x) = \frac{x^{3/2}}{3/2} - \frac{x^{1/2}}{1/2} = \frac{2}{3}x^{3/2} - 2\,x^{1/2}$

$$\int_{1}^{9} \frac{x-1}{\sqrt{x}}\, dx = F(b) - F(a) = F(9) - F(1)$$

$$= \left[\frac{2}{3}(9)^{3/2} - 2\,(9)^{1/2}\right] - \left[\frac{2}{3} - 2\right]$$

$$= (18-6) - \left(-\frac{4}{3}\right) = 12 + \frac{4}{3} = \frac{40}{3}$$

Example:

What is wrong with the calculation $\int_{0}^{\pi} sec^2\, x\; dx = 0$

Solution:

Given $f(x) = sec^2\, x = \frac{1}{cos^2\, x}\; 0 \leq x \leq \pi$

The fundamental theorem of calculus applies to continuous function.

Here, $f(x) = sec^2 x = \frac{1}{cos^2 x}$ is not continuous at $x = \frac{\pi}{2}$.

Since $f\left(\frac{\pi}{2}\right) = \frac{1}{cos^2 \frac{\pi}{2}} = \frac{1}{0} = \infty$

At $x = \frac{\pi}{2}$ the function $f(x) = sec^2 x$ is discontinuous.

So $\int_0^{\pi} sec^2 x \; dx$ does not exist.

Example:

What is wrong with the calculation $\int_{-1}^{3} \frac{dx}{x^2} = -\frac{4}{3}$

Solution:

The fundamental theorem of calculus applies to continuous function.

Here, $f(x) = \frac{1}{x^2}$ is not continuous at $[-1, 3]$.

That is f(x) is discontinuous at $x = 0$. So $\int_{-1}^{3} \frac{dx}{x^2}$ does not exist.

Example:

What is wrong with the calculation $\int_{\pi/3}^{\pi} sec\,\theta\, tan\,\theta\, d\theta = -3$

Solution:

Given $\int_{\pi/3}^{\pi} sec\,\theta\, tan\,\theta\, d\theta$

$$\int_{\pi/3}^{\pi} sec\,\theta\, tan\,\theta\, d\theta = [sec\,\theta]_{\pi/3}^{\pi} = -3$$

The fundamental theorem of calculus applies to continuous function.

Here, $f(\theta) = \sec\theta \tan\theta$ is not continuous on the interval $\left[\frac{\pi}{3}, \pi\right]$, since $\tan\frac{\pi}{2} = \infty$

Indefinite Integral

$\int g(x)dx = f(x) + C$ where C is the arbitrary constant of integration. By taking different values C we get any number of solution. Therefore $f(x) + C$ is called the indefinite integral of $g(x)$.

For convenience, we normally omit C when we evaluate an indefinite integral. As the fundamental theorem of calculus establish a connection between anti derivative and integrals. Thus $\int g(x)dx = f(x)\; means\; f'(x) = g(x)$.

Formulae

1. $\int k\, dx = kx + C$

2. $\int e^x \, dx = e^x + C$

3. $\int x^n \, dx = \frac{x^{n+1}}{n+1} + C \; (n \neq 1)$

4. $\int \frac{dx}{x} = \log x + C$

5. $\int a^x \, dx = a^x \log a + C$

6. $\int sinx \, dx = -\cos x + C$

7. $\int cosx \, dx = \sin x + C$

8. $\int sec^2 x \, dx = tanx + C$

9. $\int cosec^2 x \, dx = -cotx + C$

10. $\int sec\, x \, tan\, x \, dx = \sec x + C$

11. $\int cosec\, x \, cot\, x \, dx = -\operatorname{cosec} x + C$

12. $\int \tan x \, dx = \log \sec x + C$

13. $\int \cot x \, dx = \log \sin x + C$

14. $\int \sec x \, dx = \log(\sec x + \tan x) + C$

15. $\int cosec\, x \, dx = \log(\, cosec\, x - \cot x\,) + C$

16. $\int \frac{dx}{1+x^2} = tan^{-1} x + C$

17. $\int \frac{dx}{\sqrt{1-x^2}} = sin^{-1} x + C$

18. $\int sinhx \, dx = \cos hx + C$

19. $\int coshx \, dx = \sin \mathrm{h}\, x + C$

Example:

Evaluate $\int \frac{x^3 + 2x + 1}{x^4} \, dx$

Solution:

Given $\int \frac{x^3 + 2x + 1}{x^4} \, dx$

$$= \int \left(\frac{1}{x} + \frac{2}{x^3} + \frac{1}{x^4}\right) dx \quad = \int \left(\frac{1}{x} + 2x^{-3} + x^{-4}\right) dx$$

$$= logx + 2\frac{x^{-2}}{(-2)} + \frac{x^{-3}}{(-3)} + C$$

$$= logx - \frac{1}{x^2} - \frac{1}{3x^3} + C$$

Example:

Evaluate $\int \frac{x^3 - 2\sqrt{x}}{x}\, dx$

Solution:

Given $\int \frac{x^3 - 2\sqrt{x}}{x}\, dx$

$$= \int \left(x^2 - \frac{2}{\sqrt{x}}\right) dx \quad = \int \left(x^2 - 2x^{-1/2}\right) dx$$

$$= \frac{x^3}{3} - 2\,\frac{x^{1/2}}{1/2} + C \quad = \frac{1}{3}\,x^3 - 4\sqrt{x} + C$$

Example:

Evaluate $\int \left(x^{2/5} - x^{-3/5}\right)^2\, dx$

Solution:

Given $\int \left(x^{2/5} - x^{-3/5}\right)^2\, dx$

$$= \int \left[\left(x^{2/5}\right)^2 + \left(x^{-3/5}\right)^2 - 2\left(x^{2/5}\right)\left(x^{-3/5}\right) dx\right]$$

$$= \int \left[x^{4/5} + x^{-6/5} - 2\left(x^{-1/5}\right)\, dx\right]$$

$$= \frac{x^{\frac{4}{5}+1}}{\left(\frac{4}{5}+1\right)} + \frac{x^{\frac{-6}{5}+1}}{\left(-\frac{6}{5}+1\right)} - \frac{x^{\frac{-1}{5}+1}}{\left(-\frac{1}{5}+1\right)} + C$$

$$= \frac{5}{9}\,x^{9/5} - 5x^{-1/5} - \frac{5}{2}\,x^{4/5} + C$$

Example:

Evaluate $\int x^2\,(1-x)^2\, dx$

Solution:

Given $\int x^2\,(1-x)^2\, dx$

$$= \int x^2\,(1 + x^2 - 2x)\, dx$$

$$= \int (x^2 + x^4 - 2x^3)\, dx$$

$$= \frac{x^3}{3} + \frac{x^5}{5} - 2\,\frac{x^4}{4} + C$$

Example:

Evaluate $\int \frac{1}{1+\sin x}\, dx$

Solution:

Given $\int \frac{1}{1+\sin x}\, dx$

$$\int \frac{1}{1+\sin x}\, dx = \int \frac{1}{1+\sin x}\, \frac{1-\sin x}{1-\sin x}\, dx$$

$$= \int \frac{1-\sin x}{1-\sin^2 x}\, dx = \int \frac{1-\sin x}{\cos^2 x}\, dx$$

$$= \int [\sec^2 x - \sec x \tan x]\, dx$$

$$\left[\because \frac{1}{\cos x} = \sec x\, ;\ \frac{\sin x}{\cos x} = \tan x\right]$$

$$= \tan x - \sec x + C$$

Example:

Evaluate $\int \frac{\sin^2 x}{1+\cos x}\, dx$

Solution:

Given $\int \frac{\sin^2 x}{1+\cos x}\, dx = \int \frac{1-\cos^2 x}{1+\cos x}\, dx$ $\quad [\because \sin^2 x = 1 - \cos^2 x]$

$$= \int \frac{(1-\cos x)\,(1+\cos x)}{(1+\cos x)}\, dx$$

$$[\because a^2 - b^2 = (a-b)\,(a+b)]$$

$$= \int (1-\cos x)\, dx$$

$$= x - \sin x + C$$